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Homogeneous polynomial approximation on convex and star like domains

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Abstract

In the present paper we consider the following central problem on the approximation by homogeneous polynomials: *For which 0-symmetric star like domains $K \subset \mathbb{R}^d$ and which $f \in C(\partial K)$ there exist homogeneous polynomials h_n, h_{n+1} of degree n and $n+1$, respectively, so that uniformly on ∂K*

$$f = \lim_{n\to\infty}(h_n + h_{n+1})?$$

This question is the analogue of the Weierstrass approximation problem when polynomials of total degree are replaced by the homogeneous polynomials. A survey of various recent results on the above question is given with some relevant open problems being included, as well.

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1 Introduction

A basic question of approximation theory concerns the possibility of uniformly approximating continuous functions by elements of the given family of special functions. The first fundamental density results were those of Weierstrass [17] who proved in 1885 the density of algebraic polynomials in the class of continuous real-valued functions on a compact interval, and the density of trigonometric polynomials in the class of 2π-periodic continuous real-valued functions. These classical Weierstrass approximation theorems are the foundations of the modern Approximation Theory, they subsequently led to numerous generalizations and extensions. In particular, the celebrated Stone-Weierstrass theorem (see e.g. [2], p.13) generalizes the above Weierstrass theorem to subalgebras of $C(K)$ of continuous functions on the compact set K: if A is a subalgebra in $C(K)$ (that is a linear subspace closed relative to multiplication) which separates points of K then A is dense in

$$\{f \in C(K) : f(x) = 0,\ x \in Z_A\},$$

where

$$Z_A := \{\mathbf{x} \in K : g(\mathbf{x}) = 0, \forall g \in A\}$$

is the **zero set** of A.

This yields the multivariate version of the classical Weierstrass theorem (proved originally by Pickard [11]) asserting that for any compact set $K \subset \mathbb{R}^d$ and any continuous real valued function $f \in C(K)$ there is a sequence of polynomials $p_n \in P_n^d$ of d variables and degree at most n such that $\lim_{n\to\infty} p_n = f$ uniformly on K. Here and in what follows P_n^d denotes the set of algebraic polynomials of degree at most n in d real variables. For a comprehensive treatment of density results in Approximation Theory see the nice survey by Pinkus [12].

Of course, the most interesting density problems correspond to those situations when the subalgebra property fails and thus the Stone-Weierstrass theorem is not applicable. For instance, consider the space M :=span$\{x^{\lambda_j}, 0 = \lambda_0 < \lambda_1 < \dots \uparrow \infty\}$. Then M is a linear subspace of $C[0,1]$ which is not a subalgebra, because it is not close relative to multiplication and therefore the Stone-Weierstrass theorem is not applicable. By the famous Müntz theorem [10] M is dense in $C[0,1]$ if and only if $\sum_j \frac{1}{\lambda_j} = \infty$. Another relevant example is the subspace of Lorentz type θ-*incomplete* polynomials $p_n(x) = \sum_{n\theta \le k \le n} a_k x^k$, $n \in \mathbb{N}$ where $0 < \theta < 1$

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is a fixed number, see [9]. This time the set of all incomplete polynomials is closed relative to multiplication, but clearly it is not linear, i.e., the subalgebra condition fails again. It was shown by G.G. Lorentz [9], von Golitschek [3] and Saff and Varga [13] that given $f \in C[0,1]$ there exists a sequence of θ-incomplete polynomials which converges to f uniformly on $[0,1]$ if and only if the function vanishes on $[0,\theta^2]$. Thus in order to compensate for the lack of the subalgebra property one needs to impose an additional restriction that the functions vanish on a certain set. As shown in [14] these exceptional zero sets are typical in general in case when approximating by algebraic polynomials with varying weights. This phenomena of *exceptional zero sets* will be a reoccurring theme in our considerations below.

The space of **homogeneous** multivariate polynomials appears as an important tool in various areas of Analysis and Approximation Theory. The set of homogeneous polynomials of d variables and total degree n is defined by

$$H_n^d := \{\sum_{|\mathbf{k}|=n} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R}\},\ \ \mathbf{x} \in \mathbb{R}^d.$$

The question of density of homogeneous polynomials received a considerable attention in the past two decades. A substantial progress on this interesting and difficult problem has been achieved but many intriguing problems stay open. In this paper our goal is to give a comprehensive survey of the most important results in this area and also list various interesting remaining open problems.

The first crucial question we need to consider concerns the *domain of approximation*. Homogeneous polynomials $h \in H_n^d$ of degree n contain only monomials of exact degree n, and therefore they evidently satisfy the property $h(t\mathbf{x}) = t^n h(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Hence if $h_n(\mathbf{x}) \to a, n \to \infty$ converge to a nonzero value a at some $\mathbf{x} \in \mathbb{R}^d$, then $h_n(t\mathbf{x})$ tend to zero if $|t| < 1$ and tend to infinity for $|t| > 1$. Thus we must assume that each line passing through the origin intersects the underlying domain in at most two points which are symmetric about the origin. Since homogeneous polynomials are either even or odd depending on their degree it is natural to consider compact sets symmetric with respect to the origin. So in view of the above comments it is natural to restrict our attention to *0-symmetric star like domains* $K \subset \mathbb{R}$ which satisfy the property that for every $\mathbf{x} \in K$ we have $(-\mathbf{x}, \mathbf{x}) \in \text{IntK}$. Furthermore, we will study the approximation problem on the boundary ∂K of this 0-symmetric star like domain K. Such boundary sets ∂K will be called *0-symmetric star like surfaces*. In addition, we need to observe that $h \in H_n^d$ has the same parity as n, it is even for even n and odd when n is odd. Since a continuous function f is not necessarily even or odd, clearly at least **two homogeneous polynomials are needed** for approximating f! Thus in general the density problem must be considered for sums of pairs of homogeneous polynomials.

