Dolomites Research Notes on Approximation
Special Issue *ATMA2021 - Approximation: Theory, Methods, and Applications*, Volume 15 · 2022 · Pages 96–112

A Nyström method for integral equations of the second kind with fixed singularities based on a Gauss-Jacobi-Lobatto quadrature rule

Concetta Laurita[a]

Abstract

The Gauss-Lobatto quadrature rule for integration over the interval $[-1,1]$, relative to a Jacobi weight function $w^{\alpha,\beta}(t)=(1-t)^{\alpha}(1+t)^{\beta}$, $\alpha,\beta>-1$, is considered and an error estimate for functions belonging to some Sobolev-type subspaces of the weighted space $L^1_{w^{\alpha,\beta}}([-1,1])$ is proved. Then, a Nyström type method based on a modified version of this quadrature formula is proposed for the numerical solution of integral equations of the second kind with kernels having fixed singularities at the endpoints of the integration interval and satisfying proper assumptions. The stability and the convergence of the proposed modified Nyström method in suitable weighted spaces are proved and confirmed through some numerical tests.

1 Introduction

We consider the following integral equation of the second kind with fixed singularities of Mellin convolution type

$$f(t)+\chi^-(t)\int_{-1}^{1}\mathbf{k}^-\left(\frac{1+t}{1+s}\right)\frac{f(s)}{1+s}ds+\chi^+(t)\int_{-1}^{1}\mathbf{k}^+\left(\frac{1-t}{1-s}\right)\frac{f(s)}{1-s}ds+\int_{-1}^{1}h(t,s)f(s)ds=g(t),\quad t\in(-1,1). \tag{1}$$

In this equation by $\chi^-,\chi^+\in C^\infty([-1,1])$ we denote cut-off functions, i.e. smooth functions such that $0\le\chi^-(t)\le 1$ for $-1\le t\le 1$ with $\chi^-(t)=1$ for $-1\le t\le -0.5$ and $\chi^-(t)=0$ for $0\le t\le 1$, and that $\chi^+(t)=\chi^-(-t)$.
The functions $\mathbf{k}^-$ and $\mathbf{k}^+$ are real valued functions over the half axis $\mathbb{R}^+=(0,+\infty)$ satisfying proper assumptions and define the associated Mellin kernels

$$k^-(t,s)=\frac{1}{1+s}\mathbf{k}^-\left(\frac{1+t}{1+s}\right),\qquad k^+(t,s)=\frac{1}{1-s}\mathbf{k}^+\left(\frac{1-t}{1-s}\right) \tag{2}$$

having fixed singularities at the points $(-1,-1)$ and $(1,1)$ of the square $[-1,1]^2$, respectively.
The bivariate function $h(t,s)$ and the right-hand side $g(t)$ are assumed sufficiently smooth on $[-1,1]^2$ and $[-1,1]$, respectively, while $f(t)$ denotes the unknown function.
For simplicity, we restrict ourselves throughout this paper to Mellin convolutions with singularities fixed at the points $(-1,-1)$ and $(1,1)$ but the case of Mellin kernels with fixed singularities at $(-1,1)$ and $(1,-1)$ can be treated in a similar manner.

Integral equations naturally occur in many areas of mathematical physics. Many engineering and applied science problems arising in water waves, potential theory and electrostatics are reduced to solving integral equations. In particular, integral equations with fixed singularities in the kernel encounter rather often in theory of elasticity and boundary value problems for elliptic equations in domains with non-smooth boundary (see e.g. [5]).

The aim of this work is to propose a stable numerical method for the solution of integral equations of type (1) when the Mellin kernels satisfy the following assumptions

$$\int_0^\infty x^{-1+\sigma^\pm}|\mathbf{k}^\pm(x)|dx<\infty, \tag{3}$$

for some $0<\sigma^-,\sigma^+<1$. Let us remark that under these conditions the Mellin integral operators

$$(K^-f)(t)=\int_{-1}^{1}k^-(t,s)f(s)ds \tag{4}$$

and

$$(K^+f)(t)=\int_{-1}^{1}k^+(t,s)f(s)ds \tag{5}$$

[a]Department of Mathematics, Computer Science and Economics,
University of Basilicata,
Via dell'Ateneo Lucano 10, 85100 Potenza, Italy

are not bounded with respect to the uniform norm and the solution f of (1) could be singular at both the endpoints of the interval $[-1,1]$. Due to the noncompactness of such operators, the standard stability theory for the numerical methods does not apply.

Systems of integral equations of type (1) with the Mellin kernels $k^{\pm}(t,s)$ as in (2) satisfying conditions of type (3) arise, for instance, using the single layer representation of the potential in the exterior Neumann problem for the Laplace's equation in a plane region with corners. In such a case the functions $\mathbf{k}^{\pm}$ both take the following expression

$$\mathbf{k}^{\pm}(x) = \frac{1}{\pi}\frac{\sin\theta}{1-2x\cos\theta+x^2},$$

θ being the interior angle at a corner point of the boundary.

Many different methods have been proposed for the numerical solution of integral equations of Mellin type with one fixed-point strong singularity in the kernel (see [2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 16, 17] and the references therein). Most of them are based on piecewise polynomial approximations with graded meshes but, more recently, also numerical methods based on global approximation have been considered, sometimes in special cases.

In particular in [2, 3, 4, 13, 14] the numerical solution of integral equations of the second kind with only one fixed singularity of Mellin type at the origin is addressed by means of modified Nyström type methods proposed under different assumptions on the Mellin kernel $k(t,s) = \frac{1}{t}\bar{k}\left(\frac{t}{s}\right)$. More precisely, in [13] the case when $\bar{k}(t)$ is non-negative and fulfills

$$\int_0^{+\infty}\frac{\bar{k}(t)}{t}dt < +\infty$$

was considered. In such a case the solution of the integral equation is continuous. A modified Nyström method based on the Gauss-Legendre quadrature rule was introduced and its stability and convergence were proved. In [3] we treated the case where the following condition

$$\int_0^{+\infty} t^{\frac{1}{p}-1+\sigma}|\bar{k}(t)|dt < +\infty \tag{6}$$

holds true for some $1 \le p < +\infty$ and $\sigma \in [-\frac{1}{p}, 1-\frac{1}{p}]$. According to the corresponding mapping properties of the Mellin integral operator, we approximate the solution of the problem in suitable weighted L^p spaces by using a Nyström type method based on a proper Gauss-Jacobi quadrature formula. A modification of the classical method is introduced near the singularity as well as preconditioning strategy is employed in order to obtain satisfactory numerical results showing the stability and convergence of the method. Nevertheless, a theoretical proof of the stability was remained an open challenge. In [4] the concern over the stability was addressed when the Mellin kernel satisfies the previous condition with $p = +\infty$ and $\sigma > 0$, i.e.

$$\int_0^{+\infty} t^{-1+\sigma}|\bar{k}(t)|dt < +\infty, \quad \sigma > 0. \tag{7}$$

In this case the Mellin integral operator is not necessarily bounded with respect to the uniform norm, therefore the authors studied the problem in weighted spaces of continuous functions. Then, they reduced to solve a new Mellin integral equation whose solution is at least continuous and vanishes at the origin. For its numerical solution they proposed a modified Nyström method employing a suitable Gauss-Jacobi quadrature formula, whose stability and convergence were studied in the space of the continuous functions vanishing at the point 0. In the more recent paper [14] the solution f was searched in a proper weighted space of continuous functions, too. A new integral equation was obtained multiplying the original one by a Jacobi weight of the type $v^{\sigma}(t) = t^{\sigma}$, chosen according to the singularity of the solution. It admits exactly the same solution of the original problem and was numerically solved by applying a Nyström type method based on a proper Gauss-Radau type quadrature rule for the discretization of the weighted integral operators. Also in this case, when applied to the Mellin operator, the quadrature formula needs to be modified near the singularity in order to achieve the stability. The stability and convergence of the method were proved, from both the theoretical and numerical point of view, directly in the weighted space where the solution f lives. Error estimates in weighted uniform norm were also provided.

In the case under consideration, due to the presence of two fixed strong singularities of the Mellin convolution kernels $k^{\pm}(t,s)$ (defined by (2) and satisfying (3)), one has to take into account the singular behavior of the solution near both the endpoints ± 1. For this reason, the study of the solvability of equation (1) as well as of the stability and convergence of the numerical method we are going to propose will be carried out in proper weighted function spaces with the weight of Jacobi type having the form $w^{\sigma^+,\sigma^-}(t) = (1-t)^{\sigma^+}(1+t)^{\sigma^-}$. This approach, in the same fashion of [14], allows us to avoid recurring to smoothing transformation in order to handle the singularities.

The numerical method is of Nyström type and it will use a certain Gauss-Jacobi-Lobatto quadrature rule, suitably modified near both singularities for achieving stability and convergence results. For the proof of such properties, under proper assumptions, we will use an error estimate for the adopted quadrature rule which is demonstrated in the present paper, too. Moreover, we highlight that resorting to preconditioning techniques is not necessary since the linear systems arising from the discretization are well conditioned.

The plan of this paper is as follows. In the first section we give some notation and useful preliminary results. Section 3 is devoted to prove an error estimate for the Gauss-Lobatto quadrature formula relative to a Jacobi weight $w^{\alpha,\beta}(t) = (1-t)^{\alpha}(1+t)^{\beta}$, $\alpha, \beta > -1$ holding true for functions belonging to some Sobolev-type subspaces of the weighted space $L^1_{w^{\alpha,\beta}}$. In Section 4 we propose a modified Nyström method for the numerical solution of equation (1) and show its stability and convergence in some weighted spaces of continuous functions. An error estimate is provided and the well-conditioning of the involved linear systems is also proved. Finally, some numerical tests will be presented in Section 5.

2 Notation and preliminary results

2.1 Notation

For a general weight function $w(t)$ on $[-1,1]$ and $1 \le p < +\infty$, let $L^p_w \equiv L^p_w([-1,1])$ denote the weighted space of all real-valued measurable functions f on $[-1,1]$ such that

$$\|f\|_{L^p_w} = \|f\|_{w,p} = \|fw\|_p = \left(\int_{-1}^{1} |f(t)w(t)|^p dt\right)^{\frac{1}{p}} < +\infty.$$

For $p=\infty$ and w continuous on the interval $[-1,1]$, we set $L^\infty_w \equiv C_w$ with $C_w \equiv C_w([-1,1])$ the Banach space of all continuous functions $f:(-1,1)\to\mathbb{R}$ such that fw is continuous in $[-1,1]$ (i.e. $fw \in C \equiv C([-1,1])$), equipped with the weighted norm

$$\|f\|_{w,\infty} = \sup_{t\in[-1,1]} |f(t)|w(t).$$

With $1 \le p \le \infty$, let $W^p_r(w)$ be the following weighted Sobolev-type subspaces of L^p_w

$$W^p_r(w) = \left\{f \in L^p_w : \ f^{(r-1)} \in AC(-1,1), \ \|f^{(r)}\varphi^r\|_{w,p} < +\infty\right\},$$

where $r\in\mathbb{N}$, $r \ge 1$, $\varphi(t) = \sqrt{1-t^2}$ and $AC(-1,1)$ is the collection of all functions which are absolutely continuous on every closed subset of $(-1,1)$, equipped with the norm

$$\|f\|_{W^p_r(w)} = \|f\|_{w,p} + \|f^{(r)}\varphi^r\|_{w,p}.$$

For a function $f \in L^p_w$, the error of the best approximation of f in L^p_w by polynomials of degree at most n is defined as

$$E_n(f)_{w,p} = \inf_{P\in\mathbb{P}_n} \|f-P\|_{w,p}.$$

Fixed a Jacobi weight $w^{\gamma,\delta}(t) = (1-t)^\gamma(1+t)^\delta$, $\gamma,\delta > -1$, we will denote by $x^{\gamma,\delta}_{n,k}$ and $\lambda^{\gamma,\delta}_{n,k}$, $k=1,\dots,n$, the nodes and coefficients of the corresponding n–point Gauss-Jacobi quadrature rule on $[-1,1]$ and by

$$p^{\gamma,\delta}_n(t) = \gamma^{\gamma,\delta}_n t^n + \text{lower degree terms}, \qquad \gamma^{\gamma,\delta}_n > 0,$$

the Jacobi polynomial of degree n orthonormal w.r.t. $w^{\gamma,\delta}(t)$ having positive leading coefficient.
In the sequel $\mathcal{C}$ will denote a positive constant which may assume different values in different formulas. We write $\mathcal{C}=\mathcal{C}(a,b,\dots)$ to say that $\mathcal{C}$ is dependent on the parameters $a,b,\dots$. and $\mathcal{C}\ne\mathcal{C}(a,b,\dots)$ to say that $\mathcal{C}$ is independent of them. Moreover, we will write $A\sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B such that $1/\mathcal{C} \le A/B \le \mathcal{C}$.

