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Variation on a Theme of Chebyshev: Sharp Estimates for the Leading Coefficients of Bounded Polynomials

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Abstract

The n-th Chebyshev polynomial of the first kind, T_n, maximizes various functionals on $\mathbf{B_n}$, the unit ball of real polynomials with respect to the uniform norm on $[-1,1]$, see e.g. [3], [18], [20], [37], [41]. The earliest example (1854) is Chebyshev's inequality [6] for the leading coefficient of $P_n \in \mathbf{B_n}$ (where $P_n(x) = \sum_{k=0}^{n} a_k x^k$ and degree $\le n$): (i) $|a_n| \le 2^{n-1}$. In 1892 V.A. Markov [16] found analogous sharp estimates for $|a_{n-1}|$ and for $|a_{n-2}|$, and Szegö did likewise for $|a_{n-1}|+|a_n|$, as published by Erdös in 1947 [12]. Only recently we have provided in [34] the sharp estimate for $|a_{n-2}|+|a_{n-1}|$ and have announced in [32] the exact upper bound for $|a_{n-2}+a_{n-1}+a_n|$. In Theorem 2.1 we solve the encompassing extremal problem of finding the sharp estimates for <u>all</u> possible compositions of the first three leading coefficients a_n, a_{n-1}, a_{n-2} of $P_n \in \mathbf{B_n}$ and even of $P_n \in \mathbf{C_n} = \{P_n : |P_n(\cos\frac{(n-i)\pi}{n})| \le 1 \text{ for } 0 \le i \le n\}$, where $\mathbf{C_n} \supset \mathbf{B_n}$ if $n \ge 2$. In Theorem 3.1 we furthermore provide the sharp estimates for selected compositions which additionaly contain the fourth leading coefficient, a_{n-3}. Altogether we so obtain a substantial amplification of (i) comprised of more than forty known and new estimates for leading coefficients of $P_n \in \mathbf{B_n}$ or $P_n \in \mathbf{C_n}$. It adds to the classical Approximation Theory and solves special cases of V.A. Markov's general extremal coefficient problem of 1892, see e.g. [2], [13], [16]. For all but four of these inequalities an extremizer within $\mathbf{C_n}$ is T_n (from some initial $n = n_0$ on). The four exceptional compositions include $|a_{n-1}|$ and $|a_{n-3}|$, whose extremizer is the implicitly defined Rogosinski polynomial $\Pi_{n-1} \in \mathbf{C_n}$, see [42] (1955). We reveal here (presumably for the first time in print) the explicit expressions for the maximizing two leading coefficients of Π_{n-1}. This complements a result of V.A. Markov [16] who determined the maximum of $|a_{n-1}|$ and $|a_{n-3}|$ by means of the two known leading coefficients of T_{n-1}, provided P_n is restricted to vary in $\mathbf{B_n}$. Finally, we turn to limitations of the extremizers T_n and Π_{n-1}: They both fail to maximize the composition $|a_{n-3}|+|a_{n-2}|+|a_{n-1}|$, if P_n varies in $\mathbf{C_n}$ (Theorem 4.1).

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1 Introduction and Historical Remarks

This paper is based on the author's contributed talk presented at the 3rd Dolomites Workshop on Constructive Approximation and Applications (Alba di Canazei (Trento), Italy, September 9 - 14, 2012).

Let $\mathbf{\Phi_n}$ denote the linear space of real algebraic (univariate) polynomials of degree $\le n$ with elements P_n given in power form by $P_n(x) = \sum_{k=0}^{n} a_k x^k$ $(n \ge 1, a_k \in \mathbb{R})$, and let $\mathbf{B_n}$ denote the unit ball in $\mathbf{\Phi_n}$ with respect to the uniform norm $||P_n||_{\mathbf{I},\infty} = \sup_{x\in\mathbf{I}} |P_n(x)|$ on the interval $\mathbf{I} = [-1,1]$:

$$\mathbf{B_n} = \{P_n \in \mathbf{\Phi_n} : ||P_n||_{\mathbf{I},\infty} \le 1\}. \tag{1}$$

The n-th Chebyshev polynomial of the first kind with respect to $\mathbf{I}$, T_n with $T_n(x) = \sum_{k=0}^{n} t_{n,k} x^k$, belongs to $\mathbf{B_n}$. It is recursively defined by

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),\ n \ge 2, \text{ with } T_0(x) = 1 \text{ and } T_1(x) = x, \tag{2}$$

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and is hence an even resp. odd polynomial, depending on the parity of n, so that $t_{n,k}=0$, if $n-k$ is odd, whereas, if $n-k$ is even, its coefficients $t_{n,k}$ are nonzero integers given by

$$t_{n,k}=t_{n,n-2q}=\frac{(-1)^q}{n-q}n2^{n-2q-1}\binom{n-q}{q},\ 0\le q\le\lfloor n/2\rfloor. \tag{3}$$

The extremal points of T_n are the alternation points $x^*_{n,i}$ where $T_n(x^*_{n,i})=(-1)^{n-i}$ holds, i.e,

$$x^*_{n,i}=\cos((n-i)\pi/n),\ 0\le i\le n,\ \text{ with } x^*_{n,i}+x^*_{n,n-i}=0 \tag{4}$$

(symmetry with respect to zero), and with ordering

$$-1=x^*_{n,0}<x^*_{n,1}<\ldots<x^*_{n,n-1}<x^*_{n,n}=1. \tag{5}$$

See the dedicated books [18], [24], and [41] for more information on T_n.
The following convex set $\mathbf{C_n}$ encompasses $\mathbf{B_n}$ for $n\ge 2$, see [41, p. 139], and hence contains T_n:

$$\mathbf{C_n}=\{P_n\in\mathbf{\Phi_n}:|P_n(x^*_{n,i})|\le 1 \text{ for } 0\le i\le n\},\ \text{ where } x^*_{n,i} \text{ is defined in (4)}. \tag{6}$$

Various extremal problems for polynomials have been solved first within $\mathbf{B_n}$ and were later extended to the superset $\mathbf{C_n}$, with $\pm T_n$ as the mutual extremizer, see e.g. the books [37, pp. 672], [41, pp. 107]. A well-known example is V.A. Markov's inequality [16, p. 93] for the uniform norm of the k-th derivative of $P_n\in\mathbf{B_n}$ on $\mathbf{I}$ and its refinement to $P_n\in\mathbf{C_n}$ by R.J. Duffin and A.C. Schaeffer [10] (1941), see also D.P. Dryanov [7] (2004). On the other hand, there are extremal problems where this pattern fails: The extremizer within $\mathbf{B_n}$ may be different from the extremizer within $\mathbf{C_n}$. We will encounter both instances below.

Our point of departure is P.L. Chebyshev's classical inequality for the magnitude of the leading coefficient of $P_n\in\mathbf{B_n}$:

Corollary 1.1. *Let $P_n\in\mathbf{B_n}$ with $P_n(x)=\sum_{k=0}^{n}a_kx^k$ be arbitrary. Then,*

$$|a_n|\le|t_{n,n}|=t_{n,n}=2^{n-1}\ \ (n\ge 1,\ \textit{equality if } P_n=\pm T_n). \tag{7}$$

This result follows from Chebyshev's pioneering approximation theorem of 1854 [6, p. 123] stating that among all $P_n\in\mathbf{\Phi_n}$ with $a_n=1$ the polynomial $2^{1-n}T_n$ deviates least from zero on $\mathbf{I}$, or equivalently, that the best approximation to x^n from the span of the monomials $1,x,x^2,\ldots,x^{n-1}$ is given by $x^n-2^{1-n}T_n(x)$, see e.g. the books [23, p. 39], [41, p. 98], and [43, p. 10; pp. 161]. In that paper [6, p. 112] ("Theórie des mécanismes...") Chebyshev refers to the work of the famous mechanical engineer J. Watt who, in his improvements to the steam engine, studied a mechanical linkage mechanism which converts rotational motion to approximate rectilinear motion. Chebyshev himself constructed several such linkages as well as related practical devices, for example, a wheel-chair and a mechanical spline ("adjustable arc curve ruler"), see the website `http://tcheb.ru` and the books [5], [21]. It was the famous chemist D.I. Mendeleev who in 1887, in his book on aqueous solutions [19, p. 289], explicitly asked for the extremal magnitude of all three coefficients of $P_2\in\mathbf{\Phi_2}$, assuming that $\mathbf{P_2}$ is uniformly bounded on a given compact interval, see [30] for details of Mendeleev's question and of A.A. Markov's solution as reported in a largely unknown footnote in Mendeleev's book [19, pp. 289]. Five years later, A.A.

Markov's younger half-brother, V.A. Markov, answered Mendeleev's question for arbitrary n, see e.g. [16, pp. 80], [17, p. 248], [23, p. 56], [43, p. 167]. In particular, he obtained for the second, third, and fourth leading coefficient of P_n the following sharp estimates (the estimate for the first leading coefficient is already covered by (7)):

Corollary 1.2. *Let $P_n\in\mathbf{B_n}$ with $P_n(x)=\sum_{k=0}^{n}a_kx^k$ be arbitrary. Then,*

$$|a_{n-1}|\le t_{n-1,n-1}=2^{n-2}\ \ (n\ge 2,\ \textit{equality if } P_n=\pm T_{n-1}), \tag{8}$$

$$|a_{n-2}|\le|t_{n,n-2}|=n2^{n-3}\ \ (n\ge 2,\ \textit{equality if } P_n=\pm T_n), \tag{9}$$

$$|a_{n-3}|\le|t_{n-1,n-3}|=(n-1)2^{n-4}\ \ (n\ge 3,\ \textit{equality if } P_n=\pm T_{n-1}). \tag{10}$$

In that same celebrated paper [16, p. 93] V.A. Markov also published the above mentioned inequality for the uniform norm of the k-th derivative of $P_n\in\mathbf{B_n}$ on $\mathbf{I}$. But both his classical results on extremal coefficients and on extremal pointwise k-th derivatives of $P_n\in\mathbf{B_n}$ (which imply the uniform norm of $P_n^{(k)}$) can be viewed as particular solutions to a quite general extremal problem which V.A. Markov himself had posed in 1892 [16, p. 79], see also [2, p. 4], [13, p. 39], [17, p. 246], [25, p. 701]:

V.A. Markov's Extremal Problem (Maximal Coefficient Functional Version).
Find the maximum of the coefficient functional

$$|a_0\beta_0+a_1\beta_1+a_2\beta_2+\ldots+a_n\beta_n|, \tag{11}$$

if P_n (with $P_n(x)=\sum_{k=0}^{n}a_kx^k$) varies in $\mathbf{B_n}$, where the β_k's are given real parameters. In fact, the two mentioned theorems of V.A. Markov yield as the desired maximum

(i) $|t_{n,k}|$ (if $n-k$ is even) resp. $|t_{n-1,k}|$ (if $n-k$ is odd), if one considers the functional $|\sum_{j=0}^{n} a_j\beta_j| = |a_k|$ in (11), with $\beta_k = 1$ and $\beta_j = 0$ if $j \neq k$. Obviously, (7) is the case $k = n$, i.e., $\max|a_n| = |t_{n,n}| = t_{n,n}$.

(ii) $|P_{*,n}^{(k)}(x_0)|$ ($P_{*,n}$ being a Chebyshev or a Zolotarev polynomial), if one considers the functional $|\sum_{j=0}^{n} a_j\beta_j| = |P_n^{(k)}(x_0)|$ in (11), with $\beta_j = \dfrac{x_0^{j-k}j!}{(j-k)!}$ $(1 \le k \le j \le n)$ and $\beta_j = 0$ else, where $x_0 \in \mathbf{I}$ is a given point. V.A. Markov additionally showed how the local maximal k-th derivative at x_0 (among all $P_n \in \mathbf{B_n}$) is dominated by the global one: $|P_{*,n}^{(k)}(x_0)| \le T_n^{(k)}(1)$ for all $x_0 \in \mathbf{I}$.

The alternative dual version of V.A. Markov's extremal problem reads as follows:

V.A. Markov's Extremal Problem (Least Deviation from Zero Version).
Among all $P_n \in \mathbf{\Phi_n}$ (with $P_n(x) = \sum_{k=0}^{n} a_k x^k$) whose coefficients satisfy the linear constraint

$$a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + \ldots + a_n\beta_n = 1, \tag{12}$$

where the β_k's are given real parameters, find the one which best approximates the zero function on $\mathbf{I}$.
Chebyshev's inequality (7) corresponds to the solution $2^{1-n}T_n$, if one sets $\sum_{j=0}^{n} a_j\beta_j = a_n = 1$ in (12). V.A. Markov's extremal problem can be generalized by assuming $P_n \in \mathbf{C_n}$ in place of $P_n \in \mathbf{B_n}$.

The inequalities compiled in Theorems 2.1 and 3.1 below contribute to the solution of this classical extremal problem (in its generalized form) for a special choice of parameters: $\beta_j \in \{-1, 0, 1\}$.