So now let us formulate the central problem related to approximation by homogeneous polynomials which was first proposed by the author of the present survey at the Functional Analysis and Approximation Theory Conference held in Maratea in 2004.

For which 0-symmetric star like domains $K \subset \mathbb{R}^d$ it holds that for every $f \in C(\partial K)$ there exist homogeneous polynomials $h_n, h_{n+1} \in H_n^d, H_{n+1}^d$ so that uniformly on ∂K

$$f = \lim_{n\to\infty}(h_n + h_{n+1})? \tag{1}$$

2 Some examples

It is not hard to show that (1) is equivalent to $f = \lim_{n\to\infty} h_{2n}, h_{2n} \in H_{2n}^d$ being true for every even function $f \in C(\partial K)$. Indeed, any odd function f can be written as $f(\mathbf{x}) = \sum_{1\le j\le d} x_j g_j(\mathbf{x})$ where $g_j(\mathbf{x}) := \frac{x_j}{|\mathbf{x}|^2} f(\mathbf{x}) \in C(\partial K)$ are all even. Thus the ability to approximate even function and usual even-odd decomposition yields required approximation for any function.

Moreover, in order to verify (1) it even suffices to show that $h_{2n} \to 1$ uniformly on ∂K for certain $h_{2n} \in H_{2n}^d$, because if such a sequence of homogeneous polynomials tending to unity exists then the general case can be easily handled with the help of the Weierstrass approximation theorem, see [16], Proposition 1.2 for details.

Before proceeding further let us look at some specific examples.

Example A. Unit sphere. Consider the unit sphere in $\mathbb{R}^d$ given by

$$S^{d-1} = \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : |\mathbf{x}|^2 = x_1^2 + ... + x_d^2 = 1\}.$$

Obviously, when j is an even integer

$$|\mathbf{x}|_{l_2}^j = (x_1^2 + ... + x_d^2)^{\frac{j}{2}} \in H_j^d.$$

Hence on the unit sphere S^{d-1} we have for every $p \in P_n^d$

$$p = h_n + h_{n-1} + ... + h_0 = \sum_{n-j=\text{even}} h_j + \sum_{n-j=\text{odd}} h_j =$$

$$\sum_{n-j=\text{even}} |\mathbf{x}|_{l_2}^{n-j} h_j + \sum_{n-j=\text{odd}} |\mathbf{x}|_{l_2}^{n-1-j} h_j \subset H_n^d + H_{n-1}^d.$$

Thus on S^{d-1} the relation $P_n^d = H_n^d + H_{n-1}^d$ holds. Now by the Weierstrass theorem it follows that (1) must be true for every function continuous on the unit sphere.

Example B. Bivariate unit square. Consider the unit sphere in $\mathbb{R}^2$ which we define as

$$D := \{(x,y) \in \mathbb{R}^2 : |x| + |y| = 1\}.$$

Let $h(x,y) = \sum_{k=0}^n a_k x^{2k} y^{2n-2k}, a_k \in \mathbb{R}$ be a bivariate homogeneous polynomial even with respect to both variables. Then on the line segment $x + y = 1, x, y \geq 0$ (which is the first quarter part of the square D) $h(x,y) = \sum_{k=0}^n a_k x^{2k}(1-x)^{2n-2k} = g(x)$ where $g(x) \in P_{2n}^1$ is a so called self-reciprocal univariate polynomial of degree $2n$ satisfying the property $x^{2n} g(\frac{1}{x}) = g(x), x \in \mathbb{R}$. And vice-versa it is easy to see that any self reciprocal polynomial of degree $2n$ can be written as a linear combination of $x^{2k}(1-x)^{2n-2k}, 0 \leq k \leq n$. Furthermore, it has been proved in Kroó-Szabados [6], Theorem 3, p. 475 that there exists a sequence g_{2n} of self-reciprocal polynomials of degree $2n$ tending to 1 uniformly on $[0,1]$. This means that a certain sequence of homogeneous polynomials $h_{2n} \in H_{2n}^2$ even in both variables tends to 1 uniformly on the segment $x + y = 1, x, y \geq 0$. Since h_{2n} is even in both variables convergence to 1 holds on all of the square D. As it was observed above this yields the desired approximation property for every continuous function on D. Moreover, since homogeneous polynomials are rotation and dilation invariant the same holds for the standard unit square $\{(x,y) \in \mathbb{R}^2 : \max\{|x|,|y|\} = 1\}$.

Thus the density property (1) holds for both the unit sphere and unit bivariate square.

Now let us give a simple example when the required density fails.