2.2 Preliminary results

It is well known that for functions f belonging to $W^p_1(w)$, the following Favard inequality

$$E_n(f)_{w,p} \le \frac{\mathcal{C}}{n} E_{n-1}(f')_{\varphi w,p}, \tag{8}$$

is fulfilled for a positive constant $\mathcal{C}$ independent of n and f (see, for example, [15, (2.5.22), p. 172]). By iteration of inequality (8), it follows that, for $f \in W^p_r(w)$, $r \ge 1$, the estimate

$$E_n(f)_{w,p} \le \frac{\mathcal{C}}{n^r} E_{n-r}(f^{(r)})_{\varphi^r w,p}, \qquad \mathcal{C}\ne\mathcal{C}(n,f) \tag{9}$$

holds true.
Let us recall that the knots $x^{\gamma,\delta}_{n,k}$ and Christoffel numbers $\lambda^{\gamma,\delta}_{n,k}$ of the Gaussian quadrature formula corresponding to the Jacobi weight $w^{\gamma,\delta}$ satisfy the following properties (see, for instance, [15, (4.2.4), p. 249])

$$x^{\gamma,\delta}_{n,k+1} - x^{\gamma,\delta}_{n,k} \sim \frac{\sqrt{1-t^2}}{n}, \qquad x^{\gamma,\delta}_{n,k} \le t \le x^{\gamma,\delta}_{n,k+1}, \tag{10}$$

and (see [20, (14), p. 673])

$$\lambda^{\gamma,\delta}_{n,k} \sim \frac{\sqrt{1-\left(x^{\gamma,\delta}_{n,k}\right)^2}}{n} w^{\gamma,\delta}\left(x^{\gamma,\delta}_{n,k}\right) \tag{11}$$

uniformly for $1\le k\le n$, $n\in\mathbb{N}$. Moreover, for the orthonormal polynomials $\left\{p^{\gamma,\delta}_n(t)\right\}_n$ one has that (see [21, (12.7.2), p. 309])

$$\frac{\gamma^{\gamma,\delta}_n}{\gamma^{\gamma,\delta}_{n-1}} \sim 1, \qquad \text{as} \quad n\to\infty, \tag{12}$$

and (see, for instance, [19, p. 170]) the equality

$$\frac{1}{p^{\gamma,\delta}_{n-1}\left(x^{\gamma,\delta}_{n,k}\right)} = \frac{\gamma^{\gamma,\delta}_{n-1}}{\gamma^{\gamma,\delta}_n} \lambda^{\gamma,\delta}_{n,k} \left(p^{\gamma,\delta}_n\right)' \left(x^{\gamma,\delta}_{n,k}\right) \tag{13}$$

holds true. Furthermore (see [19, Corollary 9.34, p. 171])

$$\left|p_n^{\gamma,\delta}(1)\right| \sim n^{\gamma+\frac{1}{2}}, \qquad \left|p_n^{\gamma,\delta}(-1)\right| \sim n^{\delta+\frac{1}{2}}, \tag{14}$$

uniformly for $n \in \mathbb{N}$ and, more generally, [15, (4.2.29)-(4.2.30), p. 255])

$$\left|p_n^{\gamma,\delta}(t)\right| \sim n^{\gamma+\frac{1}{2}}, \qquad 1-\frac{\mathcal{C}}{n^2} \le t \le 1, \tag{15}$$

$$\left|p_n^{\gamma,\delta}(t)\right| \sim n^{\delta+\frac{1}{2}}, \qquad -1 \le t \le -1+\frac{\mathcal{C}}{n^2}. \tag{16}$$

3 An error estimate for the Gauss-Jacobi-Lobatto quadrature rule

For a Jacobi weight function $w^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$, $\alpha,\beta > -1$, on the interval $[-1,1]$, we consider the $(n+2)$-point Gauss-Jacobi-Lobatto rule

$$\int_{-1}^{1} f(t)w^{\alpha,\beta}(t)dt = w_0 f(-1) + \sum_{k=1}^{n} w_k f(t_k) + w_{n+1} f(1) + e_n(f) \tag{17}$$

which is exact for polynomials of degree at most $2n+1$, i.e.

$$e_n(f) = 0, \qquad \forall f \in \mathbb{P}_{2n+1}$$

$\mathbb{P}_n$ being the set of all algebraic polynomials on $[-1,1]$ of degree at most n.
It is well known that the interior quadrature nodes t_k, $k=1,\dots,n$, are the zeros of the Jacobi polynomial of degree n orthonormal with respect to the Jacobi weight $w^{\alpha+1,\beta+1}(t) = (1-t)^2 w^{\alpha,\beta}(t)$. The weights of the formula (17) are given by

$$w_k = \int_{-1}^{1} l_k(t) w^{\alpha,\beta}(t)dt, \qquad k = 0,1,\dots,n+1 \tag{18}$$

where, setting from now on $t_0 = -1$ and $t_{n+1} = 1$, $l_k(t)$ denotes the $(k+1)$-th Lagrange fundamental polynomial associated to the system of nodes $\{t_0, t_1, \dots, t_{n+1}\}$.
By standard arguments, it can be easily proved that the quadrature error $e_n(f)$ satisfies the following estimate

$$|e_n(f)| \le 2\left(\int_{-1}^{1} w^{\alpha,\beta}(t)\,dt\right) E_{2n+1}(f)_\infty, \qquad \forall f \in C([-1,1]),$$

where

$$E_n(f)_\infty = \inf_{P\in\mathbb{P}_n} \|f-P\|_\infty$$

denotes the error of best approximation of a function $f \in C([-1,1])$ by means of polynomials of degree at most n with respect to the uniform norm.
The aim of the present section is to provide a new error estimate for less regular functions belonging to some Sobolev-type subspaces of the weighted space $L^1_{w^{\alpha,\beta}}([-1,1])$. A similar estimate is proved in [15] for the classical Gauss-Jacobi quadrature formula, in [10] for the Gauss-Lobatto rule with respect to the Legendre weight $w^{0,0}$ and, more recently, in [14] for the Gauss-Radau formula with respect to a general Jacobi weight $w^{\alpha,\beta}$.

Lemma 3.1. *The nodes and the weights of the Gauss-Jacobi-Lobatto quadrature formula (17) satisfy the following relations*

$$w_0 \le \mathcal{C}\,\Delta t_0 w^{\alpha,\beta}(t_1), \tag{19a}$$

$$w_k \sim \begin{cases} \Delta t_k\, w^{\alpha,\beta}(t_k), & k=1,\dots,n-1, \\ \Delta t_{k-1}\, w^{\alpha,\beta}(t_k) & k=n, \end{cases} \tag{19b}$$

$$w_{n+1} \le \mathcal{C}\,\Delta t_n w^{\alpha,\beta}(t_n), \tag{19c}$$

where $\Delta t_k = t_{k+1} - t_k$, $k = 0,1,\dots,n$ *and* $\mathcal{C} \ne \mathcal{C}(n)$.

Proof. First, let us observe that the weights of the formula (17), given in (18), have the following alternative representation

$$w_0 = \frac{1}{1-t_0}\int_{-1}^{1} \frac{p_n^{\alpha+1,\beta+1}(t)}{p_n^{\alpha+1,\beta+1}(t_0)} w^{\alpha+1,\beta}(t)dt, \tag{20a}$$

$$w_k = \frac{1}{\left(1-t_k^2\right)}\int_{-1}^{1} \frac{p_n^{\alpha+1,\beta+1}(t)}{(t-t_k)\left(p_n^{\alpha+1,\beta+1}\right)'(t_k)} w^{\alpha+1,\beta+1}(t)dt, \qquad k=1,\dots,n, \tag{20b}$$

$$w_{n+1} = \frac{1}{1+t_{n+1}}\int_{-1}^{1} \frac{p_n^{\alpha+1,\beta+1}(t)}{p_n^{\alpha+1,\beta+1}(t_{n+1})} w^{\alpha,\beta+1}(t)dt. \tag{20c}$$

For $k = 1, \dots, n$, since

$$w_k = \frac{\lambda_{n,k}^{\alpha+1,\beta+1}}{1-\left(x_{n,k}^{\alpha+1,\beta+1}\right)^2},$$

using (11) and (10) for $\gamma = \alpha+1$ and $\delta = \beta+1$, we can deduce that

$$w_k \sim \begin{cases} w^{\alpha,\beta}(t_k)\Delta t_k & k = 1,\dots,n-1, \\ w^{\alpha,\beta}(t_k)\Delta t_{k-1} & k = n, \end{cases}$$

i.e. (19b). Now, in order to prove (19a), let us observe that, since

$$\int_{-1}^{1}\left[\frac{p_n^{\alpha+1,\beta+1}(t)}{p_n^{\alpha+1,\beta+1}(t_0)} - \left(\frac{p_n^{\alpha+1,\beta+1}(t)}{p_n^{\alpha+1,\beta+1}(t_0)}\right)^2\right] w^{\alpha+1,\beta}(t)dt = 0,$$

we can rewrite the first coefficient w_0 in (20a) as follows

$$w_0 = \frac{1}{2}\left[\frac{\left(p_n^{\alpha+1,\beta+1}\right)'(t_1)}{p_n^{\alpha+1,\beta+1}(t_0)}\right]^2 \int_{-1}^{1}\left[\frac{p_n^{\alpha+1,\beta+1}(t)}{(t-t_1)\left(p_n^{\alpha+1,\beta+1}\right)'(t_1)}\right]^2 \frac{(t-t_1)^2}{1+t} w^{\alpha+1,\beta+1}(t)dt.$$