One solution, corresponding to $\beta_{k-1} = 1, \beta_k = \pm 1$, (where $n-k$ is even) and $\beta_j = 0$ else, was already provided by G. Szegö as published in a paper by P. Erdös [12, p. 1176] (1947), see also [30], [31], [37, p. 673; p. 679]. It covers consecutive pairs of coefficients of $P_n \in \mathbf{B_n}$ and in particular gives a striking refinement of (7) and (9):

Corollary 1.3. *Let $P_n \in \mathbf{B_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be arbitrary. Then,*

$$|a_{n-1} \pm a_n| \le |a_{n-1}| + |a_n| \le t_{n,n} \ \ (n \ge 1, \textit{ equality if } P_n = \pm T_n), \tag{13}$$

$$|a_{n-3} \pm a_{n-2}| \le |a_{n-3}| + |a_{n-2}| \le |t_{n,n-2}| \ \ (n \ge 3, \textit{ equality if } P_n = \pm T_n). \tag{14}$$

Only recently, we have provided in [34] a solution to (11) for similar special cases ($\beta_k = 1, \beta_{k+1} = \pm 1$, (where $n-k$ is even) and $\beta_j = 0$ else). It covers alternative pairs of consecutive coefficients of $P_n \in \mathbf{B_n}$ and in particular implies:

Corollary 1.4. *Let $P_n \in \mathbf{B_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be arbitrary. Then,*

$$|a_{n-2} \pm a_{n-1}| \le |a_{n-2}| + |a_{n-1}| \le |t_{n,n-2}| \ \ (n \ge 6, \textit{ equality if } P_n = \pm T_n), \tag{15}$$

$$|a_{n-4} \pm a_{n-3}| \le |a_{n-4}| + |a_{n-3}| \le t_{n,n-4} = n(n-3)2^{n-6} \ \ (n \ge 13, \textit{ equality if } P_n = \pm T_n). \tag{16}$$

Yet another such particular solution to V.A. Markov's extremal problem concerning the j-th backward partial sums of coefficients of $P_n \in \mathbf{B_n}$ (that is, $\sum_{k=j}^{n} a_k$, corresponding to $\beta_k = 1$ for $j \le k$ and $\beta_k = 0$ else) we have announced in [32]. If we confine ourselves to the backward partial sums of the first three ($j = n-2$) and four ($j = n-3$) leading coefficients of $P_n \in \mathbf{B_n}$, then this solution reads as follows:

Corollary 1.5. *Let $P_n \in \mathbf{B_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be arbitrary. Then,*

$$|a_{n-2} + a_{n-1} + a_n| \le |t_{n,n-2} + t_{n,n}| = (n-4)2^{n-3} \ \ (n \ge 10, \textit{ equality if } P_n = \pm T_n), \tag{17}$$

$$|a_{n-3} + a_{n-2} + a_{n-1} + a_n| \le |t_{n,n-2} + t_{n,n}| \ \ (n \ge 6, \textit{ equality if } P_n = \pm T_n). \tag{18}$$

Observe that

$$|t_{n,n-2} + t_{n,n}| = (n-4)2^{n-3} < |t_{n,n-2}| + t_{n,n} = (n+4)2^{n-3} \tag{19}$$

so that the trivial estimate $|a_{n-2} + a_{n-1} + a_n| \le |a_{n-2}| + |a_{n-1}| + |a_n| \le |t_{n,n-2}| + t_{n,n}$ (according to (7), (15) or (9), (13)) would not be sharp for $|a_{n-2} + a_{n-1} + a_n|$. An analogous remark applies to inequality (18).

We point out that related but weaker inequalities than those in Corollary 1.5 are contained in a result given by M. Reimer [39] (1968), see also [41, p. 112], which was derived under the stronger assumption that $P_n \in \mathbf{B_n}$ is even or odd (implying here: $a_{n-3} = a_{n-1} = 0$), according to the parity of n.

We now collect known liftings of the preceding coefficient estimates, from $P_n \in \mathbf{B_n}$ (i.e., $|P_n(x)| \le 1$ for $|x| \le 1$) to $P_n \in \mathbf{C_n}$ (i.e., $|P_n(x^*_{n,i})| \le 1$ for $0 \le i \le n$, so that P_n may attain values > 1 or < -1 between two consecutive alternation points of T_n):

The generalizations of (7) and (9) to $P_n \in \mathbf{C_n}$ are contained in a result of J.A. Shohat [44, p. 687] (1929), see also [37, pp. 672]. The generalizations of (13), (14) to $P_n \in \mathbf{C_n}$ are contained in a result of O.J. Munch [22, p. 26] (1960) (see also [30], [37, p. 673]), and have been rediscovered by several authors, e. g. [40]. The versions of (15), (16) for $P_n \in \mathbf{C_n}$ are contained in [34]. We will prove the generalization of Corollary 1.5 to $P_n \in \mathbf{C_n}$ in Theorem 2.1 resp. 3.1 below, but it is still a special case of a more general result announced in [32], which encompasses Reimer's [39]. In all these liftings from $\mathbf{B_n}$ to $\mathbf{C_n}$ the Chebyshev polynomial $T_n \in \mathbf{B_n}$ prevails its extremal property within $\mathbf{C_n}$. However, as for (8) and (10), simple examples will show that T_{n-1} is not extremal within $\mathbf{C_n}$ for $|a_{n-1}|$ or $|a_{n-3}|$. It was W.W. Rogosinski [42, p. 10] who, in 1955, determined the even respectively odd polynomial (depending on the parity of $n-1$) $\Pi_{n-1} \in \mathbf{C_n}$ whose leading coefficients maximize $|a_{n-1}|$ and $|a_{n-3}|$, if P_n varies in $\mathbf{C_n}$. But the given interpolatory definition of Π_{n-1} does not reveal the explicit expressions for the maximizing leading coefficients in the power form representation

$$\Pi_{n-1}(x) = \sum_{k=0}^{n-1} c_{n-1,k} x^k, n \ge 3. \tag{20}$$

Rogosinski's interpolatory definition of $\Pi_{n-1} \in \mathbf{C_n}$ reads (where $x^*_{n,i}$ is from (4)):

$$\begin{aligned} \Pi_{n-1}(x^*_{n,i}) &= (-1)^i, \text{ if } 0 \le i \le \frac{n-1}{2}, n \text{ odd}, && (21)\\ &= (-1)^{i+1}, \text{ if } \frac{n+1}{2} \le i \le n, n \text{ odd}; && (22) \end{aligned}$$

$$\begin{aligned} \Pi_{n-1}(x^*_{n,i}) &= (-1)^{i+1}, \text{ if } 0 \le i \le \frac{n}{2} - 1, n \text{ even}, && (23)\\ &= 0, \text{ if } i = \frac{n}{2}, n \text{ even}, && (24)\\ &= (-1)^i, \text{ if } \frac{n}{2} + 1 \le i \le n, n \text{ even}. && (25) \end{aligned}$$

The first few Rogosinski polynomials Π_{n-1} are displayed in power form in Example 3.2 below. We notice that Rogosinski [42] has actually considered an even more general setting than $P_n \in \mathbf{C_n}$.

Against the background of the preceding findings we now pose two encompassing extremal problems. The first one is to determine the sharp estimates for the magnitude of all compositions that can be generated from the first three leading coefficients a_n, a_{n-1}, a_{n-2} of a pointwise bounded $P_n \in \mathbf{C_n}$. For the sake of completeness we include appropriate compositions already considered in Corollaries 1.1 - 1.5 (valid for $P_n \in \mathbf{B_n}$), and in their liftings to $P_n \in \mathbf{C_n}$.

By a composition of leading coefficients, we mean the following coefficient functionals:

(i) the modulus of a single leading coefficient, or

(ii) the modulus of sums or differences of two or more distinct leading coefficients, or

(iii) a sum of compositions generated according to (i) and / or (ii), see Theorems 2.1 - 4.1.

Problem 1.6. Suppose $P_n \in \mathbf{C_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$. Find the sharp upper bounds for all compositions of the first three leading coefficients a_n, a_{n-1} and a_{n-2}.

We will obtain the complete solution to this problem in Theorem 2.1 below as a mixture of twenty known and new inequalities. It will turn out that, except for $|a_{n-1}|$, all remaining compositions are maximized by the corresponding compositions of the leading coefficients of $\pm T_n$ (from some initial $n = n_0$ on). If this is the case then, since $T_n \in \mathbf{B_n}$ and $T_n \in \mathbf{C_n}$, the resulting coefficient inequalities in particular hold sharply for all P_n from the subset $\mathbf{B_n}$ of $\mathbf{C_n}$. The modulus of the exceptional coefficient a_{n-1} of $P_n \in \mathbf{C_n}$ will be maximized by the corresponding leading coefficient $c_{n-1,n-1}$ of $\Pi_{n-1} \in \mathbf{C_n}$, according to [42]. We reveal here, presumably for the first time in print, the explicit expression for $c_{n-1,n-1}$ (with the aid of trigonometric functions). This is to be contrasted with V.A. Markov's result (8) stating that the modulus of a_{n-1} will be maximized by the (known) leading coefficient $t_{n-1,n-1}$ of $T_{n-1} \in \mathbf{B_{n-1}} \subset \mathbf{B_n}$, if P_n varies in the subset $\mathbf{B_n}$ of $\mathbf{C_n}$.

The second extremal problem is to determine the sharp estimates for the magnitude of selected compositions of the first four leading coefficients $a_n, a_{n-1}, a_{n-2}, a_{n-3}$ of a pointwise bounded $P_n \in \mathbf{C_n}$. By selected, we mean that

(i) the composition includes the fourth leading coefficient a_{n-3};

(ii) the composition does not contain the minus-sign;

(iii) the extremizer of the composition is either $\pm T_n$ or $\pm \Pi_{n-1}$.

For the sake of completeness we include appropriate compositions already considered in Corollaries 1.2-1.5 (valid for $P_n \in \mathbf{B_n}$), and in their liftings to $P_n \in \mathbf{C_n}$. This second more complex extremal problem leads to twenty-three coefficient inequalities.

Problem 1.7. Suppose $P_n \in \mathbf{C_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$. Find the sharp upper bounds for selected compositions of the first four leading coefficients $a_n, a_{n-1}, a_{n-2}, a_{n-3}$.

We will obtain the solution to this problem in Theorem 3.1 below, again as a mixture of known and new inequalities. Except for $|a_{n-3}|, |a_{n-3}+a_{n-1}|$, and $|a_{n-3}|+|a_{n-1}|$, all remaining compositions are maximized by the corresponding compositions of the leading coefficients of $\pm T_n$ (from some initial $n = n_0$ on), so that the resulting coefficient inequalities in particular hold sharply for all $P_n \in \mathbf{B_n}$. The modulus of the exceptional coefficient a_{n-3} of $P_n \in \mathbf{C_n}$ will be maximized by the modulus of the corresponding leading coefficient $c_{n-1,n-3}$ of $\Pi_{n-1} \in \mathbf{C_n}$, according to [42]. What again is novel here is our explicit expression for $|c_{n-1,n-3}|$ (with the aid of trigonometric functions), and this is to be contrasted with V.A. Markov's result (10) stating that $|a_{n-3}|$ will be maximized by the modulus of the (known) second nonzero leading coefficient $t_{n-1,n-3}$ of $T_{n-1} \in \mathbf{B_n}$, if P_n varies in the subset $\mathbf{B_n}$ of $\mathbf{C_n}$.

Altogether, the more than forty coefficient inequalities compiled in Theorems 2.1 and 3.1 constitute a substantial amplification of Chebyhev's classical inequality (7) and they reveal new extremal properties of the leading coefficients of T_n.

Finally, we turn to limitations of the extremizers $\pm T_{n-1}$ and $\pm \Pi_{n-1}$, and prove in Theorem 4.1 that they both fail to maximize the particular composition $|a_{n-3}|+|a_{n-2}|+|a_{n-1}|$, if P_n varies in $\mathbf{C_n}$.

Suggested additional reading on Chebyshev's circle of ideas: [1],[3],[14],[15],[20],[38], and [47].

2 Theorem 2.1, Proof, and Example

We now provide the solution to Problem 1.6. It covers <u>all</u> compositions that can be generated from the first three leading coefficients a_n, a_{n-1}, a_{n-2} of $P_n \in \mathbf{C_n}$ (expressions $|\gamma|$ and $|-\gamma|$ are not considered as different):

Theorem 2.1. (Variation on Chebyshev's Coefficient Inequality, Part 1) *Let $P_n \in \mathbf{C_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be arbitrary. Let $T_n \in \mathbf{C_n}$ with $T_n(x) = \sum_{k=0}^{n} t_{n,k} x^k$ denote the Chebyshev polynomial of the first kind, and let $\Pi_{n-1} \in \mathbf{C_n}$ with $\Pi_{n-1}(x) = \sum_{k=0}^{n-1} c_{n-1,k} x^k$ denote the Rogosinski polynomial. Then the following sharp estimates for compositions generated out of the first three leading coefficients of P_n hold true:*

$$\begin{aligned}
|a_n| &\le t_{n,n}, n \ge 1, && (26)\\
|a_{n-1}| &\le c_{n-1,n-1} = \frac{2^{n-1}}{n\tan\frac{\pi}{2n}}, n \ge 4 \textit{ even, equality if } P_n = \pm\Pi_{n-1}, && (27)\\
|a_{n-1}| &\le c_{n-1,n-1} = \frac{2^{n-1}}{n\sin\frac{\pi}{2n}}, n \ge 3 \textit{ odd, equality if } P_n = \pm\Pi_{n-1}, && (28)\\
|a_{n-2}| &\le |t_{n,n-2}|, n \ge 2, && (29)\\
|a_{n-2} \pm a_{n-1}| &\le |t_{n,n-2}|, n \ge 6, && (30)\\
|a_{n-2} + a_n| &\le |t_{n,n-2} + t_{n,n}|, n \ge 6, && (31)\\
|a_{n-2} - a_n| &\le |t_{n,n-2}| + t_{n,n}, n \ge 2, && (32)\\
|a_{n-1} \pm a_n| &\le t_{n,n}, n \ge 1, && (33)\\
|a_{n-2}| + |a_{n-1}| &\le |t_{n,n-2}|, n \ge 6, && (34)\\
|a_{n-2}| + |a_n| &\le |t_{n,n-2}| + t_{n,n}, n \ge 2, && (35)\\
|a_{n-1}| + |a_n| &\le t_{n,n}, n \ge 1, && (36)\\
|a_{n-2} + a_{n-1} + a_n| &\le |t_{n,n-2} + t_{n,n}|, n \ge 10, && (37)\\
|a_{n-2} + a_{n-1} - a_n| &\le |t_{n,n-2}| + t_{n,n}, n \ge 2, && (38)\\
|a_{n-2} - a_{n-1} + a_n| &\le |t_{n,n-2} + t_{n,n}|, n \ge 10, && (39)\\
|-a_{n-2} + a_{n-1} + a_n| &\le |t_{n,n-2}| + t_{n,n}, n \ge 2, && (40)\\
|a_{n-1}| + |a_{n-2} + a_n| &\le |t_{n,n-2} + t_{n,n}|, n \ge 10, && (41)\\
|a_{n-1}| + |a_{n-2} - a_n| &\le |t_{n,n-2}| + t_{n,n}, n \ge 2, && (42)\\
|a_{n-2}| + |a_{n-1} \pm a_n| &\le |t_{n,n-2}| + t_{n,n}, n \ge 2, && (43)\\
|a_n| + |a_{n-2} \pm a_{n-1}| &\le |t_{n,n-2}| + t_{n,n}, n \ge 2, && (44)\\
|a_{n-2}| + |a_{n-1}| + |a_n| &\le |t_{n,n-2}| + t_{n,n}, n \ge 2. && (45)
\end{aligned}$$

Except for (27) and (28), equality occurs if $P_n = \pm T_n$. If this is the case, then the corresponding coefficient estimates in particular hold sharply for $P_n \in \mathbf{B_n}$, since $T_n \in \mathbf{B_n}$. The upper bounds for $|a_{n-1}|$ as given in (27) and (28) can be replaced by the tighter sharp upper bound given in (8), if $P_n \in \mathbf{B_n}$. Sharp upper bounds for marginal low-degree constellations not covered by (30), (31), (34), (37), (39), (41) are provided in Remark 1 below.