Example C. Interlacing discs. Let $R = \{(x,y) \in \mathbb{R}^2 : (x \pm a)^2 + y^2 \leq 1\}$ be the union of two interlacing unit discs centered at $(\pm a, 0)$ with $0 < a < 1$ to be chosen below. Evidently, R is a 0-symmetric star like domain. Furthermore, it is well known that for every univariate polynomial $p \in P_n^1$ of degree $\leq n$ such that $|p(x)| \leq 1, |x| \leq 1$ we have $|p(b)| \leq (2b)^n$ whenever $b > 1$. Now consider any $h_{2n}(x,y) \in H_{2n}^2$ normalized by $\|h_{2n}\|_{C(\partial R)} = 1$. Then applying the previous estimate to the univariate polynomial $h_{2n}(at, 1-t), t \in \mathbb{R}$ of degree $\leq 2n$ implies

$$|h_{2n}(0,1)| \leq (2\sqrt{1+a^2})^{2n}. \tag{2}$$

Note that $(0, \sqrt{1-a^2}) \in \partial R$ and in addition by (2)

$$|h_{2n}(0, \sqrt{1-a^2})| = (\sqrt{1-a^2})^{2n} |h_{2n}(0,1)| \leq 4^n (1-a^4)^n.$$

Thus choosing arbitrary $\frac{3}{4} < a^4 < 1$ will lead to $|h_{2n}(0, \sqrt{1-a^2})| \to 0, n \to \infty$ for any sequence of homogeneous polynomials $h_{2n} \in H_{2n}^2$ normalized by $\|h_{2n}\|_{C(\partial R)} = 1$. Since $(0, \sqrt{1-a^2}) \in \partial R$ this clearly yields that (1) can not hold on ∂R with $f = 1$.

3 Density of homogeneous polynomials on 0-symmetric convex bodies

The examples exhibited in the previous section show that the density property (1) holds for both the unit sphere and square even though these two domains are quite different from the point of view of smoothness. On the other hand an important common feature of these domains is their *convexity*. This observation lead the author of this survey to the formulation of the following conjecture.

Conjecture.(2004) *For any 0-symmetric **convex** body $K \subset \mathbb{R}^d$ and every $f \in C(\partial K)$ there exist homogeneous polynomials $h_n, h_{n+1} \in H_n^d, H_{n+1}^d$ so that $f = \lim_{n\to\infty}(h_n + h_{n+1})$ holds uniformly on ∂K.*

The above conjecture attracted considerable attention and soon several important partial cases have been resolved affirmatively. Namely, Varjú [16] settled the case $d = 2$, and verified the above conjecture when K is a convex polytope or a C_+^2 convex domain in $\mathbb{R}^d$. Independently, Benko and Kroó [1] also verified the case $d = 2$ and proved the conjecture for convex bodies under the more general $C^{1+\epsilon}$ smoothness condition with any $\epsilon > 0$. Subsequently, the $C^{1+\epsilon}$ smoothness condition was relaxed further in a paper by Kroó and Szabados [7] where the above conjecture was shown to hold for any *regular* convex body in $\mathbb{R}^d$. Recall that regularity of convex body means that it possesses a *unique supporting hyperplane* at every point on its boundary.

Thus summarizing these results it can be observed that the density conjecture for convex bodies has been resolved in the following three important cases:

$$(i)\ d = 2;\ (ii)\ \textbf{convex polytopes};\ (iii)\ \textbf{regular convex bodies}. \tag{3}$$

A significant contribution to resolving the above conjecture was made by Totik [15] using the concept of *ϵ-regularity*. A convex body $K \subset \mathbb{R}^d$ is called ϵ-regular if the angle between any two normals at every point on its boundary is at most ϵ. Based on this notion the following result is essentially verified in [15]:

Let $K \subset \mathbb{R}^d$ be a 0-symmetric convex body and assume that there exists an integer η_K depending only on K such that for every $\epsilon > 0$ the set K is the intersection of at most η_K ϵ-regular 0-symmetric convex bodies. Then for every $f \in C(\partial K)$ there exist homogeneous polynomials $h_n, h_{n+1} \in H_n^d, H_{n+1}^d$ so that (1) holds uniformly on ∂K.

Above result includes all three main cases (3) when the density conjecture is known to hold. For instance, if $K \subset \mathbb{R}^d$ is a regular convex body then clearly the regularity condition holds with $\eta_K = 1$ $(\epsilon = 0)$. Furthermore, any 0-symmetric convex polytope in $\mathbb{R}^d$ with $2m$ faces of dimension $d-1$ is an intersection of m 0-symmetric regular convex bodies, so $\eta_K = m$ for convex polytopes. Finally, it can be shown, see [15] that for any 0-symmetric convex body $K \subset \mathbb{R}^2$ we have $\eta_K = 4$ yielding the full conjecture in case $d = 2$.

The concept of ϵ-regularity on one hand leads to a unified treatment of all major known cases (3) when the density conjecture is known to hold, but on the other hand it does not lead to new significant classes of convex bodies for which the density property can be verified. In particular, this concept is not applicable for such standard convex bodies like for example 0-symmetric *circular cones*.

Nevertheless, using the density property established on squares together with the fact that the set of homogeneous polynomials is closed relative to *composition* allows to handle circular cones, as well.

Consider the 0-symmetric circular cone

$$\partial\Omega := \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : \sqrt{x_1^2 + ... + x_{d-1}^2} + |x_d| = 1\} \subset \mathbb{R}^d.$$

Obviously, we have that $\eta_\Omega = \infty$ for this convex body if $d > 2$ and hence circular cones in $\mathbb{R}^d, d > 2$ are not covered by any of the above cases (3). Nevertheless, the relation (1) can be shown to hold for any function f continuous on $\partial\Omega$.