Taking into account that $(t-t_1)^2/(1+t) \le \mathcal{C}$ and using (13) (with $\gamma = \alpha+1$ and $\delta = \beta+1$) for $k = 1$ (we recall that $x_{n,1}^{\alpha+1,\beta+1} \equiv t_1$ in our notation), we get

$$\begin{aligned} w_0 &\le \mathcal{C}\left[\frac{\left(p_n^{\alpha+1,\beta+1}\right)'(t_1)}{p_n^{\alpha+1,\beta+1}(t_0)}\right]^2 \lambda_{n,1}^{\alpha+1,\beta+1} \\ &= \mathcal{C}\frac{1}{\left[p_n^{\alpha+1,\beta+1}(t_0)\right]^2}\left(\frac{\gamma_n^{\alpha+1,\beta+1}}{\gamma_{n-1}^{\alpha+1,\beta+1}}\right)^2 \frac{1}{\lambda_{n,1}^{\alpha+1,\beta+1}\left[p_{n-1}^{\alpha+1,\beta+1}(t_1)\right]^2}. \end{aligned}$$

Now, in virtue of (11), it is

$$\lambda_{n,1}^{\alpha+1,\beta+1} \sim \frac{\left(1-x_{n,1}^{\alpha+1,\beta+1}\right)^{\alpha+\frac{3}{2}}\left(1+x_{n,1}^{\alpha+1,\beta+1}\right)^{\beta+\frac{3}{2}}}{n} \sim \frac{1}{n^{2\beta+4}}. \tag{21}$$

Taking into account (21), (12), (14) and (16) (all applied with $\gamma = \alpha+1$ and $\delta = \beta+1$), we deduce the following estimate of w_0

$$w_0 \le \mathcal{C}\frac{n^{2\beta+4}}{n^{4\beta+6}} = \frac{\mathcal{C}}{n^{2\beta+2}}. \tag{22}$$

On the other hand,

$$\Delta t_0 w^{\alpha,\beta}(t_1) = \left(1-x_{n,1}^{\alpha+1,\beta+1}\right)^{\alpha}\left(1+x_{n,1}^{\alpha+1,\beta+1}\right)^{\beta+1} \sim \frac{1}{n^{2\beta+2}}$$

which, combined with (22), gives (19a). In order to prove (19c), proceeding in an analogous way, we start writing w_{n+1} as follows

$$w_{n+1} = \frac{1}{2}\left[\frac{\left(p_n^{\alpha+1,\beta+1}\right)'(t_n)}{p_n^{\alpha+1,\beta+1}(t_{n+1})}\right]^2 \int_{-1}^{1}\left[\frac{p_n^{\alpha+1,\beta+1}(t)}{(t-t_n)\left(p_n^{\alpha+1,\beta+1}\right)'(t_n)}\right]^2 \frac{(t-t_n)^2}{1-t} w^{\alpha+1,\beta+1}(t)dt.$$

Then, using $(t-t_n)^2/(1-t) \le \mathcal{C}$, (11) and (13) for $\gamma = \alpha+1$, $\delta = \beta+1$ and $k = n$, (12), (14) and (15) also with $\gamma = \alpha+1$ and $\delta = \beta+1$, we deduce the estimate

$$w_{n+1} \le \mathcal{C}\left[\frac{\left(p_n^{\alpha+1,\beta+1}\right)'(t_n)}{p_n^{\alpha+1,\beta+1}(t_{n+1})}\right]^2 \lambda_{n,n}^{\alpha+1,\beta+1} \sim \frac{1}{n^{2\alpha+2}}$$

with $\mathcal{C} \ne \mathcal{C}(n)$. Since

$$\Delta t_n w^{\alpha,\beta}(t_n) \sim \left(1-x_{n,n}^{\alpha+1,\beta+1}\right)^{\alpha+1}\left(1+x_{n,n}^{\alpha+1,\beta+1}\right)^{\beta} \sim \frac{1}{n^{2\alpha+2}}$$

we can, finally, conclude that also (19c) holds true. □

Using the previous lemma we are able to prove our main result.

Theorem 3.2. *For $f \in W_r^1(w^{\alpha,\beta})$, $r \ge 1$, the error of the Gauss-Jacobi-Lobatto quadrature formula (17) satisfies the following estimate*

$$|e_n(f)| \le \frac{\mathcal{C}}{(2n)^r} E_{2n+1-r}(f^{(r)})_{\varphi^r w^{\alpha,\beta},1} \tag{23}$$

where $\varphi(t) = \sqrt{1-t^2}$ and $\mathcal{C} \ne \mathcal{C}(f,n)$ is a positive constant.

 ISSN 2035-6803

Proof. We start by proving the estimate (23) in the case $r = 1$. First, we are going to show that

$$\sum_{k=0}^{n+1} w_k |f(t_k)| \le \mathcal{C}\|f w^{\alpha,\beta}\|_1 + \frac{\mathcal{C}}{n}\int_{-1}^{1} |f'(t)|\varphi(t) w^{\alpha,\beta}(t)dt. \tag{24}$$

Taking into account (19a), (19b) and (19c) we have

$$\begin{aligned}\sum_{k=0}^{n+1} w_k |f(t_k)| \quad &\le \quad \mathcal{C}\Big[\,\Delta t_0 w^{\alpha,\beta}(t_1)|f(t_0)| + \sum_{k=1}^{n-1}\Delta t_k w^{\alpha,\beta}(t_k)|f(t_k)| + \\ &+ \quad \Delta t_{n-1} w^{\alpha,\beta}(t_n)|f(t_n)| + \Delta t_n w^{\alpha,\beta}(t_n)|f(t_{n+1})|\,\Big].\end{aligned}$$

In virtue of the following inequality

$$\left.\begin{array}{l}(b-a)|f(a)| \\ (b-a)|f(b)|\end{array}\right\} \le \int_a^b |f(t)|dt + (b-a)\int_a^b |f'(t)|dt,$$

it follows that

$$\begin{aligned}\sum_{k=0}^{n+1} w_k |f(t_k)| \quad &\le \quad \mathcal{C}\left[w^{\alpha,\beta}(t_1)\left(\int_{t_0}^{t_1}|f(t)|dt + \Delta t_0\int_{t_0}^{t_1}|f'(t)|dt\right)\right. \\ &+ \quad \sum_{k=1}^{n-1} w^{\alpha,\beta}(t_k)\left(\int_{t_k}^{t_{k+1}}|f(t)|dt + \Delta t_k\int_{t_k}^{t_{k+1}}|f'(t)|dt\right) \\ &+ \quad w^{\alpha,\beta}(t_n)\left(\int_{t_{n-1}}^{t_n}|f(t)|dt + \Delta t_{n-1}\int_{t_{n-1}}^{t_n}|f'(t)|dt\right) \\ &+ \quad \left. w^{\alpha,\beta}(t_n)\left(\int_{t_n}^{t_{n+1}}|f(t)|dt + \Delta t_n\int_{t_n}^{t_{n+1}}|f'(t)|dt\right)\right],\end{aligned}$$

from which, being for $k = 0, 1, \ldots, n$,

$$1 \pm t_k \sim 1 \pm t \sim 1 \pm t_{k+1}, \qquad t_k \le t \le t_{k+1}, \tag{25}$$

and, taking into account (10) and also that

$$\begin{aligned}\Delta t_0 \quad &\sim \quad \frac{\sqrt{1-t^2}}{n}, \qquad -1 < t \le t_1, \\ \Delta t_n \quad &\sim \quad \frac{\sqrt{1-t^2}}{n}, \qquad t_n \le t < 1,\end{aligned}$$

we deduce

$$\begin{aligned}\sum_{k=0}^{n+1} w_k |f(t_k)| \quad &\le \quad \mathcal{C}\left[\int_{t_0}^{t_1}|f(t)|w^{\alpha,\beta}(t)dt + \frac{1}{n}\int_{t_0}^{t_1}|f'(t)|\varphi(t)w^{\alpha,\beta}(t)dt\right. \\ &+ \quad \sum_{k=1}^{n-1}\left(\int_{t_k}^{t_{k+1}}|f(t)|w^{\alpha,\beta}(t)dt + \frac{1}{n}\int_{t_k}^{t_{k+1}}|f'(t)|\varphi(t)w^{\alpha,\beta}(t)dt\right) \\ &+ \quad \left.\int_{t_n}^{t_{n+1}}|f(t)|w^{\alpha,\beta}(t)dt + \frac{1}{n}\int_{t_n}^{t_{n+1}}|f'(t)|\varphi(t)w^{\alpha,\beta}(t)dt\right] \\ &= \quad \mathcal{C}\left[\int_{-1}^{1}|f(t)|w^{\alpha,\beta}(t)dt + \frac{1}{n}\int_{-1}^{1}|f'(t)|\varphi(t)w^{\alpha,\beta}(t)dt\right]\end{aligned}$$

i.e. (24). Since the $(n+2)$-point quadrature formula (17) is exact for any polynomial of degree at most $2n+1$, for $P \in \mathbb{P}_{2n+1}$ we have

$$\begin{aligned}|e_n(f)| \quad &= \quad |e_n(f-P)| \\ &\le \quad \left|\int_{-1}^{1}(f-P)(t)w^{\alpha,\beta}(t)dt\right| + \left|\sum_{k=0}^{n+1} w_k (f-P)(t_k)\right| \\ &\le \quad \|(f-P)w^{\alpha,\beta}\|_1 + \sum_{k=0}^{n+1} w_k\,|(f-P)(t_k)|\,.\end{aligned}$$

Now, combining the previous inequality with (24) and the following relation (see, for instance, [15, p. 339], [18, p. 286])

$$\|(f-P)'\varphi w^{\alpha,\beta}\|_1 \le \mathcal{C}(2n+2)\|(f-P)w^{\alpha,\beta}\|_1 + \mathcal{C}_1 E_{2n}(f')_{\varphi w^{\alpha,\beta},1},$$

we can deduce

$$\begin{aligned} |e_n(f)| &\le \mathcal{C}\|(f-P)w^{\alpha,\beta}\|_1 + \frac{\mathcal{C}}{n}\|(f-P)'\varphi w^{\alpha,\beta}\|_1 \\ &\le \mathcal{C}\|(f-P)w^{\alpha,\beta}\|_1 + \frac{\mathcal{C}_1}{n}E_{2n}(f')_{\varphi w^{\alpha,\beta},1}. \end{aligned}$$

Taking the infimum over $P \in \mathbb{P}_{2n+1}$ and using the Favard inequality (9) we get

$$|e_n(f)| \le \mathcal{C}E_{2n+1}(f)_{w^{\alpha,\beta},1} + \frac{\mathcal{C}_1}{n}E_{2n}(f')_{\varphi w^{\alpha,\beta},1} \le \frac{\mathcal{C}}{2n}E_{2n}(f')_{\varphi w^{\alpha,\beta},1}. \tag{26}$$

The estimate (23) for $r > 1$ can be deduced from (26) by iterating the application of the Favard inequality. □