Proof. In the course of the proof of Theorem 2.1 (and of Theorems 3.1 and 4.1) we will make use of the following identities, see [33]:

Let $(V_{i,k})_{0 \le i,k \le n}$ denote the inverse of the Vandermonde matrix associated with the extremal points (4) of T_n.

Lemma 2.2. *If $n-k$ is even and $0 \le i \le \lfloor n/2 \rfloor$, then [33, p. 298]*

$$V_{i,k} = (-1)^k V_{n-i,k} \text{ and } V_{i,k-1} = x^*_{n,i} V_{i,k}.$$

Lemma 2.3. *The elements $V_{i,k}, 0 \le i,k \le n$, are given as follows [33, p. 302]:*
The elements $V_{0,k}$, if $n-k$ is even, are

$$\begin{aligned} V_{0,n} &= (-1)^n n^{-1} 2^{n-2}, \\ V_{0,n-2q} &= (-1)^n n^{-1} 2^{n-2} \sum_{t=0}^{q} \sigma_{n+1,n+1-2(q-t)}, \text{ if } 1 \le q \le \lceil (n-2)/2 \rceil, \\ V_{0,0} &= 0, \text{ if } n \text{ is even.} \end{aligned}$$

The elements $V_{i,k}$ with $1 \le i \le \lfloor (n-1)/2 \rfloor$, if $n-k$ is even, are

$$\begin{aligned} V_{i,n} &= (-1)^{n-i} n^{-1} 2^{n-1}, \\ V_{i,n-2q} &= (-1)^{n-i} n^{-1} 2^{n-1} \sum_{t=0}^{q} (x^*_{n,i})^{2t} \sigma_{n+1,n+1-2(q-t)}, \text{ if } 1 \le q \le \lceil (n-2)/2 \rceil, \\ V_{i,0} &= 0, \text{ if } n \text{ is even.} \end{aligned}$$

The elements $V_{n/2,k}$, if n is even and $n-k$ is even, are

$$\begin{aligned} V_{n/2,n} &= (-1)^{n/2} n^{-1} 2^{n-1}, \\ V_{n/2,n-2q} &= (-1)^{n/2} n^{-2} (n-2q+1)^{-1} (n^2+n-2q) t_{n,n-2q}, \text{ if } 2 \le n-2q \le n-2, \\ V_{n/2,0} &= 1. \end{aligned}$$

Those elements $V_{i,k}$ not covered by the above identities can be recovered from these identities by applying Lemma 2.2.

Lemma 2.4. *The numbers $\sigma_{n+1,p}$, which appear in Lemma 2.3, are given (with the aid of the coefficients of T_n) as follows [33, p. 301]:*

$$\begin{aligned} \sigma_{n+1,n+1} &= 1, \\ \sigma_{n+1,p} &= p^{-1} n^{-1} 2^{1-n} (n^2+p-1) t_{n,p-1}, \text{ if } n+1-p \text{ is even and } 2 \le p \le n-1, \\ \sigma_{n+1,p} &= 0, \text{ if } n-p \text{ is even and } 2 \le p \le n, \\ \sigma_{n+1,1} &= (-1)^{n/2} n 2^{1-n}, \text{ if } n \text{ is even, resp. } \sigma_{n+1,1} = 0, \text{ if } n \text{ is odd,} \\ \sigma_{n+1,0} &= (-1)^{(n+1)/2} 2^{1-n}, \text{ if } n \text{ is odd, resp. } \sigma_{n+1,0} = 0, \text{ if } n \text{ is even.} \end{aligned}$$

Lemma 2.5. *The moduli of the coefficients of T_n and Π_{n-1} can be represented as [33, p. 299],*

$$|t_{n,k}| = \sum_{i=0}^{n} |V_{i,k}|, \text{ if } n-k \text{ is even and } |c_{n-1,k}| = \sum_{i=0}^{n} |V_{i,k}|, \text{ if } n-k \text{ is odd.}$$

Lemma 2.6. *A coefficient a_k of $P_n \in \mathbf{C_n}$ can be represented as [33, p. 296]*

$$a_k = \sum_{i=0}^{n} P_n(x^*_{n,i}) V_{i,k}.$$

Lemma 2.7. *For any real numbers α and β there holds the following identity, which goes back to A. Tarski [45]:*

$$||\alpha| - |\beta|| = |\alpha+\beta| + |\alpha-\beta| - |\alpha| - |\beta|.$$

It immediately implies the identities

$$|\alpha+\beta| + |\alpha-\beta| = 2\max\{|\alpha|,|\beta|\} \text{ and } |\alpha|+|\beta| = \max\{|\alpha+\beta|,|\alpha-\beta|\}.$$

Lemma 2.8. *The first and second (nonzero) leading coefficient of Π_{n-1} can be represented as [33, p. 311]*

$$\begin{aligned} c_{n-1,n-1} &= 2\frac{2^{n-2}}{n} + \frac{2^{n-1}}{n} \sum_{i=1}^{n-1} |x^*_{n,i}|, \\ |c_{n-1,n-3}| &= \frac{2^{n-1}}{n}\left(\frac{n}{4} - \frac{1}{2}\right) + \frac{2^n}{n} \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} |x^*_{n,i}| \left(\left(\frac{n}{4} + \frac{1}{2}\right) - |x^*_{n,i}|^2\right). \end{aligned}$$

We now turn to the proof of the inequalities (26) - (45) contained in Theorem 2.1.
(26): This is Chebyshev's inequality (7), as generalized to $P_n \in \mathbf{C_n}$ by Shohat [44, p. 687].
(27) and (28): The inequality $|a_{n-1}| \le c_{n-1,n-1}$ is due to Rogosinski [42, p. 10]. The explicit two-staged expression for the upper bound $c_{n-1,n-1}$ we deduce as follows:
According to the parity of n we obtain from Lemma 2.8, in view of (4),

$$\begin{aligned} c_{n-1,n-1} &= \frac{2^{n-1}}{n} + \frac{2^{n-1}}{n} 2 \sum_{i=1}^{n/2-1} |x^*_{n,i}| && (46)\\ &= \frac{2^{n-1}}{n} + \frac{2^{n}}{n} \sum_{k=1}^{n/2-1} \cos\frac{k\pi}{n}, \text{ if } n \text{ even,} && (47) \end{aligned}$$

respectively

$$\begin{aligned} c_{n-1,n-1} &= \frac{2^{n-1}}{n} + \frac{2^{n-1}}{n} 2 \sum_{i=1}^{(n-1)/2} |x^*_{n,i}| && (48)\\ &= \frac{2^{n-1}}{n} + \frac{2^{n}}{n} \sum_{k=1}^{(n-1)/2} \cos\frac{k\pi}{n}, \text{ if } n \text{ odd.} && (49) \end{aligned}$$

We now deploy the known J.L. Lagrange identity (see e.g. [26])

$$\sum_{k=1}^{N} \cos(kx) = -\frac{1}{2} + \frac{\sin((N+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} \qquad (50)$$

to get with $x = \pi/n$ and $N = n/2-1$, if n is even, respectively $N = (n-1)/2$, if n is odd,

$$\begin{aligned} c_{n-1,n-1} &= \frac{2^{n-1}}{n}\left(1+2\left(-\frac{1}{2}+\frac{\sin(\frac{\pi}{2}-\frac{\pi}{2n})}{2\sin(\frac{\pi}{2n})}\right)\right)\\ &= \frac{2^{n-1}}{n}\left(1+2\left(-\frac{1}{2}+\frac{\cos(\frac{\pi}{2n})}{2\sin(\frac{\pi}{2n})}\right)\right)\\ &= \frac{2^{n-1}}{n}\cot\left(\frac{\pi}{2n}\right), \text{ if } n \text{ even,} && (51) \end{aligned}$$

respectively

$$\begin{aligned} c_{n-1,n-1} &= \frac{2^{n-1}}{n}\left(1+2\left(-\frac{1}{2}+\frac{\sin(\frac{\pi}{2})}{2\sin(\frac{\pi}{2n})}\right)\right)\\ &= \frac{2^{n-1}}{n}\left(1+2\left(-\frac{1}{2}+\frac{1}{2\sin(\frac{\pi}{2n})}\right)\right)\\ &= \frac{2^{n-1}}{n}\csc\left(\frac{\pi}{2n}\right), \text{ if } n \text{ odd.} && (52) \end{aligned}$$

(29): This is V.A. Markov's inequality (9), as generalized to $P_n \in \mathbf{C_n}$ by Shohat [44, p. 687].
(30): This follows from (34) by the triangle inequality, compare with (15).
(31): This follows from (41), if $n \ge 10$. The elementary proof of the marginal cases $n \in \{6,7,8,9\}$ is similar to the proof given in [33], Section 3.2, for comparable marginal cases. Therefore we only give here the straightforward proof for $n = 6$ and leave the proof for $n \in \{7,8,9\}$ to the reader. From Lemmas 2.2 and 2.6 we adopt for the here relevant coefficients of $P_6 \in \mathbf{C_6}$ the upper bound

$$\begin{aligned} |a_4 + a_6| &= |\sum_{i=0}^{6} P_6(x^*_{6,i})(V_{i,4}+V_{i,6})| \le \sum_{i=0}^{6} |P_6(x^*_{6,i})||V_{i,4}+V_{i,6}|\\ &\le \sum_{i=0}^{6} |V_{i,4}+V_{i,6}| = 2\sum_{i=0}^{2} |V_{i,4}+V_{i,6}| + |V_{3,4}+V_{3,6}|. \end{aligned}$$

From Lemma 2.3 we deduce, with $|x^*_{6,0}|^2 = 1, |x^*_{6,1}|^2 = \frac{3}{4}, |x^*_{6,2}|^2 = \frac{1}{4}$, and $|x^*_{6,3}|^2 = 0$ (see (4)): $V_{0,4} = \frac{8}{3}(\sigma_{7,5}+1) = -\frac{8}{3}, V_{1,4} = \frac{-16}{3}(\sigma_{7,5}+\frac{3}{4}) = \frac{20}{3}, V_{2,4} = \frac{16}{3}(\sigma_{7,5}+\frac{1}{4}) = \frac{-28}{3}, V_{3,4} = \frac{-16}{3}(\sigma_{7,5}+0) = \frac{32}{3}$, because $\sigma_{7,5} = -2$ (see Lemma 2.4). Furthermore we get $V_{0,6} = \frac{8}{3}, V_{1,6} = \frac{-16}{3}, V_{2,6} = \frac{16}{3}, V_{3,6} = \frac{-16}{3}$. Hence, $2\sum_{i=0}^{2}|V_{i,4}+V_{i,6}| + |V_{3,4}+V_{3,6}| = 16 = |-48+32| = |t_{6,4}+t_{6,6}|$, as claimed.