Indeed, using (3) (ii) (or Example B) the density property holds on the bivariate square

$$D := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| = 1\} \subset \mathbb{R}^2.$$

Therefore there exist $h_{2n} \in H_{2n}^2$ such that $h_{2n} \to 1, n \to \infty$ uniformly on D. Note that since the square D is symmetric relative to each coordinate the homogeneous polynomials h_{2n} can be chosen to be even in each variable x_1, x_2. Now consider the homogeneous polynomial

$$h_{2n}^*(x_1, ..., x_d) = h_{2n}(|x_d|, \sqrt{x_1^2 + ... + x_{d-1}^2}) \in H_{2n}^d.$$

Clearly $h_{2n}^*(x_1, ..., x_d)$ approximates 1 on the cone $\partial\Omega$ verifying the density on this domain.

The same approach can be used in order to verify the density of homogeneous polynomials on more general *domains of revolution*, see [4] for details.

4 Density of homogeneous polynomials on 0-symmetric star like surfaces

In the previous section we summarized all known main results on the density of homogeneous polynomial approximation on 0-symmetric convex bodies. Now we would like to address the same question for non convex domains. We will see below that while the density may hold in the non convex case as well, typically it fails for many model non convex domains.

First let us present a class of non convex 0-symmetric star like surfaces on which homogeneous polynomials are dense. The idea is somewhat similar to the Example A above which was based on the observation that the unit sphere is a level surface of a quadratic homogeneous polynomial. Now we will explore this idea further by considering level surfaces of even homogeneous polynomials of degree > 2 which evidently are not necessarily convex.

So let $h_{2r}^* \in H_{2r}^d, r > 1$ be an arbitrary even homogeneous polynomial of degree > 2 and consider its level surface $L_r := \{\mathbf{x} \in \mathbb{R}^d : h_{2r}^*(\mathbf{x}) = 1\}$. Clearly, L_r is a 0-symmetric star like surface. Let us verify now the following general statement.

For every compact 0-symmetric subset K of the level surface L_r and any $f \in C(K)$ there exist homogeneous polynomials $h_{2rn}, h_{2rn+1} \in H_{2rn}^d, H_{2rn+1}^d$ so that uniformly on K $f = \lim_{n\to\infty}(h_{2rn} + h_{2rn+1})$.

Denote by K_0 the set of ordered pairs $\mathbf{z} = (\mathbf{x}, -\mathbf{x}), \mathbf{x} \in K$ with these pairs being order so that the first nonzero coordinate of $\mathbf{x}$ is positive. The metric in K_0 is given by $|\mathbf{z} - \mathbf{w}| := |\mathbf{x} - \mathbf{y}|, \mathbf{z} = (\mathbf{x}, -\mathbf{x}),\ \mathbf{w} = (\mathbf{y}, -\mathbf{y}) \in K_0$. Clearly K_0 is compact in this topology. Furthermore, for an even function $f \in C(K)$ set $f_0(\mathbf{z}) = f(\mathbf{x}), \mathbf{z} = (\mathbf{x}, -\mathbf{x}) \in K_0$. Then $f_0 \in C(K_0)$. Consider the set of polynomials

$$P^* = \{\sum_{j=0}^{m} h_j,\ h_j \in H_{2rj}^d,\ m \in \mathbb{N}\}.$$

Clearly P^* is a subalgebra of $C(K_0)$ containing 1. In addition, it is easy to show that P^* separates points in K_0. Indeed, if $\mathbf{z}_1 = (\mathbf{x}, -\mathbf{x}), \mathbf{z}_2 = (\mathbf{y}, -\mathbf{y})$ are two distinct elements of K_0 then evidently $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} \neq -\mathbf{y}$. This implies that either $\mathbf{x} \neq t\mathbf{y}, t \in \mathbb{R}$ (points are not collinear) or $\mathbf{x} = t\mathbf{y}, t \neq \pm 1$ (points are collinear). In case if the first case holds we can choose $\mathbf{w} \perp \mathbf{y}$ so that $\mathbf{w} \cdot \mathbf{x} \neq 0$. Then setting $h^*(\mathbf{z}) := (\mathbf{z} \cdot \mathbf{w})^{2r} \in P^*$ we have $h^*(\mathbf{y}) = 0$ while $h^*(\mathbf{x}) \neq 0$ and thus the point separation holds. In case if $\mathbf{x} = t\mathbf{y}, t \neq \pm 1$ we can use $h^*(\mathbf{z}) := (\mathbf{z} \cdot \mathbf{y})^{2r} \in P^*$ yielding $h^*(\mathbf{y}) = |\mathbf{y}|^{4r} \neq h^*(\mathbf{x}) = t^{2r}|\mathbf{y}|^{4r}$.

Thus $P^* \subset C(K_0)$ satisfies all requirements of the Stone-Weierstrass theorem and therefore for any $\epsilon > 0$ and every even function $f \in C(K)$ there exists $h_n = \sum_{j=0}^{n} h_j, h_j \in H_{2rj}^d$ so that $\|f - h_n\|_{C(K)} < \epsilon$. Since $h_{2r}^*(\mathbf{x}) = 1$ on the level surface L_r we obviously have for any $\mathbf{x} \in L_r$ and every $N > n$

$$h_n = \sum_{j=0}^{n} h_j (h_{2r}^*)^{N-j} := h_N \in H_{2rN}^d, \quad \|f - h_N\|_{C(K)} < \epsilon.$$

Now applying a standard diagonalization process yields the required statement for even functions $f \in C(K)$: there exist even homogeneous polynomials $h_{2rn} \in H_{2rn}^d$ so that uniformly on K we have $h_{2rn} \to f, n \to \infty$. Moreover, given any odd function

$g \in C(K)$ we can use the approximation property for each even function $\frac{x_j}{|\mathbf{x}|^2} g(\mathbf{x}) \in C(K), 1 \le j \le d$ implying with proper $h_{2rn,j} \in H^d_{2rn}$ the uniform on K convergence

$$h_{2rn,j} \to \frac{x_j}{|\mathbf{x}|^2} g(\mathbf{x}), \;\; n \to \infty, \; 1 \le j \le d.$$

Thus setting $h_{2rn+1} := \sum_{1 \le j \le d} x_j h_{2rn,j} \in H^d_{2rn+1}$ it follows that $h_{2rn+1} \to g, n \to \infty$ uniformly on K. Finally invoking the standard even+odd decomposition of functions yields the required general statement.