4 The numerical method

4.1 A new integral equation

In order to propose a numerical method for the solution of integral equation (1), we start from giving a new formulation of the problem. Using the cut-off functions $\chi^{\pm}$ occurring in (1) and described in Section 1, by the trivial equalities

$$f(s) = \chi^{\pm}(s)f(s) + (1-\chi^{\pm}(s))f(s),$$

we can split each of the operators $K^{\pm}$ into the sum of two operators as follows

$$(K^- f)(t) = \int_{-1}^{1} \frac{1}{1+s}\mathbf{k}^-\left(\frac{1+t}{1+s}\right)\chi^-(s)f(s)ds + \int_{-1}^{1} \frac{1}{1+s}\mathbf{k}^-\left(\frac{1+t}{1+s}\right)(1-\chi^-(s))f(s)ds =: (K_1^- f)(t) + (K_2^- f)(t) \tag{27}$$

$$(K^+ f)(t) = \int_{-1}^{1} \frac{1}{1-s}\mathbf{k}^+\left(\frac{1-t}{1-s}\right)\chi^+(s)f(s) + \int_{-1}^{1} \frac{1}{1-s}\mathbf{k}^+ ds\left(\frac{1-t}{1-s}\right)(1-\chi^+(s))f(s)ds =: (K_1^+ f)(t) + (K_2^+ f)(t). \tag{28}$$

Let us note that while the operators K_1^- and K_1^+ have Mellin convolution type kernels with fixed singularities at the points $(-1,-1)$ and $(1,1)$, respectively, the remaining operators K_2^- and K_2^+ have kernels which are continuous in $[-1,1]^2$.
Using (27)-(28) and setting

$$(Hf)(t) = \int_{-1}^{1} h(t,s)f(s)ds,$$

we can rewrite equation (1) in a compact form as follows

$$(I + \chi^- K_1^- + \chi^+ K_1^+ + \chi^- K_2^- + \chi^+ K_2^+ + H)f(t) = g(t), \qquad t \in (-1,1), \tag{29}$$

I being the identity operator. Now, let us analyze some mapping properties of the integral operators involved in (29).
To this end, we consider the Jacobi weight function $w(t) = w^{\sigma^+,\sigma^-}(t) = (1-t)^{\sigma^+}(1+t)^{\sigma^-}$, with the exponents $0 < \sigma^{\pm} < 1$ such that (3) is satisfied. Then, for any function f belonging to the weighted space C_w, we can extend by continuity the definition of the weighted functions $(wK^- f)(t)$ and $(wK^+ f)(t)$ as follows

$$(wK^- f)(t) = \begin{cases} (wf)(-1)\displaystyle\int_0^{\infty} x^{-1+\sigma^-}\mathbf{k}^-(x)dx, & t = -1 \\ w(t)\displaystyle\int_{-1}^{1} k^-(t,s)f(s)ds, & t \in (-1,1] \end{cases}, \tag{30}$$

and

$$(wK^+ f)(t) = \begin{cases} w(t)\displaystyle\int_{-1}^{1} k^+(t,s)f(s)ds, & t \in [-1,1) \\ (wf)(1)\displaystyle\int_0^{\infty} x^{-1+\sigma^+}\mathbf{k}^+(x)dx, & t = 1 \end{cases}. \tag{31}$$

For the weighted integral operators wK^- and wK^+ defined by (30) and (31), respectively, the following result holds true.

Theorem 4.1. *If the kernels $k^{\pm}(t,s)$ defined by (2) satisfy conditions (3) for some $0 < \sigma^{\pm} < 1$, then the operators $wK^{\pm}$ are linear maps from C_w to C.*

Proof. We start by proving that the operators $wK^{\pm}$ are linear maps from C_w to C. The linearity of $wK^{\pm}$ trivially follows from the linearity of the integral operators $K^{\pm}$ defined in (4)-(5). Now, let us prove that for any $f \in C_w$, the function $wK^- f$ is continuous in $[-1,1]$. Similar arguments apply to $wK^+ f$. For $f \in C_w$, the continuity of $(wK^- f)(t)$ in $(-1,1]$ is a consequence of the continuity of the kernel $k^-(t,s)$ for $t+s>-2$. Moreover, since by definition (30), using first the change of variable $1+s=(1+t)y$ and then $x=y^{-1}$, we get

$$\begin{aligned}
\lim_{t\to-1}(wK^-f)(t) &= \lim_{t\to-1} w(t)\int_0^{\frac{2}{1+t}} \frac{1}{y}\mathbf{k}^-\left(\frac{1}{y}\right) f((1+t)y-1)dy \\
&= \lim_{t\to-1} w(t)\int_0^{\infty} \frac{1}{y}\mathbf{k}^-\left(\frac{1}{y}\right) \chi_{[0,\frac{2}{1+t}]}(y)(2-(1+t)y)^{-\sigma^+}(1+t)^{-\sigma^-}y^{-\sigma^-}f((1+t)y-1)w((1+t)y-1)dy \\
&= (fw)(-1)\int_0^{\infty} y^{-1-\sigma^-}\mathbf{k}^-\left(\frac{1}{y}\right)dy \\
&= (fw)(-1)\int_0^{\infty} x^{-1+\sigma^-}\mathbf{k}^-(x)dx = (wK^-f)(-1),
\end{aligned}$$

with $\chi_{[0,\frac{2}{1+t}]}$ the characteristic function of the interval $\left[0,\frac{2}{1+t}\right]$, we can deduce that $(wK^-f)(t)$ is continuous at $t=-1$, too. □

The second main step of our procedure consists in solving the following new weighted integral equation, with the same unknown function f,

$$\left(wI+\chi^-wK_1^-+\chi^+wK_1^++\chi^-wK_2^-+\chi^+wK_2^++wH\right)f=wg, \tag{32}$$

obtained by multiplying both sides of (29) by the weight function $w(t)$. The weighted Mellin operators wK_1^- and wK_1^+ occuring in (32) are defined as

$$(wK_1^{\pm}f)(t)=(wK^{\pm}\chi^{\pm}f)(t), \tag{33}$$

with the operators $wK^{\pm}$ given by (30)-(31), and satisfy the following

Theorem 4.2. *If the kernels $\mathrm{k}^{\pm}(t,s)$ defined by (2) satisfy conditions (3) for some $0<\sigma^{\pm}<1$, then the operators $\chi^{\pm}wK_1^{\pm}:C_w\longrightarrow C$ are bounded linear maps with*

$$\|\chi^{\pm}wK_1^{\pm}\|_{C_w\to C}\le 2^{\sigma^{\mp}}\int_0^{\infty} x^{-1+\sigma^{\pm}}\left|\mathbf{k}^{\pm}(x)\right|dx. \tag{34}$$

Proof. We prove the theorem for the operator $\chi^-wK_1^-$. In a similar way one can prove the analogous result for $\chi^+wK_1^+$. For $f\in C_w$ the continuity of the function $\chi^-wK_1^-f$ in $[-1,1]$ is a consequence of Theorem 4.1 and (33), taking also into account that the cut off function χ^- is assumed to be very smooth. Then, it remains to show that the operators $\chi^-wK_1^-:C_w\longrightarrow C$ is bounded and satisfies (34). For $-1\le t\le 0$, proceeding as in the proof of Theorem 4.1, we can write

$$\begin{aligned}
\left|\left(\chi^-wK_1^-f\right)(t)\right| &\le \chi^-(t)w(t)\int_0^{\frac{1}{1+t}} \frac{1}{y}\mathbf{k}^-\left(\frac{1}{y}\right)(\chi^-f)((1+t)y-1)dy \\
&\le \chi^-(t)(1-t)^{\sigma^+}\int_0^{\infty} y^{-1-\sigma^-}\left|\mathbf{k}^-\left(\frac{1}{y}\right)\right|\chi_{[0,\frac{1}{1+t}]}(y)(2-(1+t)y)^{-\sigma^+}\left|(\chi^-fw)((1+t)y-1)\right|dy \\
&\le 2^{\sigma^+}\|f\|_{w,\infty}\int_0^{\infty} x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|dx,
\end{aligned}$$

being $0\le\chi^-(t)\le 1$ and, for $0\le y\le\frac{1}{1+t}$, $2^{-\sigma^+}\le(2-(1+t)y)^{-\sigma^+}\le 1$. Finally, since for $0\le t\le 1$ one has $\left(\chi^-wK_1^-f\right)(t)=0$, the thesis immediately follows. □

Now, we are able to prove the following theorem establishing the solvability of equation (32).

Theorem 4.3. *Let us assume that the conditions*

$$\int_0^{\infty} x^{-1+\sigma^{\pm}}\left|\mathbf{k}^{\pm}(x)\right|dx<2^{-\sigma^{\mp}} \tag{35}$$

are satisfied and that $h(t,s)w(t)$ is continuous on $[-1,1]^2$. Then, if the equation $(wI+\chi^-wK^-+\chi^+wK^++wH)f=0$ has only the trivial solution in the space C_w, equation (32) has a unique solution $f\in C_w$ for each given function $g\in C_w$.

Proof. First, we note that the operator $wI:C_w\longrightarrow C$ is linear, bounded and invertible with the inverse $w^{-1}I:C\longrightarrow C_w$ such that $\|w^{-1}I\|_{C\to C_w}=1$, where $w^{-1}(t)=(1-t)^{-\sigma^+}(1+t)^{-\sigma^-}$ denotes the reciprocal of the weight $w(t)$.
Now, let us estimate the norm of the operator $\chi^-wK_1^-+\chi^+wK_1^+:C_w\longrightarrow C$. For $f\in C_w$, taking into account (34), we have

$$\begin{aligned}
\left\|\left(\chi^-wK_1^-+\chi^+wK_1^+\right)f\right\|_\infty &= \max_{t\in[-1,1]}\left|\left(\chi^-wK_1^-+\chi^+wK_1^+\right)f(t)\right| \\
&= \max\left\{\max_{t\in[-1,0]}\left|\chi^-wK_1^-f(t)\right|,\max_{t\in[0,1]}\left|\chi^+wK_1^+f(t)\right|\right\} \\
&\le \|f\|_{w,\infty}\max\left\{2^{\sigma^+}\int_0^{\infty} x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|dx,2^{\sigma^-}\int_0^{\infty} x^{-1+\sigma^+}\left|\mathbf{k}^+(x)\right|dx\right\}
\end{aligned}$$

from which, by assumptions (35), we can deduce

$$\left\| \chi^- w K_1^- + \chi^+ w K_1^+ \right\|_{C_w \to C} < 1. \tag{36}$$

Then, in virtue of (36), we can apply the Neumann series theorem and deduce that the operator $wI + \chi^- wK_1^- + \chi^+ wK_1^+$ is invertible as a map from C_w to C. Finally, observing that the kernels

$$k_2^-(t,s) := \chi^-(t) \frac{1}{1+s} \mathbf{k}^- \left(\frac{1+t}{1+s} \right) (1 - \chi^-(s))$$

of the operator $\chi^- K_2^-$ and

$$k_2^+(t,s) := \chi^+(t) \frac{1}{1-s} \mathbf{k}^+ \left(\frac{1-t}{1-s} \right) (1 - \chi^+(s))$$

of $\chi^+ K_2^+$ are continuous on the square $[-1,1]^2$ as well as, by assumption, the product $h(t,s)w(t)$, we can conclude that the operator $\chi^- wK_2^- + \chi^+ wK_2^+ + wH : C_w \longrightarrow C$ is compact being sum of compact operators (see [14, Theorem 2]). Then, the thesis easily follows if one uses [11, Corollary 3.8]. □