(32): This follows by the triangle inequality and from (26) and (29).
(33): This follows by the triangle inequality and from (36), compare with (13).
(34): This is our result from [34], compare with (15).
(35): This follows from (26) and (29).
(36): This is Szegö's inequality (13), as generalized to $P_n \in \mathbf{C_n}$ by Munch [22, p. 26].
(37): This follows by the triangle inequality and from (41).
(38): This follows by the triangle inequality and from (29) and (33).
(39): This follows by the triangle inequality and from (41).
(40): This follows by the triangle inequality and from (29) and (33).
(41): It will suffice to consider the case of $n \geq 11$ odd since the proof is quite similar for $n \geq 10$ even. From Lemma 2.6 we adopt for the here relevant coefficients of $P_n \in \mathbf{C_n}$ the upper bound

$$\begin{aligned} |a_{n-2}+a_{n-1}+a_n| &= |\sum_{i=0}^{n} P_n(x^*_{n,i})(V_{i,n-2}+V_{i,n-1}+V_{i,n})| \qquad (53) \\ &\leq \sum_{i=0}^{n} |P_n(x^*_{n,i})||V_{i,n-2}+V_{i,n-1}+V_{i,n}| \\ &\leq \sum_{i=0}^{n} |V_{i,n-2}+V_{i,n-1}+V_{i,n}|. \qquad (54) \end{aligned}$$

We now take advantage of the symmetries in Lemma 2.2 to rewrite this upper bound as

$$|a_{n-2}+a_{n-1}+a_n| \leq \sum_{i=0}^{(n-1)/2} |V_{i,n-2}+V_{i,n-1}+V_{i,n}| + \sum_{i=0}^{(n-1)/2} |-V_{i,n-2}+V_{i,n-1}-V_{i,n}|. \qquad (55)$$

In view of Lemma 2.7 we obtain the modified upper bound

$$|a_{n-2}+a_{n-1}+a_n| \leq 2\sum_{i=0}^{(n-1)/2} \max\{|V_{i,n-1}|, |V_{i,n-2}+V_{i,n}|\}. \qquad (56)$$

Suppose we had, for $n \geq 11$ odd, $|V_{i,n-1}| \leq |V_{i,n-2}+V_{i,n}|$ for $i = 0, 1, ..., (n-1)/2$. This would imply

$$|a_{n-2}+a_{n-1}+a_n| \leq 2\sum_{i=0}^{(n-1)/2} |V_{i,n-2}+V_{i,n}| = \sum_{i=0}^{n} |V_{i,n-2}+V_{i,n}|, \qquad (57)$$

where the last identity again follows from Lemma 2.2. For i odd we rewrite $|V_{i,n-2}+V_{i,n}| = |-V_{i,n-2}+(-V_{i,n})|$ and observe, based on Lemma 2.3, that $V_{i,n-2} > 0$ if i is even and $(-V_{i,n-2}) > 0$ if i is odd, and likewise $V_{i,n} < 0$ if i is even and $(-V_{i,n}) < 0$ if i is odd. We would obtain

$$|a_{n-2}+a_{n-1}+a_n| \leq \sum_{i\ even} |V_{i,n-2}+V_{i,n}| + \sum_{i\ odd} |-V_{i,n-2}+(-V_{i,n})| = \sum_{i=0}^{n} ||V_{i,n-2}|-|V_{i,n}||. \qquad (58)$$

Suppose further we had $|V_{i,n-2}|-|V_{i,n}| \geq 0$ for $i = 0, 1, ..., n$. We could then continue to write

$$|a_{n-2}+a_{n-1}+a_n| \leq \sum_{i=0}^{n}(|V_{i,n-2}|-|V_{i,n}|) = \sum_{i=0}^{n}|V_{i,n-2}| - \sum_{i=0}^{n}|V_{i,n}| = |t_{n,n-2}|-|t_{n,n}|, \qquad (59)$$

where the last identity follows from Lemma 2.5. From (3) we deduce that $t_{n,n-2} < 0$ and $t_{n,n} > 0$, and $|t_{n,n-2}| - t_{n,n} > 0$ for $n \geq 11$, so that we eventually would get

$$|a_{n-2}+a_{n-1}+a_n| \leq |t_{n,n-2}|-|t_{n,n}| = ||t_{n,n-2}|-t_{n,n}| = |-|t_{n,n-2}|+t_{n,n}| = |t_{n,n-2}+t_{n,n}|, \text{ for } n \geq 11. \qquad (60)$$

If this were true for $P_n \in \mathbf{C_n}$, then there would also hold

$$|a_{n-2}-a_{n-1}+a_n| \leq |t_{n,n-2}+t_{n,n}| \text{ for } P_n \in \mathbf{C_n} \text{ since } Q_n \text{ with } Q_n(x) = P_n(-x) \text{ belongs to } \mathbf{C_n}. \qquad (61)$$

Hence, in view of Lemma 2.7, $|a_{n-1}|+|a_{n-2}+a_n| \leq |t_{n,n-2}+t_{n,n}|$, for $n \geq 11$, as claimed.
We now proceed to verify the two assumptions made above:
(A) To show that $|V_{i,n-1}| \leq |V_{i,n-2}+V_{i,n}|$ for $i = 0, 1, ..., (n-1)/2$ and $n \geq 11$ odd, we recall from Lemmas 2.2, 2.3, and 2.4 that

$$\begin{aligned} |V_{0,n-1}| &= |x^*_{n,0}||V_{0,n}| = |V_{0,n}| = n^{-1}2^{n-2}, \text{ and} \qquad (62) \\ |V_{0,n-2}+V_{0,n}| &= n^{-1}2^{n-2}\left|\frac{n}{4}-\frac{3}{2}\right|, \qquad (63) \end{aligned}$$

since $V_{0,n-2}=(-1)^n n^{-1}2^{n-2}(\sigma_{n+1,n-1}+\sigma_{n+1,n+1})=-n^{-1}2^{n-2}((n-1)^{-1}n^{-1}2^{1-n}(n^2+n-2)t_{n,n-2}+1)$ with $t_{n,n-2}=-n2^{n-3}$, and $V_{0,n}=(-1)^n n^{-1}2^{n-2}$. As $\left|\frac{n}{4}-\frac{3}{2}\right|>1$ for $n\geq 11$, we get $|V_{0,n-1}|<|V_{0,n-2}+V_{0,n}|$. We furthermore recall from Lemmas 2.2 and 2.3 that

$$\begin{aligned} |V_{i,n-1}| &= |x^*_{n,i}||V_{i,n}|=|x^*_{n,i}|n^{-1}2^{n-1}<n^{-1}2^{n-1} \quad \text{for } i=1,2,\ldots,(n-1)/2, && (64)\\ |V_{i,n-2}+V_{i,n}| &= n^{-1}2^{n-1}\left|\frac{n}{4}-\frac{1}{2}-|x^*_{n,i}|^2\right|. && (65)\end{aligned}$$

Since $0<|x^*_{n,i}|<1$, we have $\left|\frac{n}{4}-\frac{1}{2}-|x^*_{n,i}|^2\right|>1$ for $n\geq 11$ which implies $|V_{i,n-1}|<|V_{i,n-2}+V_{i,n}|$.
(B) To show that $|V_{i,n-2}|-|V_{i,n}|\geq 0$ for $0\leq i\leq n$ and $n\geq 11$ odd, we recall from the lines after (63),

$$|V_{0,n}|=n^{-1}2^{n-2} \text{ and } |V_{0,n-2}|=n^{-1}2^{n-2}|\frac{n}{4}-\frac{1}{2}|. \tag{66}$$

From these identities it is obvious that $|V_{0,n}|\leq|V_{0,n-2}|$ for $n\geq 11$ odd. We furthermore recall from Lemmas 2.2 and 2.3 that

$$\begin{aligned} |V_{i,n}| &= n^{-1}2^{n-1} \text{ and}\\ |V_{i,n-2}| &= n^{-1}2^{n-1}\left|\frac{n}{4}+\frac{1}{2}-|x^*_{n,i}|^2\right|, \quad \text{for } i=1,2,\ldots,(n-1)/2. && (67)\end{aligned}$$

Since $0<|x^*_{n,i}|<1$, we have $\left|\frac{n}{4}+\frac{1}{2}-|x^*_{n,i}|^2\right|>1$ for $n\geq 11$ which implies $|V_{i,n}|\leq|V_{i,n-2}|$. The remaining cases $(n-1)/2<i\leq n$ follow by symmetry in view of Lemma 2.2. This concludes the proof of (41).
(42): This follows by the triangle inequality and from (29) and (36).
(43): This follows from (29) and (33).
(44): This follows by the triangle inequality and from (45).
(45): This follows from (29) and (36).
Except for (27) and (28), the condition for equality is due to the fact that T_n is even or odd, depending on the parity of n, and that the nonzero coefficients of T_n alternate in sign, see (3). If $P_n\in\mathbf{B_n}$, then the sharp upper bound for $|a_{n-1}|$ is $t_{n-1,n-1}$ as follows from (8). □

Example 2.1. For the least common initial value $n=n_0=10$ in Theorem 2.1 the coefficient inequalities read:

$$\begin{aligned}
|a_{10}| &\leq t_{10,10}=512\\
|a_9| &\leq c_{9,9}=\frac{51.2}{\tan\frac{\pi}{20}}=323.26407\ldots\\
|a_8| &\leq |t_{10,8}|=1280\\
|a_8\pm a_9| &\leq |t_{10,8}|=1280\\
|a_8+a_{10}| &\leq |t_{10,8}+t_{10,10}|=768\\
|a_8-a_{10}| &\leq |t_{10,8}|+t_{10,10}=1792\\
|a_9\pm a_{10}| &\leq t_{10,10}=512\\
|a_8|+|a_9| &\leq |t_{10,8}|=1280\\
|a_8|+|a_{10}| &\leq |t_{10,8}|+t_{10,10}=1792\\
|a_9|+|a_{10}| &\leq t_{10,10}=512\\
|a_8+a_9+a_{10}| &\leq |t_{10,8}+t_{10,10}|=768\\
|a_8+a_9-a_{10}| &\leq |t_{10,8}|+t_{10,10}=1792\\
|a_8-a_9+a_{10}| &\leq |t_{10,8}+t_{10,10}|=768\\
|-a_8+a_9+a_{10}| &\leq |t_{10,8}|+t_{10,10}=1792\\
|a_9|+|a_8+a_{10}| &\leq |t_{10,8}+t_{10,10}|=768\\
|a_9|+|a_8-a_{10}| &\leq |t_{10,8}|+t_{10,10}=1792\\
|a_8|+|a_9\pm a_{10}| &\leq |t_{10,8}|+t_{10,10}=1792\\
|a_{10}|+|a_8\pm a_9| &\leq |t_{10,8}|+t_{10,10}=1792\\
|a_8|+|a_9|+|a_{10}| &\leq |t_{10,8}|+t_{10,10}=1792.
\end{aligned}$$

3 Theorem 3.1, Proof, and Example

Next we will provide the solution to Problem 1.7. It covers selected compositions generated from the first four leading coefficients $a_n,a_{n-1},a_{n-2},a_{n-3}$ of $P_n\in\mathbf{C_n}$. As a_{n-3} is contained in each such composition, there will be no intersection with Theorem 2.1 (again, expressions $|\gamma|$ and $|-\gamma|$ are not considered as different):

Theorem 3.1. (Variation on Chebyshev's Coefficient Inequality, Part 2) *Let $P_n \in \mathbf{C_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be arbitrary. Let $T_n \in \mathbf{C_n}$ with $T_n(x) = \sum_{k=0}^{n} t_{n,k} x^k$ denote the Chebyshev polynomial of the first kind, and let $\Pi_{n-1} \in \mathbf{C_n}$ with $\Pi_{n-1}(x) = \sum_{k=0}^{n-1} c_{n-1,k} x^k$ denote the Rogosinski polynomial. Then the following sharp estimates for compositions generated out of the first four leading coefficients of P_n hold true:*

$$|a_{n-3}| \leq |c_{n-1,n-3}| = \frac{2^{n-3}}{n}\left(\frac{1}{\tan\frac{3\pi}{2n}} + \frac{n-1}{\tan\frac{\pi}{2n}}\right), n \geq 4 \text{ even, equality if } P_n = \pm\Pi_{n-1}, \tag{68}$$

$$|a_{n-3}| \leq |c_{n-1,n-3}| = \frac{2^{n-3}}{n}\left(\frac{1}{\sin\frac{3\pi}{2n}} + \frac{n-1}{\sin\frac{\pi}{2n}}\right), n \geq 3 \text{ odd, equality if } P_n = \pm\Pi_{n-1}, \tag{69}$$

$$|a_{n-3}+a_{n-2}| \leq |t_{n,n-2}|, n \geq 3, \tag{70}$$

$$|a_{n-3}+a_{n-1}| \leq |c_{n-1,n-3}+c_{n-1,n-1}|, n \geq 6, \text{ equality if } P_n = \pm\Pi_{n-1}, \tag{71}$$

$$|a_{n-3}|+|a_{n-2}| \leq |t_{n,n-2}|, n \geq 3, \tag{72}$$

$$|a_{n-3}|+|a_{n-1}| \leq |c_{n-1,n-3}|+|c_{n-1,n-1}|, n \geq 3, \text{ equality if } P_n = \pm\Pi_{n-1}, \tag{73}$$

$$|a_{n-3}+a_{n-2}+a_{n-1}| \leq |t_{n,n-2}|, n \geq 4, \tag{74}$$

$$|a_{n-2}|+|a_{n-3}+a_{n-1}| \leq |t_{n,n-2}|, n \geq 4, \tag{75}$$

$$|a_{n-2}|+|a_{n-3}+a_n| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{76}$$

$$|a_n|+|a_{n-3}+a_{n-2}| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{77}$$

$$|a_{n-3}|+|a_{n-2}|+|a_n| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{78}$$

$$|a_{n-3}+a_{n-2}+a_{n-1}+a_n| \leq |t_{n,n-2}+t_{n,n}|, n \geq 6, \tag{79}$$

$$|a_{n-2}|+|a_{n-3}+a_{n-1}+a_n| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{80}$$

$$|a_n|+|a_{n-3}+a_{n-2}+a_{n-1}| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{81}$$

$$|a_{n-3}+a_{n-2}|+|a_{n-1}+a_n| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{82}$$

$$|a_{n-3}+a_{n-1}|+|a_{n-2}+a_n| \leq |t_{n,n-2}+t_{n,n}|, n \geq 6, \tag{83}$$

$$|a_{n-3}+a_n|+|a_{n-2}+a_{n-1}| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{84}$$

$$|a_{n-3}|+|a_{n-2}|+|a_{n-1}+a_n| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{85}$$

$$|a_{n-3}|+|a_n|+|a_{n-2}+a_{n-1}| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{86}$$

$$|a_{n-2}|+|a_{n-1}|+|a_{n-3}+a_n| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{87}$$

$$|a_{n-2}|+|a_n|+|a_{n-3}+a_{n-1}| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{88}$$

$$|a_{n-1}|+|a_n|+|a_{n-3}+a_{n-2}| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3, \tag{89}$$

$$|a_{n-3}|+|a_{n-2}|+|a_{n-1}|+|a_n| \leq |t_{n,n-2}|+t_{n,n}, n \geq 3. \tag{90}$$

Except for (68), (69), (71), and (73), equality occurs if $P_n = \pm T_n$. If this is the case, then the corresponding coefficient estimates in particular hold sharply for $P_n \in \mathbf{B_n}$, since $T_n \in \mathbf{B_n}$. The upper bounds for $|a_{n-3}|$ as given in (68) and (69) can be replaced by the tighter sharp upper bound given in (10), if $P_n \in \mathbf{B_n}$. The upper bound for $|a_{n-3}|+|a_{n-1}|$ as given in (73) can be replaced by the tighter sharp upper bound $|t_{n-1,n-3}|+|t_{n-1,n-1}| = (n+3)2^{n-4}$ (according to (8) and (10)), if $P_n \in \mathbf{B_n}$. The upper bound for $|a_{n-3}+a_{n-1}|$ as given in (71) can be replaced by the tighter sharp upper bound $|t_{n-1,n-3}+t_{n-1,n-1}| = (n-5)2^{n-4}$ $(n \geq 7)$, if $P_n \in \mathbf{B_n}$. Sharp upper bounds for marginal low-degree constellations not covered by (71), (74), (75), (79), (83) are provided in Remark 1 below.