Example. Consider the bivariate polynomial $h_4(x,y) := x^4 + y^4 + ax^2y^2 \in H^2_4$. It can be easily seen that its level curve $L_4 := \{(x,y) \in \mathbb{R}^2 : h_4(x,y) = 1\}$ incloses a bounded non convex set whenever $-2 < a < -1$. Nevertheless by the above result the homogeneous polynomials are dense in $C(L_4)$. This provides an example of a smooth non convex 0-symmetric closed curve for which the homogeneous density property holds. Similar examples can be exhibited in higher dimensions, too.

Now we proceed by presenting a unified approach to the study of density of homogeneous polynomials on 0-symmetric star like surfaces. It turns out that the boundary of every 0-symmetric star like domain contains a 0-symmetric *exceptional zero set* so that a continuous function can be uniformly approximated on this star like surface by a sum of two homogeneous polynomials if and only if the function vanishes on this set. This means that the Weierstrass type approximation problem on non convex star like domains amounts to the study of these exceptional zero sets.

The existence of exceptional zero sets is verified in Kroó [4].

For every 0-symmetric star like domain K in $\mathbb{R}^d$ there exists a 0-symmetric set $Z(K) \subset \partial K$ so that for any given $f \in C(\partial K)$ the following statements are equivalent

(i) there exist $h_n + h_{n+1} \in H^d_n + H^d_{n+1}$ such that $f = \lim_{n\to\infty}(h_n + h_{n+1})$ uniformly on ∂K

(ii) $f = 0$ on $Z(K)$.

Clearly, the conjecture concerning the density of homogeneous polynomials on every 0-symmetric convex surface stated in the previous section can now be reformulated in terms of the exceptional zero sets.

Conjecture. *For every 0-symmetric convex body $K \subset \mathbb{R}^d$ we have that $Z(K) = \emptyset$.*

In addition the density results for convex bodies listed in the previous section can be now restated as well:

Let $K \subset \mathbb{R}^d$ be a 0-symmetric convex body. Then $Z(K) = \emptyset$ whenever K is regular, or K is a polytope, or $d = 2$.

Finding this mysterious exceptional zero set $Z(K)$ is a rather nontrivial matter. This question has been settled recently for the model case of the so called l_α *spheres*.

Consider the l_α ball B^d_α and sphere S^{d-1}_α in $\mathbb{R}^d$ given for any $\alpha > 0$ by

$$B^d_\alpha := \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : |x_1|^\alpha + ... + |x_d|^\alpha \le 1\},$$

$$S^{d-1}_\alpha := \partial B^d_\alpha = \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : |x_1|^\alpha + ... + |x_d|^\alpha = 1\}.$$

When $\alpha = 1$ B^d_α is a convex polytope, while for $\alpha > 1$ it is a regular convex domain, hence by the results listed in Section 3 the density of homogeneous polynomials holds on $S^{d-1}_\alpha, d \ge 2$ whenever $\alpha \ge 1$. On the other hand for $0 < \alpha < 1$ the set B^d_α is **not convex** which leads to the question of finding exceptional sets of l_α spheres when $0 < \alpha < 1$.

It was recently shown by Kroó and Totik [8] that in the bivariate case the exceptional zero set of the l_α spheres when $0 < \alpha < 1$ coincides with the set of all non smooth points of the boundary. That is it consists of four points

$$Z(S^2_\alpha) = \{(\pm 1, 0), (0, \pm 1)\}.$$

Thus when $d = 2$ and $0 < \alpha < 1$ a function continuous on the l_α sphere S^2_α can be uniformly approximated on it by pairs of homogeneous polynomials if and only if this function vanishes for $\{(\pm 1, 0), (0, \pm 1)\}$.

In case when $0 < \alpha < 1$ is *rational* the proof of sufficiency in [8] includes an *explicit construction* of approximating homogeneous polynomials which is quite rare in this theory.

It should be also noted that the necessity part in the above result immediately extends to the case of $d > 2$ variables since the restriction of a homogeneous polynomial to any 2-dimensional plane passing through the origin is clearly a bivariate homogeneous polynomial of the same degree. This implies that when $d > 2$ the exceptional set of S^{d-1}_α must contain the set of its *nonsmooth points*, i.e.,

$$Z(S^d_\alpha) \supseteq \{\mathbf{x} = (x_1, ..., x_d) \in S^d_\alpha : x_1 \cdot ... \cdot x_d = 0\}.$$

Moreover, when $d > 2$ and $0 < \alpha < 1$ is rational it was subsequently verified in Kroó [5] that

$$Z(S^d_\alpha) = \{\mathbf{x} = (x_1, ..., x_d) \in S^d_\alpha : x_1 \cdot ... \cdot x_d = 0\}.$$

Again the construction of approximating homogeneous polynomials can be given explicitly if $0 < \alpha < 1$ is rational.