4.2 The Nyström method

In this subsection we are going to propose a numerical method for the solution of integral equation (32) (and consequently of (1)). The method is of Nyström type, hence it is based on suitable approximations of all the integrals occurring in the equation. In particular, for this purpose, we will use the Gauss-Jacobi-Lobatto quadrature rule given in (17) with respect to the Jacobi weight $w^{\alpha,\beta}(t) = w^{-\sigma^+,-\sigma^-}(t) = w^{-1}(t)$ with $0 < \sigma^\pm < 1$ satisfying (3). While all the compact integral operators involved in (32) are discretized just applying the quadrature formula (17), we need to use a modified version of it for the approximation of the Mellin convolution operators $\chi^- wK_1^-$ and $\chi^+ wK_1^+$ in order to achieve convergence and stability results for the Nyström method. First, we introduce the following discrete operators

$$(K_{1,n}^\pm f)(t) = \sum_{k=0}^{n+1} w_k k^\pm(t,t_k) \chi^\pm(t_k)(wf)(t_k), \tag{37}$$

$$(K_{2,n}^\pm f)(t) = \sum_{k=0}^{n+1} w_k k^\pm(t,t_k) \big(1 - \chi^\pm(t_k)\big)(wf)(t_k), \tag{38}$$

and

$$(H_n f)(t) = \sum_{k=0}^{n+1} w_k h(t,t_k)(wf)(t_k).$$

and approximate the weighted operators $wK_2^\pm$ and wH by means of $wK_{2,n}^\pm$ and wH_n, respectively. Moreover, fixed, in correspondence of a positive constant c and a small $\epsilon > 0$, a breaking point

$$\tau_n = \frac{c}{n^{2-2\epsilon}}, \tag{39}$$

we define the following weighted "modified" discrete operators

$$\left(K_{1,n}^{-,w} f\right)(t) = \begin{cases} \dfrac{1}{\tau_n} \Big[(1+t) \left(wK_{1,n}^- f\right)(-1+\tau_n) - (t+1-\tau_n)(wK_1^- f)(-1) \Big], & t \in [-1,-1+\tau_n) \\ \left(wK_{1,n}^- f\right)(t), & t \in [-1+\tau_n, 1] \end{cases} \tag{40}$$

and

$$\left(K_{1,n}^{+,w} f\right)(t) = \begin{cases} \left(wK_{1,n}^+ f\right)(t), & t \in [-1, 1-\tau_n] \\ \dfrac{1}{\tau_n} \Big[(1-t) \left(wK_{1,n}^+ f\right)(1-\tau_n) + (t-1+\tau_n)(wK_1^+ f)(1) \Big], & t \in (1-\tau_n, 1] \end{cases} \tag{41}$$

approximating wK_1^- and wK_1^+, respectively.
The Nyström method we propose consists in computing the solution f_n of the following approximating equation

$$\left(wI + \chi^- K_{1,n}^{-,w} + \chi^+ K_{1,n}^{+,w} + \chi^- wK_{2,n}^- + \chi^+ wK_{2,n}^+ + wH_n \right) f_n = wg, \tag{42}$$

by which we approximate the unknown solution f of equation (32).
In order to obtain f_n, we collocate (42) at the quadrature knots t_i, $i = 0, 1, \dots, n+1$, of the adopted Gauss-Jacobi-Lobatto formula reducing to solve a linear system of $n+2$ equations in the $n+2$ unknowns $a_j = (wf_n)(t_j)$, $j = 0, 1, \dots, n+1$,

$$\left[\left(wI + \chi^- K_{1,n}^{-,w} + \chi^+ K_{1,n}^{+,w} + \chi^- wK_{2,n}^- + \chi^+ wK_{2,n}^+ + wH_n \right) f_n \right](t_i) = (wg)(t_i), \qquad i = 0, 1, \dots, n+1. \tag{43}$$

Once the system (43) is solved, one can compute the weighted Nyström interpolant

$$wf_n = wg - \left(\chi^- K_{1,n}^{-,w} + \chi^+ K_{1,n}^{+,w} + \chi^- wK_{2,n}^- + \chi^+ wK_{2,n}^+ + wH_n \right) f_n \tag{44}$$

with f_n which satisfies equation (42).
The stability and convergence of the proposed method are established by the following

Theorem 4.4. *Let the hypotheses of Theorem 4.3 be satisfied. Moreover, we assume that the kernels $k^{\pm}(t,s)$ given in (2), for some $r\in\mathbb{N}$, satisfy*

$$\left\|\frac{\partial^j}{\partial s^j}\left(k^-(t,\cdot)\chi^-\right)\varphi^j w^{-\sigma^+,-\sigma^-}\right\|_1 \le \frac{\mathcal{C}}{(1+t)^{\frac{j}{2}+\sigma^-}}, \qquad j=0,1,\dots,r, \qquad t\in(-1,1], \tag{45}$$

$$\left\|\frac{\partial^j}{\partial s^j}\left(k^+(t,\cdot)\chi^+\right)\varphi^j w^{-\sigma^+,-\sigma^-}\right\|_1 \le \frac{\mathcal{C}}{(1-t)^{\frac{r}{2}+\sigma^+}}, \qquad j=0,1,\dots,r, \qquad t\in[-1,1), \tag{46}$$

with $\varphi(s)=\sqrt{1-s^2}$ and $\mathcal{C}$ a positive constant independent of t. Then, for sufficiently large n (say $n\ge n_0$), the operators

$$\left(wI+\chi^-K^{-,w}_{1,n}+\chi^+K^{+,w}_{1,n}+\chi^-wK^-_{2,n}+\chi^+wK^+_{2,n}+wH_n\right)^{-1}: C\longrightarrow C_w$$

exist and are uniformly bounded. Moreover, the solutions f of (32) and f_n of (42), under the assumption $g\in C_w$, satisfy

$$\lim_{n\to\infty}\|f-f_n\|_{w,\infty}=0. \tag{47}$$

Proof. Let us start by proving that, under the assumptions, $\chi^-K^{-,w}_{1,n}+\chi^+K^{+,w}_{1,n}: C_w\longrightarrow C$, $n\in\mathbb{N}$, are linear operators satisfying

$$\lim_{n\to\infty}\left\|\chi^-K^{-,w}_{1,n}+\chi^+K^{+,w}_{1,n}\right\|_{C_w\to C}<1. \tag{48}$$

The linearity is an immediate consequence of the definitions (40)-(41) and of the linearity of the operators $K^{\pm}_1$ and $K^{\pm}_{1,n}$ (see (27)-(28) and (37)). In order to check that for any $f\in C_w$ the image $\left(\chi^-K^{-,w}_{1,n}+\chi^+K^{+,w}_{1,n}\right)f$ is a continuous function on $[-1,1]$, one needs only to verify that $K^{-,w}_{1,n}f$ is continuous at the point $-1+\tau_n$ while $K^{+,w}_{1,n}f$ is continuous at $1-\tau_n$. From (40) it follows that

$$\lim_{t\to(-1+\tau_n)^-}\left(K^{-,w}_{1,n}f\right)(t)=\left(K^-_{1,n}f\right)(-1+\tau_n)=\lim_{t\to(-1+\tau_n)^+}\left(K^{-,w}_{1,n}f\right)(t)$$

and from (41)

$$\lim_{t\to(1-\tau_n)^-}\left(K^{+,w}_{1,n}f\right)(t)=\left(K^+_{1,n}f\right)(1-\tau_n)=\lim_{t\to(1-\tau_n)^+}\left(K^{+,w}_{1,n}f\right)(t).$$

Now, for $f\in C_w$, we consider the following norm

$$\left\|\left(\chi^-K^{-,w}_{1,n}+\chi^+K^{+,w}_{1,n}\right)f\right\|_\infty=\max\left\{\max_{t\in[-1,0]}\left|\left(\chi^-K^{-,w}_{1,n}f\right)(t)\right|,\max_{t\in[0,1]}\left|\left(\chi^+K^{+,w}_{1,n}f\right)(t)\right|\right\}. \tag{49}$$

Let us estimate the first term in the brackets. Assuming $\tau_n\le 0.5$, according to the properties of the cut off function χ^- and the definition of the operator $K^{-,w}_{1,n}$ (see (40)), we have

$$\begin{aligned}\max_{t\in[-1,0]}\left|\left(\chi^-K^{-,w}_{1,n}f\right)(t)\right| &= \max\left\{\max_{[-1,-1+\tau_n]}\left|\left(\chi^-K^{-,w}_{1,n}f\right)(t)\right|,\max_{[-1+\tau_n,0]}\left|\left(\chi^-K^{-,w}_{1,n}f\right)(t)\right|\right\}\\ &\le \max\left\{\left|\left(wK^-_1f\right)(-1)\right|,\max_{[-1+\tau_n,0]}\left|\left(wK^-_{1,n}f\right)(t)\right|\right\}.\end{aligned}$$

Now, proceeding as in the proof of Theorem 4.2, we can write

$$\left|\left(wK^-_1f\right)(-1)\right|\le 2^{\sigma^+}\|f\|_{w,\infty}\int_0^\infty x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|dx.$$

Furthermore, for $t\in[-1+\tau_n,0]$

$$\begin{aligned}\left|\left(wK^-_{1,n}f\right)(t)\right| &\le w(t)\sum_{k=0}^{n+1}w_k\left|k^-(t,t_k)\right|\chi^-(t_k)|(wf)(t_k)|\\ &\le \|f\|_{w,\infty}w(t)\left[\int_{-1}^1\left|k^-(t,s)\right|\chi^-(s)w^{-1}(s)ds+\left|e_n\left(|k^-(t,\cdot)|\chi^-\right)\right|\right]\end{aligned}$$

where

$$\begin{aligned}w(t)\int_{-1}^1\left|k^-(t,s)\right|\chi^-(s)w^{-1}(s)ds &\le 2^{\sigma^+}\int_{-1}^0\frac{1}{1+s}\left|\mathbf{k}^-\left(\frac{1+t}{1+s}\right)\right|\left(\frac{1+t}{1+s}\right)^{\sigma^-}ds\\ &\le 2^{\sigma^+}\int_0^\infty x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|dx\end{aligned}$$

while, in virtue of the error estimate (23) and condition (45),

$$w(t)\left|e_n\left(|k^-(t,\cdot)|\chi^-\right)\right|\le(1-t)^{\sigma^+}(1+t)^{\sigma^-}\frac{\mathcal{C}}{n}\left\|\frac{\partial}{\partial s}\left(k^-(t,\cdot)\chi^-\right)\varphi w^{-\sigma^+,-\sigma^-}\right\|_1\le\frac{\mathcal{C}}{n}\frac{2^{\sigma^+}}{(1+t)^{\frac{1}{2}}}\le\frac{\mathcal{C}}{n^\epsilon}.$$

Consequently, we get

$$\max_{t\in[-1,0]}\left|\left(\chi^{-}K_{1,n}^{-,w}f\right)(t)\right| \le \|f\|_{w,\infty}\left[2^{\sigma^+}\int_0^\infty x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|dx+\frac{\mathcal{C}}{n^\epsilon}\right]. \tag{50}$$