Proof. (68), (69): The inequality $|a_{n-3}| \leq |c_{n-1,n-3}|$ is due to Rogosinski [42, p. 10]. The explicit two-staged expression for the upper bound $|c_{n-1,n-3}|$ we deduce as follows: From Lemma 2.8 we adopt the representation

$$\begin{aligned} |c_{n-1,n-3}| &= \frac{2^{n-1}}{n}\left(\frac{n}{4}-\frac{1}{2}\right) + \frac{2^n}{n}\sum_{i=1}^{\lfloor (n-1)/2\rfloor} |x_{n,i}^*|\left(\left(\frac{n}{4}+\frac{1}{2}\right) - |x_{n,i}^*|^2\right) \\ &= \frac{2^{n-1}}{n}\left(\frac{n}{4}-\frac{1}{2}\right) + \frac{2^n}{n}\left(\frac{n}{4}+\frac{1}{2}\right)\sum_{i=1}^{\lfloor (n-1)/2\rfloor} |x_{n,i}^*| - \frac{2^n}{n}\sum_{i=1}^{\lfloor (n-1)/2\rfloor} |x_{n,i}^*|^3. \end{aligned} \tag{91}$$

According to the parity of n we furthermore get in view of (4):

$$\begin{aligned}|c_{n-1,n-3}| &= \frac{2^{n-1}}{n}\left(\frac{n}{4}-\frac{1}{2}\right)+\frac{2^n}{n}\left(\frac{n}{4}+\frac{1}{2}\right)\sum_{k=1}^{n/2-1}\cos\frac{k\pi}{n}-\\ &\quad -\frac{2^n}{n}\sum_{k=1}^{n/2-1}\cos^3\frac{k\pi}{n}, n \text{ even} \end{aligned}\tag{92}$$

$$\begin{aligned}|c_{n-1,n-3}| &= \frac{2^{n-1}}{n}\left(\frac{n}{4}-\frac{1}{2}\right)+\frac{2^n}{n}\left(\frac{n}{4}+\frac{1}{2}\right)\sum_{k=1}^{(n-1)/2}\cos\frac{k\pi}{n}-\\ &\quad -\frac{2^n}{n}\sum_{k=1}^{(n-1)/2}\cos^3\frac{k\pi}{n}, n \text{ odd.} \end{aligned}\tag{93}$$

We now deploy the known identities (50) and (see e.g. [26])

$$\sum_{k=1}^{N}\cos^3(kx)=\frac{3}{4}\cos\frac{(N+1)x}{2}\sin\frac{Nx}{2}\csc\frac{x}{2}+\frac{1}{4}\cos\frac{3(N+1)x}{2}\sin\frac{3Nx}{2}\csc\frac{3x}{2}\tag{94}$$

to obtain with $x=\frac{\pi}{n}$ and $N=\frac{n}{2}-1$, if n is even, respectively $N=(n-1)/2$, if n is odd,

$$\begin{aligned}|c_{n-1,n-3}| &= \frac{2^{n-1}}{n}\left(\frac{n}{4}-\frac{1}{2}\right)+\frac{2^n}{n}\left(\frac{n}{4}+\frac{1}{2}\right)\left(-\frac{1}{2}+\frac{1}{2\tan\frac{\pi}{2n}}\right)-\\ &\quad -\frac{2^n}{n}\left(\frac{3}{8\tan\frac{\pi}{2n}}-\frac{3}{8}\right)-\frac{2^n}{n}\left(\frac{-1}{8\tan\frac{3\pi}{2n}}-\frac{1}{8}\right)\\ &= \frac{2^{n-3}}{n}\left(\cot\frac{3\pi}{2n}+(n-1)\cot\frac{\pi}{2n}\right), n \text{ even}\end{aligned}\tag{95}$$

respectively

$$\begin{aligned}|c_{n-1,n-3}| &= \frac{2^{n-1}}{n}\left(\frac{n}{4}-\frac{1}{2}\right)+\frac{2^n}{n}\left(\frac{n}{4}+\frac{1}{2}\right)\left(-\frac{1}{2}+\frac{1}{2\sin\frac{\pi}{2n}}\right)-\\ &\quad -\frac{2^n}{n}\left(\frac{3}{4}\cos\frac{(n+1)\pi}{4n}\sin\frac{(n-1)\pi}{4n}\csc\frac{\pi}{2n}\right)-\\ &\quad -\frac{2^n}{n}\left(\frac{1}{4}\cos\frac{3(n+1)\pi}{4n}\sin\frac{3(n-1)\pi}{4n}\csc\frac{3\pi}{2n}\right)\\ &= \frac{2^{n-3}}{n}\left(\csc\frac{3\pi}{2n}+(n-1)\csc\frac{\pi}{2n}\right), n \text{ odd.}\end{aligned}\tag{96}$$

(70): This follows by the triangle inequality and from (72), compare with (14).
(71): It will suffice to consider the case of $n\geq 7$ odd. From Lemma 2.2 we deduce the upper bound

$$\begin{aligned}|a_{n-3}+a_{n-1}| &= |\sum_{i=0}^{n}P_n(x^*_{n,i})(V_{i,n-3}+V_{i,n-1})|\leq\sum_{i=0}^{n}|P_n(x^*_{n,i})||V_{i,n-3}+V_{i,n-1}|\\ &\leq \sum_{i=0}^{n}|V_{i,n-3}+V_{i,n-1}|.\end{aligned}\tag{97}$$

Lemmas 2.2 and 2.3 imply that the values $V_{i,n-3}$ and $V_{i,n-1}$ alternate in sign as follows:

$$sign\ V_{i,n-3}=(-1)^{i+1}, 0\leq i\leq (n-1)/2, \text{ and } sign\ V_{i,n-3}=(-1)^{i}, (n+1)/2\leq i\leq n,\tag{98}$$

so that the consecutive values $V_{(n-1)/2,n-3}$ and $V_{(n+1)/2,n-3}$ exhibit the same sign. Similarly,

$$sign\ V_{i,n-1}=(-1)^{i}, 0\leq i\leq (n-1)/2, \text{ and } sign\ V_{i,n-1}=(-1)^{i+1}, (n+1)/2\leq i\leq n,\tag{99}$$

so that the consecutive values $V_{(n-1)/2,n-1}$ and $V_{(n+1)/2,n-1}$ exhibit the same sign. We furthermore know from (64), (66), and (67) that
for $i=0: |V_{0,n-1}|=|V_{0,n}|=n^{-1}2^{n-2}\leq|V_{0,n-3}|=|V_{0,n-2}|=n^{-1}2^{n-2}\left|\frac{n}{4}-\frac{1}{2}\right|$ holds (which is false if $n=5$),
for $i>0: |V_{i,n-1}|=|x^*_{n,i}||V_{i,n}|=|x^*_{n,i}|n^{-1}2^{n-1}\leq|V_{i,n-3}|=|x^*_{n,i}||V_{i,n-2}|=$
$|x^*_{n,i}|n^{-1}2^{n-1}\left|\frac{n}{4}+\frac{1}{2}-|x^*_{n,i}|^2\right|$ holds.

With this information at hand, and invoking Lemma 2.5, we can rewrite the bound (97) as

$$\sum_{i=0}^{n}|V_{i,n-3}+V_{i,n-1}|=\sum_{i=0}^{n}(|V_{i,n-3}|-|V_{i,n-1}|)=\sum_{i=0}^{n}|V_{i,n-3}|-\sum_{i=0}^{n}|V_{i,n-1}|=|c_{n-1,n-3}|-|c_{n-1,n-1}|. \tag{100}$$

From (28) and (69) there immediately follows that $c_{n-1,n-1}>0$ and $|c_{n-1,n-3}|>|c_{n-1,n-1}|$. Suppose that we additionally had $c_{n-1,n-3}<0$. Then we could continue to rewrite the bound (100) as

$$|c_{n-1,n-3}|-|c_{n-1,n-1}|=-c_{n-1,n-3}-c_{n-1,n-1}=-(c_{n-1,n-3}+c_{n-1,n-1})=|c_{n-1,n-3}+c_{n-1,n-1}|, \tag{101}$$

as claimed. It thus remains to verify the assumption $c_{n-1,n-3}<0$. But this follows easily from Lemma 2.6 and (21), (22), (98) which imply that in the representation $c_{n-1,n-3}=\sum_{i=0}^{n}\Pi_{n-1}(x^*_{n,i})V_{i,n-3}$ each summand is negative, and this concludes the proof of (71).
(72): This is Szegö's inequality (14), as generalized to $P_n\in\mathbf{C_n}$ by Munch [22].
(73): This follows from (27), (28), and (68), (69).
(74): This follows by the triangle inequality and from (75).
(75): It will suffice to consider the case of $n\geq 5$ odd. From Lemmas 2.2, 2.6, and 2.7 we get

$$\begin{aligned}
|a_{n-3}+a_{n-2}+a_{n-1}| &= |\sum_{i=0}^{n}P_n(x^*_{n,i})(V_{i,n-3}+V_{i,n-2}+V_{i,n-1})|\\
&\leq \sum_{i=0}^{n}|P_n(x^*_{n,i})||V_{i,n-3}+V_{i,n-2}+V_{i,n-1}|\\
&\leq \sum_{i=0}^{n}|V_{i,n-3}+V_{i,n-2}+V_{i,n-1}|\\
&= \sum_{i=0}^{(n-1)/2}|V_{i,n-3}+V_{i,n-2}+V_{i,n-1}|+\sum_{i=0}^{(n-1)/2}|V_{i,n-3}-V_{i,n-2}+V_{i,n-1}|\\
&= 2\sum_{i=0}^{(n-1)/2}\max\{|V_{i,n-2}|,|V_{i,n-3}+V_{i,n-1}|\}.
\end{aligned} \tag{102}$$

Suppose we had

$$|V_{i,n-3}+V_{i,n-1}|\leq|V_{i,n-2}| \text{ for } i=0,1,\dots,(n-1)/2. \tag{103}$$

This would imply, by Lemma 2.5,

$$|a_{n-3}+a_{n-2}+a_{n-1}|\leq 2\sum_{i=0}^{(n-1)/2}|V_{i,n-2}|=\sum_{i=0}^{n}|V_{i,n-2}|=|t_{n,n-2}|. \tag{104}$$

And this would further imply $|a_{n-3}-a_{n-2}+a_{n-1}|\leq|t_{n,n-2}|$, since $Q_n\in\mathbf{C_n}$ with $Q_n(x)=P_n(-x)$, and hence altogether we would get, invoking Lemma 2.7, $|a_{n-2}|+|a_{n-3}+a_{n-1}|\leq|t_{n,n-2}|$, as claimed.
It remains to verify the assumption (103), which, by Lemma 2.2, can be rewritten as $|x^*_{n,i}||V_{i,n-2}+V_{i,n}|\leq|V_{i,n-2}|$.
If $i=0$, then in view of (63) and (66) one has to show that $|V_{0,n-2}+V_{0,n}|\leq|V_{0,n-2}|$, that is, $n^{-1}2^{n-2}\left|\frac{n}{4}-\frac{3}{2}\right|\leq n^{-1}2^{n-2}\left|\frac{n}{4}-\frac{1}{2}\right|$.
But this inequality is obviously true for all $n\geq 5$.
If $i>0$, then it is sufficient to show that $|V_{i,n-2}+V_{i,n}|\leq|V_{i,n-2}|$ since $0<|x^*_{n,i}|<1$. By (65) and (67) this amounts to showing that $n^{-1}2^{n-1}\left|\frac{n}{4}-\frac{1}{2}-|x^*_{n,i}|^2\right|\leq n^{-1}2^{n-1}\left|\frac{n}{4}+\frac{1}{2}-|x^*_{n,i}|^2\right|$. But this inequality holds true for all $n\geq 5$, as is readily seen, which completes the proof.