As a byproduct of above results we arrive at the conclusion that for the l_α sphere S_α^{d-1} in $\mathbb{R}^d$ the homogeneous approximation property (1) holds for every continuous function *if and only if* $\alpha \geq 1$. Thus at least **for the case of l_α spheres the convexity is not only sufficient but also necessary for the density of homogeneous polynomials.**

An important result proved by Varjú [16] states that whenever K_1, K_2 are 0-symmetric star-like domains in $\mathbb{R}^d$ for which density of pairs of homogeneous polynomials holds (i.e. $Z(K_1) = Z(K_1) = \emptyset$) then the same is true for $K_1 \cap K_2$. In other words in terms of the exceptional zero sets we have the following implication

$$Z(K_1) = Z(K_2) = \emptyset \ \Rightarrow \ Z(K_1 \cap K_2) = \emptyset. \tag{4}$$

It turns out that Varjú's result can be extended to a more general setting of star like domains with non empty exceptional zero sets. The next statement is verified in Kroó, [4].

Let K_1, K_2 be any 0-symmetric star-like domains in $\mathbb{R}^d$. Then

$$Z(K_1 \cap K_2) \subset (Z(K_1) \cup Z(K_2)). \tag{5}$$

It is easy to see that (4) is now a special case of (5). Moreover, we can make an essentially more general conclusion.

If K_1, K_2 are any 0-symmetric star-like domains in $\mathbb{R}^d$ then

$$Z(K_1) \cap K_2 = \emptyset, \ Z(K_2) \cap K_1 = \emptyset \ \Rightarrow \ Z(K_1 \cap K_2) = \emptyset. \tag{6}$$

The last statement easily follows from (5) because if $\mathbf{z} \in Z(K_1 \cap K_2)$ then either $\mathbf{z} \in Z(K_1)$ or $\mathbf{z} \in Z(K_2)$. But by definition we also have $\mathbf{z} \in K_1 \cap K_2$ which evidently contradicts (6).

Thus in order for density of homogeneous polynomials to hold on the intersection of two 0-symmetric star-like domains it suffices to assume that exceptional zero sets of each of them are not contained in the other domain. This observation allows us to obtain new interesting examples.

Example. "Nowhere convex" star like domains satisfying homogeneous approximation property. Consider the bivariate l_α ball given by $B_\alpha^2 := \{(x,y) \in \mathbb{R}^2 : |x|^\alpha + |y|^\alpha \leq 1\}$. As mentioned above when $0 < \alpha < 1$ then a function $f \in C(S_\alpha^1)$ is a uniform limit on S_α^1 of sums $h_n + h_{n+1}$ of homogeneous polynomials if and only if $f(\pm 1, 0) = f(0, \pm 1) = 0$. This means that the exceptional zero set of this domain is given by

$$Z(B_\alpha^2) = \{(\pm 1, 0), (0, \pm 1)\}, \quad 0 < \alpha < 1.$$

Clearly we can also rotate B_α^2 by $\frac{\pi}{4}$ and consider the domain

$$D_\alpha^2 := \{(x,y) \in \mathbb{R}^2 : |x+y|^\alpha + |x-y|^\alpha \leq 2^{\alpha/2}\}, \ 0 < \alpha < 1,$$

for which the exceptional zero set consists of the four points

$$Z(D_\alpha^2) = \{(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})\}, \quad 0 < \alpha < 1.$$

Now consider the intersection of the above domains given by

$$\Omega_\alpha := B_\alpha^2 \cap D_\alpha^2 = \{(x,y) \in \mathbb{R}^2 : |x|^\alpha + |y|^\alpha \leq 1 \text{ and } |x+y|^\alpha + |x-y|^\alpha \leq 2^{\alpha/2}\}, \ 0 < \alpha < 1.$$

Evidently, $Z(B_\alpha^2) \cap D_\alpha^2 = \emptyset$ and $Z(D_\alpha^2) \cap B_\alpha^2 = \emptyset$. Thus it follows by (6) that $Z(\Omega_\alpha) = \emptyset$, and hence the homogeneous approximation property holds for $\partial \Omega_\alpha$. It is interesting to note that the star like domain Ω_α has a "nowhere convex" boundary in the sense that discs of arbitrarily small radius centered at any point of $\partial \Omega_\alpha$ have a non convex intersection with the interior of the domain. Thus we obtain "nowhere convex" star like surfaces which nevertheless satisfy the required approximation property.

Example. Exceptional zero sets of minimal cardinality. Since every exceptional zero set is symmetric with respect to the origin any nonempty exceptional zero set must consist of at least 2 points. Do there exist such minimal exceptional zero sets in $\mathbb{R}^d$? Let us give an affirmative answer to this question.

As seen above for the bivariate l_α ball $B_\alpha^2 \subset \mathbb{R}^2$ its exceptional zero set $Z(B_\alpha^2)$ consists of 4 points $(\pm 1, 0), (0, \pm 1)$. It is shown in [4] that rotating B_α^2 yields the domain

$$\Gamma_\alpha := \{\mathbf{x} \in \mathbb{R}^d : |x_1|^\alpha + (x_2^2 + \ldots + x_d^2)^{\alpha/2} \leq 1\}, \ 0 < \alpha < 1,$$

with an exceptional zero set

$$Z(\Gamma_\alpha) = \{(\pm 1, 0, \ldots, 0), (0, \mathbf{y}), \mathbf{y} \in S^{d-2}\}.$$