Proceeding in an analogous way, using the assumption (46), the second term in (49) can be estimated as follows

$$\max_{t\in[0,1]}\left|\left(\chi^{+}K_{1,n}^{+,w}f\right)(t)\right| \le \|f\|_{w,\infty}\left[2^{\sigma^-}\int_0^\infty x^{-1+\sigma^+}\left|\mathbf{k}^+(x)\right|dx+\frac{\mathcal{C}}{n^\epsilon}\right], \tag{51}$$

hence, combining (49) with (50) and (51) and taking into account the assumption (35), the inequality (48) follows.
The next step of the proof consists in showing that

$$\lim_{n\to+\infty}\left\|\chi^{\pm}\left(K_{1,n}^{\pm,w}-wK_1^{\pm}\right)f\right\|_\infty=0, \qquad \forall f\in C_w, \tag{52}$$

using the Banach-Steinhaus theorem applied to the sequences of operators $\chi^{\pm}K_{1,n}^{\pm,w}:C_w\longrightarrow C$, $n\in\mathbb{N}$, which, as proved above (see (50) and (51)), are uniformly bounded for sufficiently large n. To this end, we consider the following dense subspace of the weighted space C_w

$$P_{w^{-1}}=\left\{w^{-1}p|p\in\mathbb{P}\right\}$$

where by $\mathbb{P}$ we have denoted the set of all algebraic polynomials defined on $[-1,1]$, and we prove that for any $f=w^{-1}p\in P_{w^{-1}}$ the limit condition (52) holds true. We limit the proof to the case "$-$".
We have that

$$\left\|\chi^-\left(K_{1,n}^{-,w}-wK_1^-\right)f\right\|_\infty=\max\left\{\max_{[-1,-1+\tau_n]}\left|\chi^-(t)\left(K_{1,n}^{-,w}-wK_1^-\right)f(t)\right|,\max_{[-1+\tau_n,0]}\left|\chi^-(t)\left(K_{1,n}^{-,w}-wK_1^-\right)f(t)\right|\right\}. \tag{53}$$

Proceeding as in [14, Proof of Lemma 1] and using the error estimate (23) for the Gauss-Jacobi-Lobatto rule proved in Section 3, from the assumption (45), for any $t\in[-1+\tau_n,0]$, it follows that

$$\begin{aligned}
\left|\chi^-(t)\left(K_{1,n}^{-,w}-wK_1^-\right)f(t)\right| &= w(t)\left|\chi^-(t)e_n\left(k^-(t,\cdot)\chi^-wf\right)\right|\\
&\le w(t)\frac{\mathcal{C}}{n^r}\sum_{j=0}^r\binom{r}{j}\int_{-1}^1\left|\frac{\partial^j}{\partial s^j}\left(k^-(t,s)\chi^-\right)\right|\varphi^j(s)\left|p^{(r-j)}(s)\right|\varphi^{r-j}(s)w^{-1}(s)ds\\
&\le 2^{\sigma^+}\frac{\mathcal{C}}{n^r}\sum_{j=0}^r\binom{r}{j}\left\|\frac{\partial^j}{\partial s^j}\left(k^-(t,\cdot)\chi^-\right)\varphi^jw^{-1}\right\|_1\\
&\le \frac{\mathcal{C}}{n^r}\sum_{j=0}^r\binom{r}{j}(1+t)^{-\frac{j}{2}}\\
&\le \frac{\mathcal{C}}{n^r}(1+t)^{-\frac{r}{2}}
\end{aligned}$$

from which we can deduce

$$\max_{[-1+\tau_n,0]}\left|\chi^-(t)\left(K_{1,n}^{-,w}-wK_1^-\right)f(t)\right|\le\frac{\mathcal{C}}{n^{r\epsilon}}. \tag{54}$$

For $t\in[-1,-1+\tau_n]$, taking into account definition (40), we have

$$\begin{aligned}
\left|\chi^-(t)\left(K_{1,n}^{-,w}-wK_1^-\right)f(t)\right| &= \chi^-(t)\left|\frac{(1+t)}{\tau_n}\left[\left(wK_{1,n}^-f\right)(-1+\tau_n)-(wK_1^-f)(-1)\right]+\left[(wK_1^-f)(-1)-\left(wK_1^-f\right)(t)\right]\right|\\
&\le \frac{(1+t)}{\tau_n}\left|\left(wK_{1,n}^-f\right)(-1+\tau_n)-\left(wK_1^-f\right)(-1+\tau_n)\right|+\\
&+\frac{(1+t)}{\tau_n}\left|\left(wK_1^-f\right)(-1+\tau_n)-(wK_1^-f)(-1)\right|+\left|(wK_1^-f)(-1)-\left(wK_1^-f\right)(t)\right|.
\end{aligned}$$

Then, wK_1^-f being a continuous function on $[-1,1]$, (see the proof of Theorem 4.1) we have

$$\lim_{n\to+\infty}\max_{t\in[-1,-1+\tau_n]}\frac{(1+t)}{\tau_n}\left|\left(wK_1^-f\right)(-1+\tau_n)-(wK_1^-f)(-1)\right|=0 \tag{55}$$

and

$$\lim_{n\to+\infty}\max_{t\in[-1,-1+\tau_n]}\left|(wK_1^-f)(-1)-\left(wK_1^-f\right)(t)\right|=0, \tag{56}$$

while, according to (54),

$$\max_{t\in[-1,-1+\tau_n]}\frac{(1+t)}{\tau_n}\left|\left(wK_{1,n}^-f\right)(-1+\tau_n)-\left(wK_1^-f\right)(-1+\tau_n)\right|\le\frac{\mathcal{C}}{n^{r\epsilon}}. \tag{57}$$

Combining (53) with (54)-(57), condition (52) is proved in the case "$-$" for any $f\in P_{w^{-1}}$. The proof can be repeated in a similar way in the case "$+$". Hence the thesis (52) follows. Moreover, using the geometric series theorem, from (48) we can deduce that, for sufficiently large n, say $n \geq n_0$, the operators $wI + \chi^- K^{-,w}_{1,n} + \chi^+ K^{+,w}_{1,n} : C_w \longrightarrow C$ are invertible and their inverses are uniformly bounded with respect to n with

$$\left\| \left(wI + \chi^- K^{-,w}_{1,n} + \chi^+ K^{+,w}_{1,n} \right)^{-1} \right\|_{C \to C_w} \leq \frac{1}{1 - \sup_{n \geq n_0} \left\| \chi^- K^{-,w}_{1,n} + \chi^+ K^{+,w}_{1,n} \right\|_{C_w \to C}}.$$

Now, we observe that the operators $\chi^- w K^{\pm}_{2,n}$ and wH_n map C_w into C and the sequences $\left\{ \chi^- w K^{\pm}_{2,n} \right\}_n$ and wH_n are collectively compact and pointwise convergent in C_w (see [14, Lemma 2]). From all the previous results it follows that (see, for instance,[11, Theorem 10.8 and Problem 10.3]) the operators $\left(wI + \chi^- K^{-,w}_{1,n} + \chi^+ K^{+,w}_{1,n} + \chi^- w K^-_{2,n} + \chi^+ w K^+_{2,n} + wH_n \right)^{-1} : C \longrightarrow C_w$ exist and are uniformly bounded for any $n \geq n_0$. Consequently, since

$$\begin{aligned} f - f_n &= \left(wI + \chi^- K^{-,w}_{1,n} + \chi^+ K^{+,w}_{1,n} + \chi^- w K^-_{2,n} + \chi^+ w K^+_{2,n} + wH_n \right)^{-1} \left[\chi^- \left(K^{-,w}_{1,n} - wK^-_1 \right) f + \chi^+ \left(K^{+,w}_{1,n} - wK^+_1 \right) f \right. \\ &+ \left. \chi^- \left(wK^-_{2,n} - wK^-_2 \right) f + \chi^+ \left(wK^+_{2,n} - wK^+_2 \right) f + (wH_n - wH) f \right] \end{aligned}$$

we have

$$\begin{aligned} \|f - f_n\|_{w,\infty} &\leq \mathcal{C} \left[\left\| \chi^- \left(K^{-,w}_{1,n} - wK^-_1 \right) f \right\|_\infty + \left\| \chi^+ \left(K^{+,w}_{1,n} - wK^+_1 \right) f \right\|_\infty \right. \\ &+ \left. \left\| \chi^- \left(wK^-_{2,n} - wK^-_2 \right) f \right\|_\infty + \left\| \chi^+ \left(wK^+_{2,n} - wK^+_2 \right) f \right\|_\infty + \|(wH_n - wH) f\|_\infty \right] \end{aligned} \tag{58}$$

from which (47) follows. □

In the last part of this section we are going to prove an estimate for the weighted error $\|f - f_n\|_{w,\infty}$, which holds true under further suitable assumptions on the kernel functions.
We shall suppose that the functions $\mathbf{k}^\pm$ defining the Mellin kernels $k^\pm(t,s)$ given in (2) are such that

$$\int_0^\infty x^{-1+\sigma^\pm - \rho^\pm} \left| x^j \left(\mathbf{k}^\pm \right)^{(j)}(x) \right| dx < \infty, \qquad j = 0, 1, \ldots, l \tag{59}$$

for some $0 < \rho^\pm < 1$ and some $l \in \mathbb{N}$. Then, using the same arguments in [14] (see also [7]), we can deduce that if $h(t,s)$ and $g(t)$ are sufficiently smooth functions, the solution $f(t)$ of (32) has the following asymptotic behavior near the endpoints of the interval $[-1,1]$

$$f(t) = \begin{cases} (1+t)^{\rho^- - \sigma^-} f_0^-(t) + (1+t)^{-\sigma^-} f_1^-(t), & t \in [-1,0] \\ (1-t)^{\rho^+ - \sigma^+} f_0^+(t) + (1-t)^{-\sigma^+} f_1^+(t), & t \in [0,1] \end{cases} \tag{60}$$

with $f_0^- \in C^l(-1,0]$ such that $(1+t)^j (f_0^-)^{(j)}(t) \in C([-1,0])$, $f_0^+ \in C^l[0,1)$ such that $(1-t)^j (f_0^+)^{(j)}(t) \in C([0,1])$, for $j = 0, 1, \ldots, l$, and f_1^- and f_1^+ smoother functions in $[-1,0]$ and $[0,1]$, respectively. The following result can be proved.

Theorem 4.5. *Under the hypotheses of Theorem 4.4, if conditions (59) are fulfilled for some $0 < \rho^\pm < 1$ and $l \in \mathbb{N}$,*

$$\sup_{-1 \leq t \leq 1} \left\| \frac{\partial^j}{\partial s^j} \left(k^-(t,\cdot)(1-\chi^-) \right) \varphi^j w^{-1} \right\|_1 < \infty, \quad j = 0, 1, \ldots, r, \tag{61}$$

$$\sup_{-1 \leq t \leq 1} \left\| \frac{\partial^j}{\partial s^j} \left(k^+(t,\cdot)(1-\chi^+) \right) \varphi^j w^{-1} \right\|_1 < \infty, \quad j = 0, 1, \ldots, r, \tag{62}$$

and the kernel $h(t,s)$ satisfies

$$\sup_{-1 \leq t \leq 1} \left\| \frac{\partial^j}{\partial s^j} h(t,\cdot) \varphi^j w^{-1} \right\|_1 < \infty, \quad j = 0, 1, \ldots, r, \tag{63}$$

and the solution f of (32) verifies

$$\left\| (wf)^{(j)} \varphi^j \right\|_\infty < \infty, \quad j = 0, 1, \ldots, r, \tag{64}$$

then the following error estimate holds true

$$\|f - f_n\|_{w,\infty} \leq \frac{\mathcal{C}}{n^\mu} \tag{65}$$

with $\mu = \min\{r\epsilon, 2(1-\epsilon)\rho^-, 2(1-\epsilon)\rho^+\}$.