(76), (77): These follow by the triangle inequality and from (78).
(78): This follows from Szegö's inequality (14), as generalized to $P_n\in\mathbf{C_n}$ by Munch [22], and from Chebyshev's inequality (7), as generalized to $P_n\in\mathbf{C_n}$ by Shohat [44].
(79): This follows by the triangle inequality and from (83).
(80), (81), (82): These follow by the triangle inequality and from Szegö's inequality (13), (14), as generalized to $P_n\in\mathbf{C_n}$ by Munch [22].
(83): We proceed similarly as in the proof of (41) and confine ourselves to $n\geq 7$ odd. Consider the sharp coefficient estimate

$$|a_{n-3}+a_{n-2}+a_{n-1}+a_n| = |\sum_{i=0}^{n}P_n(x^*_{n,i})(V_{i,n-3}+V_{i,n-2}+V_{i,n-1}+V_{i,n})|\leq$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{n} |V_{i,n-3}+V_{i,n-2}+V_{i,n-1}+V_{i,n}| \\
&= \sum_{i=0}^{(n-1)/2} |V_{i,n-3}+V_{i,n-2}+V_{i,n-1}+V_{i,n}| + \sum_{i=0}^{(n-1)/2} |V_{i,n-3}-V_{i,n-2}+V_{i,n-1}-V_{i,n}| \\
&= 2\sum_{i=0}^{(n-1)/2} \max\{|V_{i,n-3}+V_{i,n-1}|, |V_{i,n-2}+V_{i,n}|\} \\
&= 2\sum_{i=0}^{(n-1)/2} |V_{i,n-2}+V_{i,n}|. \qquad (105)
\end{aligned}
$$

where the last identity is due to $|V_{i,n-3}+V_{i,n-1}| = |x_{n,i}^*||V_{i,n-2}+V_{i,n}| \leq |V_{i,n-2}+V_{i,n}|$. We know from (57) and (60) that the bound in (105) equals $|t_{n,n-2}+t_{n,n}|$, if $n \geq 10$. Since $P_n \in \mathbf{C_n}$ is arbitrary, we obtain for $Q_n \in \mathbf{C_n}$ with $Q_n(x) = P_n(-x)$: $|-a_{n-3}+a_{n-2}-a_{n-1}+a_n| \leq |t_{n,n-2}+t_{n,n}|$. Lemma 2.7 then eventually gives

$$|a_{n-3}+a_{n-1}| + |a_{n-2}+a_n| \leq |t_{n,n-2}+t_{n,n}|, n \geq 10. \qquad (106)$$

The marginal cases $n \in \{6,7,8,9\}$ can be verified by straightforward calculation, compare the proof of (31), so that we leave the proof to the reader.
(84), (85), (86), (87), (88), (89), (90): These follow by the triangle inequality and from Szegö's inequality (13), (14), as generalized to $P_n \in \mathbf{C_n}$ by Munch [22].

Except for (68), (69), (70), and (71), the condition for equality is due to the fact that T_n is even or odd, depending on the parity of n, and that the nonzero coefficients of T_n alternate in sign, see (3). The inequality $|a_{n-3}+a_{n-1}| \leq |t_{n-1,n-3}+t_{n-1,n-1}| = (n-5)2^{n-4}$ $(n \geq 7)$, if $P_n \in \mathbf{B_n}$, follows from the following observation: Since $P_n \in \mathbf{B_n}$, the polynomial $P_n^{\sim}$, given by $P_n^{\sim}(x) = (P_n(x)+(-1)^{n+1}P_n(-x))/2 = a_{n-1}x^{n-1} + a_{n-3}x^{n-3} +$ lower-degree terms, belongs to $\mathbf{B_{n-1}}$ by the triangle inequality. Hence (31) can be applied with $n := n-1$. □

It is remarkable that the moduli of the maximizing leading coefficients $c_{n-1,n-1}$ and $c_{n-1,n-3}$ of the even resp. odd Rogosinski polynomial Π_{n-1} are two-staged, depending on the parity of $n-1$. These novel explicit formulae for $c_{n-1,n-1}$ and $|c_{n-1,n-3}|$, as given in Theorems 2.1 and 3.1, can be checked against the first few Rogosinski polynomials which read, in power form, as follows:

Example 3.1. Power form representation of $\Pi_{n-1}, 3 \leq n \leq 8$, according to (20) - (25).

$$n=3 \quad : \quad \Pi_2(x) = \sum_{k=0}^{2} c_{2,k}x^k = -\frac{5}{3} + \frac{8}{3}x^2. \qquad (107)$$

$$n=4 \quad : \quad \Pi_3(x) = \sum_{k=0}^{3} c_{3,k}x^k = -(1+2\sqrt{2})x + (2+2\sqrt{2})x^3. \qquad (108)$$

$$n=5 \quad : \quad \Pi_4(x) = \sum_{k=0}^{4} c_{4,k}x^k = \frac{1}{5}(1+4\sqrt{5}) - \frac{1}{5}(12+20\sqrt{5})x^2 + \frac{16}{5}(1+\sqrt{5})x^4. \qquad (109)$$

$$n=6 \quad : \quad \Pi_5(x) = \sum_{k=0}^{5} c_{5,k}x^k = \frac{1}{3}(15+4\sqrt{3})x - \frac{4}{3}(11+5\sqrt{3})x^3 + \frac{16}{3}(2+\sqrt{3})x^5. \qquad (110)$$

$$n=7 \quad : \quad \Pi_6(x)=\sum_{k=0}^{6} c_{6,k}x^k = c_{6,0}+c_{6,2}x^2+c_{6,4}x^4+c_{6,6}x^6 \text{ with} \tag{111}$$

$$c_{6,0}=\Big(-73+\cos\frac{\pi}{7}\Big(7-64\csc\frac{\pi}{14}\Big)+\csc^2\frac{\pi}{14}-4\csc\frac{3\pi}{14}+$$
$$+28\csc\frac{\pi}{14}\Big(1+2\cos\frac{2\pi}{7}\Big)\Big)/7(1+\cos\frac{\pi}{7})=-2.202214\ldots, \tag{112}$$

$$c_{6,2}=\Big(-\csc^2\frac{\pi}{14}+4\Big(20+\Big(5+4\cos\frac{\pi}{7}\Big)\csc\frac{3\pi}{14}\Big)+4\csc\frac{\pi}{14}(1+24\cos\frac{\pi}{7}-$$
$$-14\cos\frac{2\pi}{7})\Big)/7(1+\cos\frac{\pi}{7})=27.412029\ldots, \tag{113}$$

$$c_{6,4}=-\frac{16}{7}\left(\frac{1}{\cos\frac{2\pi}{7}}+\frac{6}{\cos\frac{3\pi}{7}}\right)=-65.297441\ldots, \tag{114}$$

$$c_{6,6}=\frac{64}{7}\left(\frac{1}{\cos\frac{3\pi}{7}}\right)=41.087627\ldots. \tag{115}$$

$$n=8 \quad : \quad \Pi_7(x)=\sum_{k=0}^{7} c_{7,k}x^k = c_{7,1}x+c_{7,3}x^3+c_{7,5}x^5+c_{7,7}x^7 \text{ with} \tag{116}$$

$$c_{7,1}=-1+2\sqrt{2}-\frac{2}{-1-\sqrt{2}+\sqrt{2(2+\sqrt{2})}}-\frac{2}{1-\sqrt{2}+\sqrt{2(2-\sqrt{2})}}$$
$$=-11.219463\ldots, \tag{117}$$

$$c_{7,3}=2-2\sqrt{2}+\frac{14}{-1-\sqrt{2}+\sqrt{2(2+\sqrt{2})}}+\frac{6}{1-\sqrt{2}+\sqrt{2(2-\sqrt{2})}}$$
$$=78.533960\ldots, \tag{118}$$

$$c_{7,5}=-\frac{28}{-1-\sqrt{2}+\sqrt{2(2+\sqrt{2})}}-\frac{4}{1-\sqrt{2}+\sqrt{2(2-\sqrt{2})}}$$
$$=-146.751928\ldots, \tag{119}$$

$$c_{7,7}=\frac{16}{-1-\sqrt{2}+\sqrt{2(2+\sqrt{2})}}=80.437431\ldots. \tag{120}$$

4 Theorem 4.1, Proof, and Example

In Theorem 3.1 we have confined ourselves to consider selected compositions of the first four leading coefficients of $P_n \in \mathbf{C_n}$ so that the extremizers were the same polynomials as in Theorem 2.1. However, if one considers arbitrary compositions of the first four leading coefficients, then new extremizers will emerge. Let us first consider, for $n=6$, the particular composition $|a_3|+|a_4|+|a_5|$:
It follows from Theorems 2.1 and 3.1 that $|t_{n,n-2}|$, the modulus of the second (nonzero) leading coefficient of T_n, is the sharp majorant for the following compositions of leading coefficients, if $P_n \in \mathbf{C_n}$ with $P_n(x)=\sum_{k=0}^{n} a_k x^k$ where $n \geq 6$: $|a_{n-2}|, |a_{n-2}|+|a_{n-1}|, |a_{n-2} \pm a_{n-1}|, |a_{n-3}|+|a_{n-2}|, |a_{n-3}+a_{n-2}|, |a_{n-2}|+|a_{n-3}+a_{n-1}|$, and $|a_{n-3}+a_{n-2}+a_{n-1}|$. However, despite these extremal properties, $|t_{n,n-2}|=|t_{n,n-3}|+|t_{n,n-2}|+|t_{n,n-1}|$ fails to majorize the particular composition $|a_{n-3}|+|a_{n-2}|+|a_{n-1}|$ (which is not covered by Theorem 3.1) for arbitrary $P_n \in \mathbf{C_n}$, if $n=6$. And likewise, the sum of the moduli of the first two (nonzero) leading coefficients of Π_{n-1}, $|c_{n-1,n-3}|+|c_{n-1,n-1}|=|c_{n-1,n-3}|+|c_{n-1,n-2}|+|c_{n-1,n-1}|$, which is the sharp majorant for the composition $|a_{n-3}|+|a_{n-1}|$, fails to majorize $|a_{n-3}|+|a_{n-2}|+|a_{n-1}|$ for arbitrary $P_n \in \mathbf{C_n}$, if $n=6$.
To verify this assertion, we will give an example below (Example 4.1).
This example can be expanded to a general theorem having a negative character: For all $n \geq 6$, neither $\pm T_n$ nor $\pm\Pi_{n-1}$ has large enough leading coefficients to maximize the composition $|a_{n-3}|+|a_{n-2}|+|a_{n-1}|$, if P_n varies in $\mathbf{C_n}$.

Theorem 4.1. (Limitations of the Extremal Polynomials $\pm T_n$ and $\pm\Pi_{n-1}$)
Let $P_n \in \mathbf{C_n}$ with $P_n(x)=\sum_{k=0}^{n} a_k x^k$ and $n \geq 6$ be arbitrary. Let $T_n \in \mathbf{C_n}$ with $T_n(x)=\sum_{k=0}^{n} t_{n,k}x^k$ denote the Chebyshev polynomial of the first kind, and let $\Pi_{n-1} \in \mathbf{C_n}$ with $\Pi_{n-1}(x)=\sum_{k=0}^{n-1} c_{n-1,k}x^k$ denote the Rogosinski polynomial.

Then, neither $|t_{n,n-2}|=|t_{n,n-3}|+|t_{n,n-2}|+|t_{n,n-1}|$ nor $|c_{n-1,n-3}|+|c_{n-1,n-1}|=|c_{n-1,n-3}|+|c_{n-1,n-2}|+|c_{n-1,n-1}|$ majorizes the composition $|a_{n-3}|+|a_{n-2}|+|a_{n-1}|$, if P_n varies in $\mathbf{C_n}$.

Proof. We consider here, for a change, the case of $n \geq 6$ even since the proof is quite similar for $n \geq 7$ odd. The composition

$|a_{n-3}+a_{n-2}-a_{n-1}|$ of an arbitrary $P_n \in \mathbf{C_n}$ with $P_n(x)=\sum_{k=0}^{n} a_k x^k$ can be estimated from above, according to Lemma 2.6, as

$$\begin{aligned} |a_{n-3}+a_{n-2}-a_{n-1}| &= |\sum_{i=0}^{n} P_n(x_{n,i}^*)(V_{i,n-3}+V_{i,n-2}-V_{i,n-1})| && (121)\\ &\le \sum_{i=0}^{n} |P_n(x_{n,i}^*)||V_{i,n-3}+V_{i,n-2}-V_{i,n-1}| \le \sum_{i=0}^{n} |V_{i,n-3}+V_{i,n-2}-V_{i,n-1}| \\ &= |a_{n-3}^*+a_{n-2}^*-a_{n-1}^*|. && (122)\end{aligned}$$

This extremal upper bound will be attained by some polynomial $P_n^* \in \mathbf{C_n}$ which satisfies $|P_n(x_{n,i}^*)|=1$ for $i=0,1,...,n$ in such a way that all (nonvanishing) summands $P_n^*(x_{n,i}^*)(V_{i,n-3}+V_{i,n-2}-V_{i,n-1})$ have the same sign, so that equality occurs in the triangle inequality. We now utilize Lemma 2.2 to obtain (note that $V_{n/2,n-3}=V_{n/2,n-1}=0$ since $x_{n,n/2}^*=0$)

$$\begin{aligned} |a_{n-3}+a_{n-2}-a_{n-1}| &\le \sum_{i=0}^{n/2-1} |V_{i,n-3}+V_{i,n-2}-V_{i,n-1}|+ \\ &\quad + \sum_{i=0}^{n/2-1} |-V_{i,n-3}+V_{i,n-2}+V_{i,n-1}|+|V_{n/2,n-2}| \\ &= \sum_{i=0}^{n/2-1} |V_{i,n-3}+V_{i,n-2}-V_{i,n-1}|+ \\ &\quad + \sum_{i=0}^{n/2-1} |V_{i,n-3}-V_{i,n-2}-V_{i,n-1}|+|V_{n/2,n-2}|\,. && (123)\end{aligned}$$