Now consider the ellipse

$$E := \{\mathbf{x} \in \mathbb{R}^d : x_1^2 + 2(x_2^2 + \ldots + x_d^2) \leq 1\}$$

and its intersection with Γ_α given by

$$\Theta_\alpha := \{\mathbf{x} \in \mathbb{R}^d : |x_1|^\alpha + (x_2^2 + \ldots + x_d^2)^{\alpha/2} \leq 1 \text{ and } x_1^2 + 2(x_2^2 + \ldots + x_d^2) \leq 1\}.$$

Since $Z(E) = \emptyset$ and $Z(\Gamma_\alpha) \cap E = (\pm 1, 0, ..., 0)$ it follows by (6) that $Z(\Theta_\alpha)$ can contain only the pair of points $(\pm 1, 0, ..., 0)$. Moreover similarly to [8], Proposition 5 it can be shown that $\{(\pm 1, 0, ..., 0)\} \subset Z(\Theta_\alpha)$. Thus $Z(\Theta_\alpha) = \{(\pm 1, 0, ..., 0)\}$ which provides the desired example of nonempty exceptional zero set consisting of exactly 2 points.

It is interesting to note that Varjú [16] constructed an example of 0-symmetric star like domain $K \subset \mathbb{R}^2$ such that $Z(K) = \partial K$, i.e., the exceptional zero set of K is *all of its boundary* and thus the homogeneous approximation does not hold anywhere on the boundary of the domain.

Now we will address the problem of approximation by homogeneous polynomials on *non convex polytopes*.

Denote by $B^d(\mathbf{x}, r)$ the ball in $\mathbb{R}^d$ of radius r centered at $\mathbf{x}$, and let $S^{d-1} := \partial B^d(0, 1)$ be the unit sphere. Given a 0-symmetric star like domain K we will say that it is locally convex at a boundary point $\mathbf{x}_0 \in \partial K$ if $B^d(\mathbf{x}_0, \epsilon) \cap K$ is convex for some sufficiently small $\epsilon > 0$.

Let $K \in \mathbb{R}^d$ be a non convex polytope. Denote by $F_{m,j}, m \leq d-1$ the m-dimensional faces of K. In addition, let $F^i_{m,j}, m \leq d-1$ denote the "inner" m-dimensional faces which are located inside the convex hull of K. Now consider the set of all non locally convex points of ∂K given by

$$\partial K^* := \{\mathbf{x} \in \partial K : B^d(\mathbf{x}, \epsilon) \cap K \text{ is not convex for any } \epsilon > 0\}.$$

In case when $K \subset \mathbb{R}^d$ is a non convex polytope it is easy to see that $\partial K^* \neq \emptyset$ is the union all "inner" $d-2$-dimensional faces of the polytope, i.e.,

$$\partial K^* = \cup_j F^i_{d-2,j}.$$

Moreover, as shown in [4] ∂K^* is always contained in the exceptional zero set of the non convex polytope K that is $\partial K^* \subset Z(K)$. Recalling that by Varjú's theorem $Z(K) = \emptyset$ if K is convex it follows now that the convexity of the polytope is necessary and sufficient for the homogeneous density condition (1) to hold. That is we arrive at the next conclusion.

Let $K \subset \mathbb{R}^d, d \geq 2$ be a 0-symmetric polytope. Then $Z(K) = \emptyset$ and thus homogeneous polynomials are dense on ∂K if and only if K is convex.

Thus in addition to l_α balls 0-symmetric polytopes present another class of domains for which convexity is not only sufficient but also necessary for the density of homogeneous polynomials.

5 Open Problems

In this last section of the paper we would like to list some open problems. Let us start first with questions related to convex bodies. Naturally, the central open problem is to verify the conjecture on density of homogeneous polynomials on arbitrary convex 0-symmetric surfaces.

Problem 1. *Prove that for any 0-symmetric convex body $K \subset \mathbb{R}^d$ and every $f \in C(\partial K)$ there exist homogeneous polynomials $h_n, h_{n+1} \in H^d_n, H^d_{n+1}, n \in \mathbb{N}$ so that (1) holds uniformly on ∂K.*

Generally speaking the main cases when the solution to Problem 1 is known are listed in (3) with some additional examples like circular cones being known, as well. Let us recall that in order to prove above problem it suffices to verify it for $f = 1$. It is worth mentioning that even the answer to the next weaker question is not known.

Problem 1A. *Prove that for any 0-symmetric convex body $K \subset \mathbb{R}^d$ there exist even homogeneous polynomials $h_{2n} \in H^d_{2n}$ so that $1 \leq \|1 - h_{2n}\|_{C(\partial K)} \leq 2, \;\; n \geq n_0(K)$.*

Finding the rate of homogeneous approximation also poses an interesting open problem. As shown above on the unit sphere S^{d-1} the relation $P^d_n = H^d_n + H^d_{n-1}$ holds and therefore sums of pairs of homogeneous polynomials $h_n + h_{n+1}$ yield the same rate of uniform approximation on S^{d-1} as polynomials of total degree n. In particular this rate is $O(\frac{1}{n})$ for Lip 1 functions on S^{d-1}. Is this phenomena true of any C^2 0-symmetric convex body $K \subset \mathbb{R}^d$? This leads to the next

Problem 2. *Show that for any C^2 0-symmetric convex body $K \subset \mathbb{R}^d$ and every $f \in$Lip 1 on ∂K there exist homogeneous polynomials $h_n, h_{n+1} \in H^d_n, H^d_{n+1}, n \in \mathbb{N}$ so that*

$$\|f - h_n - h_{n+1}\|_{C(\partial K)} = O\left(\frac{1}{n}\right).$$

Note that in Kroó-Szabados [7] the above relation can be found with an extra factor of order $\log^{d+2} n$. It is also plausible that the order of best homogeneous polynomial approximation on polytopes has only a logarithmic rate. Thus we would like to offer the following conjecture related to homogeneous approximation on the bivariate square $D^2 := \{(x, y) \in \mathbb{R}^2 : |x| + |y| = 1\}$.