Proof. We start by estimating the first term in the square brackets on the right-hand side in (58)

$$\left\| \chi^- \left(K^{-,w}_{1,n} - wK^-_1 \right) f \right\|_\infty = \max \left\{ \max_{t \in [-1,-1+\tau_n]} \left| \chi^-(t) \left(K^{-,w}_{1,n} - wK^-_1 \right) f(t) \right|, \max_{t \in [-1+\tau_n, 0]} \left| \chi^-(t) \left(K^{-,w}_{1,n} - wK^-_1 \right) f(t) \right| \right\} =: A_1 + A_2.$$

From (54) it immediately follows that

$$A_2 \leq \frac{\mathcal{C}}{n^{r\epsilon}}$$

and it remains to estimate A_1. For $t \in [-1,-1+\tau_n]$, we can write (see the proof of Theorem 4.4)

$$\left|\chi^-(t)\left(K_{1,n}^{-,w} - wK_1^-\right)f(t)\right| \le A_{1,1}(t) + A_{1,2}(t) + A_{1,3}(t)$$

with

$$A_{1,1}(t) = \frac{(1+t)}{\tau_n}\left|\left(wK_{1,n}^- f\right)(-1+\tau_n) - \left(wK_1^- f\right)(-1+\tau_n)\right|,$$

$$A_{1,2}(t) = \frac{(1+t)}{\tau_n}\left|\left(wK_1^- f\right)(-1+\tau_n) - (wK_1^- f)(-1)\right|,$$

and

$$A_{1,3}(t) = \left|(wK_1^- f)(t) - \left(wK_1^- f\right)(-1)\right|.$$

By (54), for any $t \in [-1,-1+\tau_n]$ we have

$$A_{1,1}(t) \le \frac{\mathcal{C}}{n^{r\epsilon}}.$$

Adding and subtracting the quantity

$$(w\chi^- f)(-1)\int_{\tau_n}^{\infty} x^{-1+\sigma^-}\mathbf{k}^-(x)dx,$$

and taking into account (60), for $A_{1,2}(t)$, $t \in [-1,-1+\tau_n]$, we can write

$$\begin{aligned}
A_{1,2}(t) &\le \frac{1+t}{\tau_n}\left|w(-1+\tau_n)\int_{-1}^{1}\frac{1}{1+s}\mathbf{k}^-\left(\frac{\tau_n}{1+s}\right)\chi^-(s)f(s)ds - (w\chi^- f)(-1)\int_{\tau_n}^{\infty} x^{-1+\sigma^-}\mathbf{k}^-(x)dx\right| \\
&+ \frac{1+t}{\tau_n}\left|(w\chi^- f)(-1)\right|\left|\int_0^{\tau_n} x^{-1+\sigma^-}\mathbf{k}^-(x)dx\right| \\
&\le 2^{\sigma^+}\int_{-1}^{0}\frac{1}{1+s}\left|\mathbf{k}^-\left(\frac{\tau_n}{1+s}\right)\right|\left(\frac{\tau_n}{1+s}\right)^{\sigma^-}\left|(1+s)^{\rho^-}f_0(s) + f_1(s)\right|ds + 2^{\sigma^+}|f_1(-1)|\int_{\tau_n}^{\infty} x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|dx \\
&+ \|f\|_{w,\infty}\int_0^{\tau_n} x^{-1+\sigma^- -\rho^-}\left|\mathbf{k}^-(x)\right|x^{\rho^-}dx.
\end{aligned}$$

Now, using the change of variable $x = \frac{\tau_n}{1+s}$, we get

$$\begin{aligned}
A_{1,2}(t) &\le 2^{\sigma^+}\tau_n^{\rho^-}\int_{\tau_n}^{\infty} x^{-1+\sigma^- -\rho^-}\left|\mathbf{k}^-(x)\right|\left|f_0\left(\frac{\tau_n}{x}-1\right)\right|dx + 2^{\sigma^+}\int_{\tau_n}^{\infty} x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|\left|f_1\left(\frac{\tau_n}{x}-1\right)\right|dx \\
&+ 2^{\sigma^+}|f_1(-1)|\int_{\tau_n}^{\infty} x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|dx + \|f\|_{w,\infty}\tau_n^{\rho^-}\int_0^{\tau_n} x^{-1+\sigma^- -\rho^-}\left|\mathbf{k}^-(x)\right|dx \\
&\le \mathcal{C}\tau_n^{\rho^-}\int_0^{\infty} x^{-1+\sigma^- -\rho^-}\left|\mathbf{k}^-(x)\right|dx + 2^{\sigma^+}\int_{\tau_n}^{\infty} x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|\left[\left|f_1\left(\frac{\tau_n}{x}-1\right)\right| + |f_1(-1)|\right]dx.
\end{aligned}$$

Being $\left|f_1(\xi_{\tau_n,x})\right| = \max_{x\in\left[-1,-1+\frac{\tau_n}{x}\right]}|f_1(x)|$ we have

$$\begin{aligned}
\int_{\tau_n}^{\infty} x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|\left[\left|f_1\left(\frac{\tau_n}{x}-1\right)\right| + |f_1(-1)|\right]dx &\le 2\int_{\tau_n}^{\infty} x^{-1+\sigma^-}\left|\mathbf{k}^-(x)\right|\left|f_1(\xi_{\tau_n,x})\right|\left(1+\xi_{\tau_n,x}\right)^{-\rho^-}\left(1+\xi_{\tau_n,x}\right)^{\rho^-} \\
&\le \mathcal{C}\tau_n^{\rho^-}\int_{\tau_n}^{\infty} x^{-1+\sigma^- -\rho^-}\left|\mathbf{k}^-(x)\right|dx
\end{aligned}$$

and, finally, using the assumptions (59), we can conclude that

$$A_{1,2}(t) \le \mathcal{C}\tau_n^{\rho^-} = \frac{\mathcal{C}}{n^{2(1-\epsilon)\rho^-}}.$$

Using similar tools we can estimate also the quantity $A_{1,3}(t)$, for $t \in [-1,-1+\tau_n]$, obtaining

$$\begin{aligned} A_{1,3}(t) &\leq 2^{\sigma^+}\int_{-1}^{0}\frac{1}{1+s}\left|\mathbf{k}^-\left(\frac{1+t}{1+s}\right)\right|\left(\frac{1+t}{1+s}\right)^{\sigma^-}(1+s)^{\sigma^-}|f(s)|ds + |(w\chi^- f)(-1)|\int_{1+t}^{\infty}x^{-1+\sigma^-}|\mathbf{k}^-(x)|dx \\ &+ |(w\chi^- f)(-1)|\int_0^{1+t}x^{-1+\sigma^- -\rho^-}|\mathbf{k}^-(x)|x^{\rho^-}dx \\ &\leq 2^{\sigma^+}(1+t)^{\rho^-}\int_{1+t}^{\infty}x^{-1+\sigma^- -\rho^-}|\mathbf{k}^-(x)|\left|f_0\left(\frac{1+t}{x}-1\right)\right| + 2^{\sigma^+}\int_{1+t}^{\infty}x^{-1+\sigma^-}|\mathbf{k}^-(x)|\left[\left|f_1\left(\frac{1+t}{x}\right)\right| + |f_1(-1)|\right]dx \\ &+ \|f\|_{w,\infty}(1+t)^{\rho^-}\int_0^{\infty}x^{-1+\sigma^- -\rho^-}|\mathbf{k}^-(x)|dx \\ &\leq \mathcal{C}\tau_n^{\rho^-}\int_0^{\infty}x^{-1+\sigma^- -\rho^-}|\mathbf{k}^-(x)|dx \leq \frac{\mathcal{C}}{n^{2(1-\epsilon)\rho^-}}. \end{aligned}$$

Then, we can conclude that

$$\left\|\chi^-\left(K_{1,n}^{-,w} - wK_1^-\right)f\right\|_\infty \leq \frac{\mathcal{C}}{n^{\min\{r\epsilon, 2(1-\epsilon)\rho^-\}}}. \tag{66}$$

By proceeding in an analogous way, one can prove that

$$\left\|\chi^+\left(K_{1,n}^{+,w} - wK_1^+\right)f\right\|_\infty \leq \frac{\mathcal{C}}{n^{\min\{r\epsilon, 2(1-\epsilon)\rho^+\}}}. \tag{67}$$

Now, let us estimate the last term in (58) according to the assumptions (63) and (64). Using (23), for $t \in [-1,1]$ we have

$$\begin{aligned} |(wH_n - wH)f(t)| &\leq \frac{\mathcal{C}}{n^r}w(t)\int_{-1}^{1}\left|\frac{\partial^r}{\partial s^r}[h(t,s)(wf)(s)]\right|\varphi^r(s)w^{-1}(s)ds \\ &\leq \frac{\mathcal{C}}{n^r}w(t)\sum_{j=0}^{r}\binom{r}{j}\int_{-1}^{1}\left|\frac{\partial^j}{\partial s^j}h(t,s)\right|\varphi^j(s)\left|(wf)^{(r-j)}(s)\right|\varphi^{r-j}(s)w^{-1}(s)ds \end{aligned}$$

from which it follows that

$$\|(wH_n - wH)f\|_\infty \leq \frac{\mathcal{C}}{n^r}.$$

Similar estimates hold true for the terms $\left\|\chi^-\left(wK_{2,n}^- - wK_2^-\right)f\right\|_\infty$ and $\left\|\chi^+\left(wK_{2,n}^+ - wK_2^+\right)f\right\|_\infty$ since the kernels of the integral operators $K_2^\pm$ satisfy the assumptions (61)-(62) and the proof is complete. □

Finally, following a standard scheme (see [1]), the proof of the following theorem regarding the conditioning of the linear system (43) can be carried out.

Theorem 4.6. *Under the hypotheses of Theorem 4.4, denoting by M_n the matrix of the coefficients of system (43) and by $\mathrm{cond}(M_n)$ its condition number in the uniform norm, for a sufficiently large $n_0 \in \mathbb{N}$ we have*

$$\sup_{n\geq n_0}\mathrm{cond}(M_n) \leq \sup_{n\geq n_0}\mathrm{cond}(wI + \chi^- K_{1,n}^{-,w} + \chi^+ K_{1,n}^{+,w} + \chi^- wK_{2,n}^- + \chi^+ wK_{2,n}^+ + wH_n) < \infty \tag{68}$$

with $wI + \chi^- K_{1,n}^{-,w} + \chi^+ K_{1,n}^{+,w} + \chi^- wK_{2,n}^- + \chi^+ wK_{2,n}^+ + wH_n : C_w \to C$.