In view of Lemma 2.7 we get the modified upper bound

$$|a_{n-3}+a_{n-2}-a_{n-1}| \le 2\sum_{i=0}^{n/2-1} \max\{|V_{i,n-3}-V_{i,n-1}|,|V_{i,n-2}|\}+|V_{n/2,n-2}|. \qquad (124)$$

Suppose we had, for $n \ge 6$ even, $|V_{0,n-2}|<|V_{0,n-3}-V_{0,n-1}|$. This would imply

$$|a_{n-3}+a_{n-2}-a_{n-1}| \le 2|V_{0,n-3}-V_{0,n-1}|+2\sum_{i=1}^{n/2-1} \max\{|V_{i,n-3}-V_{i,n-1}|,|V_{i,n-2}|\}+|V_{n/2,n-2}|. \qquad (125)$$

It would follow that the attainable upper bound $|a_{n-3}^*+a_{n-2}^*-a_{n-1}^*|$ is larger than (see Lemma 2.5)

$$|t_{n,n-2}|=\sum_{i=0}^{n}|V_{i,n-2}|=2|V_{0,n-2}|+2\sum_{i=1}^{n/2-1}|V_{i,n-2}|+|V_{n/2,n-2}|.$$

And this would finally give $|t_{n,n-2}|<|a_{n-3}^*+a_{n-2}^*-a_{n-1}^*| \le |a_{n-3}^*|+|a_{n-2}^*|+|a_{n-1}^*|$, so that $|t_{n,n-2}|=|t_{n,n-3}|+|t_{n,n-2}|+|t_{n,n-1}|$ cannot be extremal for $|a_{n-3}|+|a_{n-2}|+|a_{n-1}|$, if P_n varies in $\mathbf{C_n}$, $n \ge 6$ even.
It remains to verify the assumption made above: $|V_{0,n-2}|<|V_{0,n-3}-V_{0,n-1}|$, if $n \ge 6$ even. As $V_{0,n-3}=-V_{0,n-2}$ in view of Lemma 2.2, it suffices to show that $V_{0,n-2}$ and $V_{0,n-1}$ have the same (nonzero) sign, since then $|V_{0,n-3}-V_{0,n-1}|=|V_{0,n-2}+V_{0,n-1}|=|V_{0,n-2}|+|V_{0,n-1}|$ and $|V_{0,n-1}|>0$. Now, compare with the lines following (63),
$V_{0,n-2}=n^{-1}2^{n-2}((n-1)^{-1}n^{-1}2^{1-n}(n^2+n-2)t_{n,n-2}+1)<0$ since $t_{n,n-2}=-n2^{n-3}<0$, and
$V_{0,n-1}=-V_{0,n}=-n^{-1}2^{n-2}<0$, if $n \ge 6$ even, so that both $V_{0,n-2}$ and $V_{0,n-1}$ are negative.

Next we are going to show that $|c_{n-1,n-3}|+|c_{n-1,n-1}|=|c_{n-1,n-3}|+|c_{n-1,n-2}|+|c_{n-1,n-1}|$ cannot be extremal for $|a_{n-3}|+|a_{n-2}|+|a_{n-1}|$, if P_n varies in $\mathbf{C_n}$, $n \ge 6$ even. According to Lemma 2.6 we get

$$\begin{aligned} |a_{n-3}-a_{n-1}| &= |\sum_{i=0}^{n} P_n(x_{n,i}^*)(V_{i,n-3}-V_{i,n-1})| \\ &\le \sum_{i=0}^{n} |P_n(x_{n,i}^*)||V_{i,n-3}-V_{i,n-1}| \le \sum_{i=0}^{n}|V_{i,n-3}-V_{i,n-1}| \\ &= \sum_{i=0}^{n/2-1}|V_{i,n-3}-V_{i,n-1}|+\sum_{i=0}^{n/2-1}|-V_{i,n-3}-V_{i,n-1}| \\ &= 2\sum_{i=0}^{n/2-1}|V_{i,n-3}-V_{i,n-1}|, && (126)\end{aligned}$$

noting that $|V_{n/2,n-3} - V_{n/2,n-1}| = 0$. This upper bound can be enlarged to

$$\begin{aligned}
|a_{n-3} - a_{n-1}| &< 2\sum_{i=0}^{n/2-1} |V_{i,n-3} - V_{i,n-1}| + |V_{n/2,n-2}| \\
&\le 2\sum_{i=0}^{n/2-1} \max\{|V_{i,n-3} - V_{i,n-1}|, |V_{i,n-2}|\} + |V_{n/2,n-2}| \\
&= |a^*_{n-3} + a^*_{n-2} - a^*_{n-1}|. \qquad (127)
\end{aligned}$$

Since $|a_{n-3} - a_{n-1}|$ was arbitrary, we in particular have, setting $P_n = \Pi_{n-1}$ and noting that Π_{n-1} is an even or odd polynomial and $c_{n-1,n-3}$ and $c_{n-1,n-1}$ have opposite signs (compare the proof of (71)),

$$\begin{aligned}
|c_{n-1,n-3} - c_{n-1,n-1}| &= |c_{n-1,n-3}| + |c_{n-1,n-1}| = |c_{n-1,n-3}| + |c_{n-1,n-2}| + |c_{n-1,n-1}| \\
&< |a^*_{n-3} + a^*_{n-2} - a^*_{n-1}| \le |a^*_{n-3}| + |a^*_{n-2}| + |a^*_{n-1}|,
\end{aligned}$$

as claimed. □

Example 4.1. Consider, for $n = 6$, the extremal polynomial $P^*_6 \in \mathbf{C_6}$ with $P^*_6(x) = \sum_{k=0}^{6} a^*_k x^k$, already introduced in [33, p. 294]:

$$P^*_6(x) = -1 + \frac{1}{3}(3 + 4\sqrt{3})x + \frac{43}{3}x^2 + \frac{1}{3}(-16 - 20\sqrt{3})x^3 + \frac{-88}{3}x^4 + \frac{1}{3}(16 + 16\sqrt{3})x^5 + 16x^6. \qquad (128)$$

Obviously, $\pm T_6 \ne \pm P^*_6 \ne \pm\Pi_5$, and the coefficients of P^*_6 satisfy the inequalities $|a^*_3| + |a^*_4| + |a^*_5| = 40 + 12\sqrt{3} = 60.78460\ldots > |t_{6,3}| + |t_{6,4}| + |t_{6,5}| = |t_{6,4}| = 48$, and likewise $|a^*_3| + |a^*_4| + |a^*_5| = 60.78460\ldots > |c_{5,3}| + |c_{5,4}| + |c_{5,5}| = |c_{5,3}| + |c_{5,5}| = \frac{1}{3}(76 + 36\sqrt{3}) = 46.11794\ldots$. Actually, we have $|a^*_3| + |a^*_4| + |a^*_5| = 40 + 12\sqrt{3} \ge |a_3| + |a_4| + |a_5|$ for all $P_6 \in \mathbf{C_6}$ where $P_6(x) = \sum_{k=0}^{6} a_k x^k$, with equality if $P_6 = \pm P^*_6$. Observe that $P^*_6 \notin \mathbf{B_6}$ since, for example, $|P^*_6(-0.6)| > 1$.

5 Concluding Remarks

Remark 1. All inequalities in Theorems 2.1 and 3.1 are sharp, and most of them hold from the least meaningful polynomial degree n on. But some inequalities are valid only from a larger initial value $n = n_0 \in \{4, 6, 10\}$ on. This initial value is optimal, i.e., least, in the following sense: The corresponding inequality will be false if one takes $n = n_0 - 1$. This we now substantiate by (counter-) examples. We have determined extremal low-degree polynomials $P_n \in \mathbf{C_n}$ which attain the indicated upper bounds in the following inequalities if $n = n_0 - 1$:

Inequality (30) with $n = n_0 - 1 = 5 : |a_3 \pm a_4| \le 20.8 > |t_{5,3}| = 20$.
Inequality (31) with $n = n_0 - 1 = 5 : |a_3 + a_5| \le 5.6 > |t_{5,3} + t_{5,5}| = 4$.
Inequality (34) with $n = n_0 - 1 = 5 : |a_3| + |a_4| \le 20.8 > |t_{5,3}| = 20$.
Inequality (37) with $n = n_0 - 1 = 9 : |a_7 + a_8 + a_9| \le 331.24\ldots > |t_{9,7} + t_{9,9}| = 320$.
Inequality (39) with $n = n_0 - 1 = 9 : |a_7 - a_8 + a_9| \le 331.24\ldots > |t_{9,7} + t_{9,9}| = 320$.
Inequality (41) with $n = n_0 - 1 = 9 : |a_8| + |a_7 + a_9| \le 331.24\ldots > |t_{9,7} + t_{9,9}| = 320$.
Inequality (71) with $n = n_0 - 1 = 5 : |a_2 + a_4| \le 2.58\ldots > |c_{4,2} + c_{4,4}| = \frac{4}{5}(\sqrt{5} - 1) = 0.98\ldots$.
Inequality (74) with $n = n_0 - 1 = 3 : |a_0 + a_1 + a_2| \le 3.66\ldots > |t_{3,1}| = 3$.
Inequality (75) with $n = n_0 - 1 = 3 : |a_1| + |a_0 + a_2| \le 3.66\ldots > |t_{3,1}| = 3$.
Inequality (79) with $n = n_0 - 1 = 5 : |a_2 + a_3 + a_4 + a_5| \le 5.6 > |t_{5,3} + t_{5,5}| = 4$.
Inequality (83) with $n = n_0 - 1 = 5 : |a_2 + a_4| + |a_3 + a_5| \le 5.6 > |t_{5,3} + t_{5,5}| = 4$.

To give the full picture, we additionally provide for the compositions in the quoted inequalities the sharp upper bounds if n belongs to the remaining marginal range $[N_1, N_2]$, where $N_1 =$ least meaningful polynomial degree and $N_2 \le n_0 - 2$. This range is empty for inequalities (74), (75). We have again determined extremal low-degree polynomials $P_n \in \mathbf{C_n}$ which attain the indicated upper bounds if $n \in [N_1, N_2]$:

Inequalities (30), (34) with $n = n_0 - 2 = 4 : \max\{|a_2 \pm a_3|, |a_2| + |a_3|\} \le 9$.
Inequalities (30), (34) with $n = n_0 - 3 = 3 : \max\{|a_1 \pm a_2|, |a_1| + |a_2|\} \le 4$.
Inequalities (30), (34) with $n = n_0 - 4 = 2 : \max\{|a_0 \pm a_1|, |a_0| + |a_1|\} \le 2$.
Inequality (31) with $n = n_0 - 2 = 4 : |a_2 + a_4| \le 2$.
Inequality (31) with $n = n_0 - 3 = 3 : |a_1 + a_3| \le 1$.
Inequality (31) with $n = n_0 - 4 = 2 : |a_0 + a_2| \le 1$.
Inequalities (37), (39), (41) with $n = n_0 - 2 = 8 : \max\{|a_6 \pm a_7 + a_8|, |a_7| + |a_6 + a_8|\} \le 144.87\ldots$.
Inequalities (37), (39), (41) with $n = n_0 - 3 = 7 : \max\{|a_5 \pm a_6 + a_7|, |a_6| + |a_5 + a_7|\} \le 63.31\ldots$.
Inequalities (37), (39), (41) with $n = n_0 - 4 = 6 : \max\{|a_4 \pm a_5 + a_6|, |a_5| + |a_4 + a_6|\} \le 27.90\ldots$.
Inequalities (37), (39), (41) with $n = n_0 - 5 = 5 : \max\{|a_3 \pm a_4 + a_5|, |a_4| + |a_3 + a_5|\} \le 12.56\ldots$.
Inequalities (37), (39), (41) with $n = n_0 - 6 = 4 : \max\{|a_2 \pm a_3 + a_4|, |a_3| + |a_2 + a_4|\} \le 5.82\ldots$.
Inequalities (37), (39), (41) with $n = n_0 - 7 = 3 : \max\{|a_1 \pm a_2 + a_3|, |a_2| + |a_1 + a_3|\} \le 2.66\ldots$.
Inequalities (37), (39), (41) with $n = n_0 - 8 = 2 : \max\{|a_0 \pm a_1 + a_2|, |a_1| + |a_0 + a_2|\} \le 1$.

Inequality (71) with $n = n_0 - 2 = 4 : |a_1 + a_3| \le 1$.
Inequality (71) with $n = n_0 - 3 = 3 : |a_0 + a_2| \le 1$.
Inequality (79), (83) with $n = n_0 - 2 = 4 : \max\{|a_1 + a_2 + a_3 + a_4|, |a_1 + a_3| + |a_2 + a_4|\} \le 2$.
Inequality (79), (83) with $n = n_0 - 3 = 3 : \max\{|a_0 + a_1 + a_2 + a_3|, |a_0 + a_2| + |a_1 + a_3|\} \le 1$.

Remark 2. With regard to Theorem 4.1, it can be shown that actually $|c_{n-1,n-3}| + |c_{n-1,n-1}| < |t_{n,n-2}| = n2^{n-3}$ holds, if $n \ge 6$. For instance, if $n \ge 7$ is odd, then this inequality becomes equivalent to showing that $\dfrac{n+3}{\sin\frac{\pi}{2n}} + \dfrac{1}{\sin\frac{3\pi}{2n}} \le n^2$, in view of (28) and (69), and a look at the graphs of the corresponding functions in $n = x$ will furnish evidence to the claim. A formal proof can be based on the identity $\lim\limits_{t\to 0} \frac{\sin t}{t} = 1$.