Problem 3. *Prove that there exists an absolute constant $c > 0$ such that for every $h_n, h_{n+1} \in H^d_n, H^d_{n+1}$ we have*

$$\|1 - h_n - h_{n+1}\|_{C(D^2)} \geq \frac{c}{\log n}, \ \ n \in \mathbb{N}.$$

Even more ambitious would be a generalization of Problem 3 for any 0-symmetric convex polytope $K \subset \mathbb{R}^d$.

Problem 4. *Prove that for any 0-symmetric convex polytope $K \subset \mathbb{R}^d$ there exists a constant $c_K > 0$ such that for every $h_n, h_{n+1} \in H^d_n, H^d_{n+1}$ we have*

$$\|1 - h_n - h_{n+1}\|_{C(\partial K)} \geq \frac{c_K}{\log n}, \ \ n \in \mathbb{N}.$$

Now we proceed by presenting some open questions related to homogeneous polynomial approximation on non convex 0-symmetric star like domains. Our next problem concerns l_α sphere $S^{d-1}_\alpha = \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : |x_1|^\alpha + ... + |x_d|^\alpha = 1\}$ in non convex case $0 < \alpha < 1$. As it was discussed in Section 4 its exceptional zero set $Z(S^{d-1}_\alpha)$ is not empty because it contains the set $\{\mathbf{x} = (x_1, ..., x_d) \in S^{d-1}_\alpha : x_1 \cdot ... \cdot x_d = 0\}$ and coincides with this set if $0 < \alpha < 1$ is rational or $d = 2$.

Problem 5. *Show that for any $d > 2$ and every irrational $0 < \alpha < 1$ we have*

$$Z(S^{d-1}_\alpha) = \{\mathbf{x} = (x_1, ..., x_d) \in S^{d-1}_\alpha : x_1 \cdot ... \cdot x_d = 0\}.$$

Another open problem we would like to mention concerns the exceptional zero sets of non convex polytopes. As it was outlined in the previous section when $K \subset \mathbb{R}^d$ is a non convex 0-symmetric polytope its exceptional zero set contains all inner $d-2$ dimensional faces of K, i.e.,

$$(\cup_j F^i_{d-2,j}) \subset Z(K).$$

Let us show that, in addition we also have the inclusion

$$Z(K) \subset (\cup_j F^i_{d-1,j}).$$

Denote by K^c the convex hull of K. Obviously, $\partial K^c \cap \partial K = \cup_j F^o_{d-1,j}$, where $F^o_{d-1,j}$ stand for the outer $d-1$-dimensional faces of K. Recall that inner faces $F^i_{m,j}$ of K are inside its convex hull K^c, while outer faces $F^o_{m,j}$ of K belong to the boundary ∂K^c. Let $D_K := \overline{\partial K^c \setminus \partial K}$ be the closure of $\partial K^c \setminus \partial K$. Consider now any $f \in C(\partial K^c)$ such that $f = 0$ on D_K and $f > 0$ on $\partial K^c \setminus D_K$. Since K^c is a convex polytope this function can be uniformly approximated on ∂K^c by pairs of homogeneous polynomials $h_n + h_{n+1}$. Note that $\partial K^c \setminus D_K = \partial K \setminus (\cup_j F^i_{d-1,j})$. Since $f > 0$ on $\partial K^c \setminus D_K$ this immediately implies that $Z(K) \subset (\cup_j F^i_{d-1,j})$.

Thus for every non convex polytope K we have

$$(\cup_j F^i_{d-2,j}) \subset Z(K) \subset (\cup_j F^i_{d-1,j}).$$

It seems to be plausible that the union on the right hand provides the exceptional zero set in general. This leads to the next

Problem 6. *Is it true that for every non convex 0-symmetric polytope $Z(K) = \cup_j F^i_{d-1,j}$?*

The assumption of 0-symmetry of the underlying domain appears to be quite natural in approximation problems related to homogeneous polynomials since they are 0-symmetric (even or odd). But since in general we consider approximation by sums of pairs $h_n + h_{n+1}$ of homogeneous polynomials it is not clear if the 0-symmetry is indeed a necessary condition. Surprisingly, nothing is known with regard to approximation on non 0-symmetric domains. In particular, the answer to the following basic questions is not known.

Problem 7. *Does there exist a non 0-symmetric convex body $K \subset \mathbb{R}^d$ such that homogeneous approximation property (1) holds for every $f \in C(\partial K)$?*

Problem 8. *Does there exist a non 0-symmetric convex body $K \subset \mathbb{R}^d$ for which homogeneous approximation property (1) fails for some $f \in C(\partial K)$?*

Problems 7 and 8 do not appear to be trivial despite their relatively simple formulation. The most natural domain to consider is evidently a ball $B^d(\mathbf{a}, 1)$ in $\mathbb{R}^d$ of radius 1 centered at some $\mathbf{a} \neq 0$.

Problem 9. *Does the homogeneous approximation property (1) hold or fail for non 0-symmetric ball $B^d(\mathbf{a}, 1), \mathbf{a} \neq 0$.?*

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