5 Numerical results

This section is dedicated to some numerical examples showing the efficiency of the Nyström type method described in the previous one. In each test we will report the weighted absolute errors

$$err_n = \max_{i=1,\ldots,10^3} w^{\sigma^+,\sigma^-}(y_i)|f(y_i) - f_n(y_i)|,$$

with $y_1,\ldots,y_{10^3}$ equispaced points in the interval $(-1,1)$. We will retain as exact the approximate solution f_{2048} when the exact solution is unknown. We will also report the estimated order of convergence

$$eoc_n = \frac{\log(err_n/err_{2n})}{\log 2}$$

and the condition number in the infinity norm, $\mathrm{cond}(M_n)$, of the coefficient matrix M_n of the linear system (43). One can observe that the sequence $\{\mathrm{cond}(M_n)\}_n$ is bounded w.r.t. n, according to the theoretical results (see (68)).
The exponents $\sigma^\pm$ of the Jacobi weight $w(t)$, satisfying condition (35), can be chosen in such a way as to guarantee the largest values of $\rho^\pm$ which make (59) fulfilled. This allows to maximize the order of convergence, according to Theorem 4.5.
In the numerical simulations the choice of the parameters c and ϵ occurring in the definition of the breaking point τ_n (see (39)) has been made in such a way to minimize the errors, according to the numerical evidence. Anyway, we remark that some criteria proposed in [14] (see also [4]) in order to maximize the theoretical order of convergence $\mu = \min\{r\epsilon, 2(1-\epsilon)\rho^-, 2(1-\epsilon)\rho^+\}$ can be also used. More precisely, taking into account the asymptotic behavior of the solution f described by (60), one can choose $c = 10^{\rho-1}$, being $\rho = \min\{\rho^-,\rho^+\}$ and $\epsilon = \frac{2\rho}{r+2\rho}$ with r such that the assumptions (45)-(46) and (61)-(64) are fulfilled. The numerical results obtained in this case show that such convergence rate estimate is sharp.

Example 5.1. In the first example we have chosen

$$\mathbf{k}^{-}(x) = \frac{1}{2\pi}\frac{\sin\frac{\pi}{3}}{x^2 - 2x\cos\frac{\pi}{3} + 1}, \qquad \mathbf{k}^{+}(x) = \frac{1}{2\pi}\frac{\sin\frac{2}{3}\pi}{x^2 - 2x\cos\frac{2}{3}\pi + 1}, \qquad H(t,s) = s^{\frac{9}{2}}(s^2 + t^2)$$

and the right-hand side $g(t)$ such that the function $f(t) = 1$ is the exact solution of equation (1). In Table 1 we report the numerical results computed in correspondence of the specified values of the parameters involved in the applied method.

Table 1: Example 5.1, $\sigma^+ = 0.99$, $\sigma^- = 0.99$, $c = 3$, $\epsilon = 10^{-2}$

n	err_n	eoc_n	cond(M_n)
2^4	2.15e-02	2.19	6.2135e+03
2^5	4.72e-03	5.92	6.2294e+03
2^6	7.76e-05	2.79	6.2387e+03
2^7	1.11e-05	2.19	6.2378e+03
2^8	2.43e-06	2.29	6.2392e+03
2^9	4.97e-07	2.41	6.2393e+03
2^{10}	9.33e-08		6.2394e+03

Conditions (59) being verified with $\sigma^{\pm} = 0.99$ and $\rho^{\pm} = 0.98$, choosing $\epsilon = \frac{2\rho}{r+2\rho}$ with $r = 500$, we expect a theoretical order of convergence $\mu \simeq 1.9523$ which is confirmed by the estimated order eoc_n reported in the following table

Table 2: Example 5.2, $\sigma^+ = 0.99$, $\sigma^- = 0.99$, $c \simeq 9.5499e-01$, $\epsilon \simeq 3.9046e-03$

n	err_n	eoc_n
2^4	4.73e-03	3.4575
2^5	4.30e-04	1.9557
2^6	1.11e-04	1.9837
2^7	2.80e-05	1.9932
2^8	7.05e-06	1.9968
2^9	1.76e-06	1.9985
2^{10}	4.42e-07	1.9992
2^{11}	1.10e-07	1.9996
2^{12}	2.76e-08	

Finally, in order to show that modifying the quadrature formula near the singularities ±1 (when it is applied for the discretization of the Mellin integral operators) not only is necessary from a theoretical point of view for achieving stability, but it is also useful to get more accurate numerical results, in Table 3 we compare the errors err_n obtained by applying the numerical method described in Section 4 with the errors $\overline{err}_n$ obtained by using a Nyström method based on a non modified quadrature formula w.r.t. to the weight $w(t) = (1-t)^{-\sigma^+}(1+t)^{-\sigma^-}$.

Table 3: Example 5.1, $\sigma^+ = 0.99$, $\sigma^- = 0.99$, $c = 3$, $\epsilon = 10^{-2}$

n	err_n	$\overline{err}_n$
2^4	2.15e-02	8.60e-03
2^5	4.72e-03	2.35e-03
2^6	7.76e-05	6.16e-04
2^7	1.11e-05	1.57e-04
2^8	2.43e-06	3.98e-05
2^9	4.97e-07	1.00e-05
2^{10}	9.33e-08	2.51e-06
2^{11}	1.52e-08	6.29e-07
2^{12}	1.83e-09	1.57e-07
2^{13}	2.59e-10	3.94e-08
2^{14}	1.76e-10	9.85e-09

Example 5.2. In this second test we consider the following known functions

$$\mathbf{k}^-(x) = \frac{1}{\pi}\frac{x^4+1}{(x^2+1)^3}, \qquad \mathbf{k}^+(x) = \frac{x^{\frac{1}{5}}}{(5+x)^3}, \qquad H(t,s) = \sin((s-t)^2), \qquad g(t) = e^{t-1}.$$

In this case the solution f of equation (1) is unknown. Tables 4 and 5 contains the values of the chosen parameters and the obtained results.

Table 4: Example 5.2, $\sigma^+ = 0.99$, $\sigma^- = 0.99$, $c = 15$, $\epsilon = 10^{-3}$

n	err_n	eoc_n	cond(M_n)
2^4	5.93e-03	1.05	7.3990e+03
2^5	2.86e-03	2.80	7.5457e+03
2^6	4.10e-04	6.58	7.5687e+03
2^7	4.28e-06	2.93	7.5732e+03
2^8	5.59e-07	3.82	7.5759e+03
2^9	3.95e-08	3.36	7.5763e+03
2^{10}	3.83e-09		7.5765e+03

Since with $\sigma^\pm = 0.99$ and $\rho^\pm = 0.98$ conditions (59) are verified, choosing $\epsilon = \frac{2\rho^\pm}{r+2\rho^\pm}$ with $r = 100$, the expected theoretical order of convergence is $\mu \simeq 1.9223$. The values of the estimated rate of convergence eoc_n reported in table 5 show the sharpness of the error estimate (65).

Table 5: Example 5.1, $\sigma^+ = 0.99$, $\sigma^- = 0.99$, $c \simeq 9.5499e-01$, $\epsilon \simeq 1.9223e-02$

n	eoc_n
2^4	4.4540
2^5	3.2699
2^6	1.8178
2^7	1.9180
2^8	2.0070
2^9	2.2837

Example 5.3. Here we consider the following kernels and right-hand side

$$\mathbf{k}^-(x) = \frac{2}{\pi}\frac{1}{(1+x^3)^2}, \qquad \mathbf{k}^+(x) = \frac{1}{2}\sqrt{\frac{x}{(1+x^2)^3}}, \qquad H(t,s) = \frac{e^{s+t+2}}{s^2+t^2+1}, \qquad g(t) = \log(t^2+1),$$

while the solution f is unknown. We report the weighted errors, the corresponding estimated order of convergence and the condition numbers produced by our method with the given parameters in Table 6.

Table 6: Example 5.3, $\sigma^+ = 0.7$, $\sigma^- = 0.9$, $c = 5$, $\epsilon = 10^{-3}$

n	err_n	eoc_n	cond(M_n)
2^4	2.11e-02	1.67	1.2303e+02
2^5	6.63e-03	9.38	1.3116e+02
2^6	9.88e-06	3.38	1.3464e+02
2^7	9.44e-07	2.21	1.3626e+02
2^8	2.03e-07	2.05	1.3714e+02
2^9	4.88e-08	2.31	1.3756e+02
2^{10}	9.79e-09		1.3777e+02

Acknowledgements

The author is partially supported by GNCS Project 2022 “Metodi e software per la modellistica integrale multivariata”. This research has been accomplished within RITA (Research ITalian network on Approximation) and the UMI Group TAA (Approximation Theory and Applications).

References

[1] K. E. Atkinson. The Numerical Solution of Integral Equations of the Second Kind. *Cambridge University Press*, Cambridge Monographs on Applied and Computational Mathematics, 552, 1997.

[2] M. C. De Bonis, C. Laurita. A modified Nyström method for integral equations with Mellin type kernels. *J. Comp. Appl. Math.*, 296:512–527, 2016.

[3] M. C. De Bonis,C. Laurita. A Nyström method for integral equations with fixed singularities of Mellin type in weighted L^p spaces. *Appl. Math. Comput.*, 303:55–69, 2017.

[4] M. C. De Bonis,C. Laurita. On the stability of a modified Nyström method for Mellin convolution equations in weighted spaces. *Numer. Algorithms*, 79:611–631, 2018.

[5] R. Duduchava. Integral Equations in Convolution with Discontinuous Presymbols *Singular Integral Equations with Fixed Singularities, and Their Applications to Some Problems of Mechanics*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1979.

[6] D. Elliott, S. Prössdorf. An algorithm for the approximate solution of integral equations of Mellin type. *Numer. Math.*, 70:427–452, 1995.

[7] J. Elschner. On spline approximation for a class of non-compact integral equations. *Math. Nachr.*, 146:271–321, 1990.

[8] J. Elschner. The $h-p-$version of spline approximation methods for Mellin convolution equations. *J. Integral Equations Appl.*, 5:47–73, 1993.

[9] J. Elschner., I. G. Graham. Numerical methods for integral equations of Mellin type. *J. Comp. Appl. Math.*, 125:423–437, 2000.

[10] L. Fermo, C. Laurita. A Nyström method for a boundary integral equation related to the Dirichlet problem on domains with corners. *Numer. Math.*, 130:35–71, 2015.

[11] R. Kress. Linear Integral Equations. *Springer-Verlag*, Applied Mathematical Sciences, 82, 1989.

[12] R. Kress. A Nyström method for boundary integral equations in domains with corners, *Numer. Math*, 58:445–461, 1990.

[13] C. Laurita. A numerical method for the solution of integral equations of Mellin type. *Appl. Numer. Math.*, 116:215–229, 2017.

[14] C. Laurita. A new stable numerical method for Mellin integral equations in weighted spaces with uniform norm. *Calcolo*, 57:25, 2020.

[15] G. Mastroianni, G. V. Milovanović. Interpolation Processes Basic Theory and Applications. *Springer Monographs in Mathematics*, Springer-Verlag, 2009.

[16] G. Mastroianni, G. Monegato. Nyström interpolants based on the zeros of Legendre polynomials for a non-compact integral operator equation. *IMA J. Numer. Anal.*, 14:81–95, 1993.

[17] G. Monegato. A stable Nyström interpolant for some Mellin convolution equations. *Numer. Algorithms*, 11:271–283, 1996.

[18] G. Mastroianni, M. G.Russo. Lagrange interpolation in weighted Besov spaces. *Constr. Approx.*, 15:257–289, 1999.

[19] G. P. Nevai. Orthogonal Polynomials. *Amer. Math. Soc.*, 1979.

[20] G. P. Nevai. Mean convergence of Lagrange interpolation, III. *Trans. Amer. Math. Soc.*, 282:669–698, 1984.

[21] G. Szegő. Orthogonal polynomials. *Amer. Math. Soc. Colloq. Publ.*, 23,4thed., Amer. Math. Soc., 1975.