Remark 3. Proceeding similarly as in the proofs of Theorems 2.1 and 3.1, one can deduce estimates for compositions which contain even more leading coefficients of $P_n \in \mathbf{C_n}$. For example, we have determined the cases where $\pm T_n$ is the extremizer for compositions which contain even the fifth leading coefficient, a_{n-4} (unpublished).

Remark 4. It is interesting to compare, at least asymptotically, the sharp upper bounds $t_{n-1,n-1}$ for $|a_{n-1}|$ and $|t_{n-1,n-3}|$ for $|a_{n-3}|$, if $P_n \in \mathbf{B_n}$, with the sharp upper bounds $c_{n-1,n-1}$ for $|a_{n-1}|$ and $|c_{n-1,n-3}|$ for $|a_{n-3}|$, if $P_n \in \mathbf{C_n}$. The greater free moving space of the graphs of $P_n \in \mathbf{C_n}$ should result in a larger extremal magnitude of $|a_{n-1}|$ and $|a_{n-3}|$ when compared with the extremal magnitude of $|a_{n-1}|$ and $|a_{n-3}|$ in case of $P_n \in \mathbf{B_n}$, since the graphs of the latter polynomials are captured entirely within the unit square $\mathbf{I}^2$. One obtains from (8), (10), (27) resp. (28), and (68) resp. (69), by approximating $\sin\frac{\pi}{2n}$ by $\frac{\pi}{2n}$ and $\sin\frac{3\pi}{2n}$ by $\frac{3\pi}{2n}$, in view of $\lim\limits_{t\to 0}\frac{\sin t}{t} = 1$,

$$c_{n-1,n-1}/t_{n-1,n-1} \sim \frac{4}{\pi} \;\text{ and }\; c_{n-1,n-3}/t_{n-1,n-3} \sim \frac{4}{\pi} = 1.27323\ldots \;\text{ for } n \to \infty. \tag{129}$$

Thus the majorants $t_{n-1,n-1}$ and $|t_{n-1,n-3}|$ are tighter than the majorants $c_{n-1,n-1}$ and $|c_{n-1,n-3}|$, and they must be scaled, asymptotically, by the factor $\frac{4}{\pi} > 1$ in order to equalize the latter ones.

Remark 5. Extremizers in coefficient inequalities for polynomials $P_n \in \mathbf{C_n}$ need not be unique and may differ by more than just the sign. Here are two examples to substantiate this claim:

(i) Consider inequality (36) and take $n = 4$, i.e., $|a_3|+|a_4| \le 8$. The sharp upper bound 8 will be attained if one takes as an extremizer $P_4 = \pm T_4 \in \mathbf{C_4}$, with $T_4(x) = 1 - 8x^2 + 8x^4$, or $P_4 = \pm Q_4^* \in \mathbf{C_4}$, with $Q_4^*(x) = \frac{1}{5}(5 + 3x - 37x^2 - 6x^3 + 34x^4)$. Actually, in inequality (36) there are, for each $n \ge 1$, infinitely many extremizers from $\mathbf{C_n}$. A constructive proof for this claim is given in [34, Theorem 2.4].

(ii) Consider inequality (79) and take $n = 6$, i.e., $|a_3 + a_4 + a_5 + a_6| \le 16$. The sharp upper bound 16 will be attained if one takes as an extremizer $P_6 = \pm T_6 \in \mathbf{C_6}$, with $T_6(x) = -1 + 18x^2 - 48x^4 + 32x^6$, or $P_6 = \pm Q_6^* \in \mathbf{C_6}$, with $Q_6^*(x) = \frac{1}{6}(-6 + 3x + 105x^2 - 16x^3 - 272x^4 + 16x^5 + 176x^6)$.

Remark 6. To derive our coefficient inequalities, we have deployed a conversion from the monomial basis to the Lagrange interpolation formula. It was pointed out by one of the referees that a transformation from the monomial basis to the barycentric interpolation formula or to the Chebyshev polynomial basis will simplify the analysis. We will incorporate this valuable hint in future work on coefficient estimates.

Remark 7. Related papers which deal with estimates for pointwise bounded polynomials are [8], [11], and [46]. Estimates for pairs of coefficients of bounded complex polynomials are to be found in [4]. Alternative variations on Chebyshev's inequality (7) are to be found in [9] and [27].

Remark 8. Some papers of ours related to the subject-matter considered here include

(i) [33] and [34]: Estimates for non-consecutive pairs of coefficients of $P_n \in \mathbf{C_n}$.

(ii) [35] and [36]: Adaptions of V.A. Markov's and Szegö's coefficient inequalities to polynomials with two prescribed zeros (at ± 1) respectively with one prescribed zero (at -1 or at $+1$).

(iii) [28] and [29]: Estimates for j-th forward partial sums of coefficients, $\sum\limits_{k=0}^{j} a_k$.

(iv) [30] and [31]: Estimates for leading coefficients of bounded multivariate polynomials.

References

[1] S.N. Bernstein. Extremal Properties of Polynomials and the Best Approximation of Continuous Functions of a Real Variable. ONTI, Moscow, 1937 (in Russian)

[2] A. Bogatyrev. Extremal Polynomials and Riemann Surfaces. Springer Monographs in Mathematics, Berlin, 2012 (Russian original 2005)

[3] P. Borwein and T. Erdélyi. Polynomials and Polynomial Inequalities. Springer Graduate Texts in Mathematics Vol. 161, New York, 1995

[4] L. Brickman, Q.I. Rahman, and St. Ruscheweyh. On pairs of coefficients of bounded polynomials. *Proc. Am. Math. Soc.*, 92:533 - 537, 1984

[5] J. Bryant and Ch. Sangwin. How Round is Your Circle? Where Engineering and Mathematics Meet. Princeton University Press, Princeton, 2008

[6] P.L. Chebyshev. Théorie des mécanismes connus sous le nom de parallélogrammes. *Mem. Acad. Sci. St. Petersburg*, 7:111 - 143, 1854
Also available at http://www.math.technion.ac.il/hat/fpapers/cheb11.pdf

[7] D.P. Dryanov. A refinement of an inequality of R.J. Duffin and A.C. Schaeffer, 165 - 176. In: Advances in Constructive Approximation: Vanderbilt 2003 (M. Neamtu and E.B. Saff, Eds.), Proc. Conf. Nashville, TN (USA), 2003. Nashboro Press, Brentwood, TN (USA), 2004

[8] D.P. Dryanov, M.A. Qazi, and Q.I. Rahman. Certain extremal problems for polynomials. *Proc. Am. Math. Soc.*, 131:2741 - 2751, 2003

[9] D. Dryanov and R. Fournier. Refinement of an inequality of P.L. Chebyshev. *Acta Math. Hung.*, 122:59 - 69, 2009

[10] R.J. Duffin and A.C. Schaeffer. A refinement of an inequality of the brothers Markoff. *Trans. Am. Math. Soc.*, 50:517 - 528, 1941

[11] H. Ehlich and K. Zeller. Schwankung von Polynomen zwischen Gitterpunkten. *Math. Zeitschr.*, 86:41 - 44, 1964

[12] P. Erdös. Some remarks on polynomials. *Bull. Am. Math. Soc.*, 53:1169 - 1176, 1947

[13] V.L. Goncharov. The theory of best approximation of functions. *J. Approx. Theory*, 106:2 - 57, 2000 (Russian original 1945)
Also available at http://www.math.technion.ac.il/hat/fpapers/gon.pdf

[14] A.N. Kolmogorov and A.P. Yushkevich (Eds.). Mathematics of the 19th Century Vol. 3. Birkhäuser, Basel, 1998 (Russian original 1987)

[15] M.G. Krein and A.B. Nudelman. The Markov Moment Problem and Extremal Problems: Ideas and Problems of P. L. Čebyšev and A. A. Markov and Their Further Development. Translations of Mathematical Monographs Vol. 50, AMS, Providence, RI (USA), 1977 (Russian original 1973)

[16] V.A. Markov. On functions deviating least from zero in a given interval. *Izdat. Akad. Nauk St. Petersburg*, Foreword and 1 - 111, 1892 (in Russian)
Also available at http://www.math.technion.ac.il/hat/fpapers/vmar.pdf

[17] W. Markoff. Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen. *Math. Ann.*, 77:213 - 258, 1916 (slightly abridged translation to German of [16], with a foreword by S. N. Bernstein)

[18] J.C. Mason and D.C. Handscomb. Chebyshev Polynomials. Chapman & Hall / CRC, Boca Raton, 2002

[19] D.I. Mendeleev. Research on Aqueous Solutions According to their Specific Weight. V. Demakov, St. Petersburg, 1887 (in Russian)

[20] G.V. Milovanović, D.S. Mitrinović, and Th.M. Rassias. Topics in Polynomials - Extremal Problems, Inequalities, Zeros. World Scientific, Singapore, 1994

[21] F.C. Moon. The Machines of Leonardo Da Vinci and Franz Reuleaux - Kinematics of Machines from the Renaissance to the 20th Century. Springer, Dordrecht, 2007

[22] O.J. Munch. Om nogle uligheder af W. A. Markoff. *Nordisk Mat. Tidskr.*, 8:21 - 29, 1960

[23] I.P. Natanson. Constructive Function Theory Vol. I, Uniform Approximation. F. Ungar Publ., New York, 1964 (Russian original 1949)

[24] S. Paszkowski, Zastosowania Numeryczne Wielomianow i Szeregow Czebyszewa. PWN, Warsaw, 1975

[25] R. Pierre and Q.I. Rahman. On a problem of Turán about polynomials II. *Can. J. Math.*, 33:701 - 733, 1981

[26] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev. Integrals and Series Vol. 1. Gordon and Breach, New York, 1992 (Russian original 1981)

[27] M.A. Qazi and Q.I. Rahman. Some coefficient estimates for polynomials on the unit interval. *Serdica Math. J.*, 33:449 - 474, 2007

[28] H.-J. Rack. Abschätzung der Teilsummen reeller Polynome. *Math. Zeitschr.*, 182:549 - 552, 1983

[29] H.-J. Rack. Further extremal properties of the Rogosinski polynomials. *Glas. Mat.*, 19 (39):245 - 253, 1984

[30] H.-J. Rack. On V. A. Markov's and G. Szegö's inequality for the coefficients of polynomials in one and several variables. *East J. Approx.*, 14:319 - 352, 2008

[31] H.-J. Rack. On the length and height of Chebyshev polynomials in one and two variables. *East J. Approx.*, 16:35 - 91, 2010

[32] H.-J. Rack. Maximizing forward and backward partial sums of coefficients of bounded polynomials (Contribution to an extremal problem of V.A. Markov).
Talk delivered at the Paul Turán Memorial Conference, Budapest, 2011
Abstract available at http://www.renyi.hu/~turan100/abstracts.pdf

[33] H.-J. Rack. On an extremal property of the Chebyshev polynomials: Maximizing pairs of coefficients of bounded polynomials, 286 - 317. In: Constructive Theory of Functions, Proc. Conference Sozopol 2010 (G. Nikolov and R. Uluchev, Eds.), Prof. Marin Drinov Academic Publishing House, Sofia, 2012

[34] H.-J. Rack. Companion theorems to G. Szegö's inequality for pairs of coefficients of bounded polynomials. To appear in *Period. Math. Hung.*

[35] H.-J. Rack. Extensions of Schur's inequality for the leading coefficient of bounded polynomials with two prescribed zeros, 107 - 116. In: Advances in Applied Mathematics and Approximation Theory (G.A. Anastassiou and O. Duman, Eds.), Springer Proceedings in Mathematics & Statistics Vol. 41, Berlin, 2013

[36] H.-J. Rack. Extensions of Schur's inequality for the leading coefficient of bounded polynomials with one prescribed zero. Unpublished manuscript, to be submitted.
Abstract available at http://kangro100.ut.ee/dsec-part_list.php

[37] Q.I. Rahman and G. Schmeisser. Analytic Theory of Polynomials. London Math. Soc. Monographs New Series Vol. 26, Clarendon Press, Oxford, 2002

[38] Th.M. Rassias, H.M. Srivastava and A. Yanushauskas (Eds.). Topics in Polynomials of One and Several Variables and their Applications - Volume Dedicated to the Memory of P.L. Chebyshev (1821 - 1894). World Scientific, Singapore, 1993

[39] M. Reimer. Bounds for the Horner sums. *SIAM J. Numer. Anal.*, 5:461-469, 1968

[40] Th.J. Rivlin. Proposal of advanced problem 6084. *Am. Math. Mon.*, 83:292, 1976

[41] Th.J. Rivlin. Chebyshev Polynomials - From Approximation Theory to Algebra and Number Theory, Second Edition. J. Wiley & Sons, New York, 1990

[42] W.W. Rogosinski. Some elementary inequalities for polynomials. *Math. Gaz.*, 39:7 - 12, 1955

[43] A. Schönhage. Approximationstheorie. Walter de Gruyter, Berlin, 1971

[44] J.A. Shohat. On some properties of polynomials. *Math. Zeitschr.*, 29:684 - 695, 1929

[45] A. Tarski. Problem 83. *Parametr*, 1:231, 1930. Reprinted on p. 688 in: Alfred Tarski. Collected Papers Vol. 4 (S.R. Givant and R.N. McKenzie, Eds.), Birkhäuser, Basel, 1986

[46] A.H. Tureckii. Some inequalities for algebraic polynomials. *Izv. Vyssh. Uchebn. Zaved. Mat.*, 4:116 - 131, 1977 (in Russian) Also available at http://www.mathnet.ru/links/c036d78cb9718b690fa8409875fd379b/ivm5952.pdf

[47] E.V. Voronovskaja. The Functional Method and its Applications. Translations of Mathematical Monographs Vol. 28, AMS, Providence, RI (USA), 1970 (Russian original 1963)