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On Urysohn type Generalized Sampling Operators

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Abstract

The aim of this work is to define and study Urysohn type integral form of generalized sampling operators by using the Urysohn type interpolation of the given function f. The basis used in this construction are the Fréchet and Prenter Denstity theorems together with Urysohn type operator values instead of the rational sampling values of the function. After that, we investigate some properties of this operators in some function spaces. At the end of this study, some graphical representations for the various examples are given related with the Urysohn type sampling operators.

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1 Introduction

Whereas the Bernstein polynomials $B_n f$ are known to give the most elegant proof of the Weierstrass approximation theorem for algebraic polynomials on the space $C[a,b]$, the classical Whittaker-Kotel'nikov-Shannon (WKS) Sampling theorem gives an exact reconstruction of a continuous function f defined over the real line in terms of the interpolation series

$$(S_n^{\sin c} f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \sin c(nt-k), \qquad (t \in \mathbb{R}).$$

where

$$\sin c(t) := \frac{\sin \pi t}{\pi t} \text{ for } t \neq 0, \ \sin c(0) := 1.$$

This series has many important applications to signal analysis since it provides an exact reconstruction for the band-limited signals. However, the WKS sampling theorem does not hold for functions which are not band-limited and it holds only in an approximate version (see e.g., [13, 14, 15]).

To obtain a positive answer to the approximation problem on $C(\mathbb{R})$, in 1919 Maria Theis used another special kernel function $(\sin c(t))^2$ which belongs to $L_1(\mathbb{R})$ for the sampling operators [31] .

But, the general theory for the approximation problem on $C(\mathbb{R})$ is due to the great mathematician P. L. Butzer. Namely, a counterpart for the approximation of functions $f \in C(\mathbb{R})$ on the whole real line, for the functions which are not necessarily band-limited, are the generalized sampling series.

One and two dimensional case of the Generalized Sampling Series are defined as follows.

Let f be a bounded function defined on $\mathbb{R}$, then the one dimensional generalized sampling series $(S_n f)$ is given by

$$(S_n f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \varphi(nt-k), \qquad (t \in \mathbb{R}, \ n \in \mathbb{N}), \tag{1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is the kernel function satisfying

$$\varphi \in L^1, \ \sum_{k=-\infty}^{\infty} \varphi(u-k) = 1 \text{ for every } u \in \mathbb{R}, \tag{2}$$

and

$$A_\varphi := \sup_{u \in \mathbb{R}} \sum_{k=-\infty}^{\infty} |\varphi(u-k)| < \infty \tag{3}$$

where the convergence of the series (3) is uniform on each compact subintervals of $\mathbb{R}$.

Similarly, given a bounded function f defined on $\mathbb{R}^2$ and a "kernel" function $\varphi : \mathbb{R} \to \mathbb{R}$, the two dimensional Generalized sampling series are defined as:

$$(S_n f)(x,y) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f\left(\frac{k}{n}, \frac{j}{n}\right) \varphi(nx-k)\varphi(ny-j) \tag{4}$$

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where $n \in \mathbb{N}$, $(x, y) \in \mathbb{R}^2$.

These operators $(S_n f)$ allow us to reconstruct (in some sense) a given signal f by a sequence of its sample values, which are of the form $f(k/n)$ or $f\left(\frac{k}{n}, \frac{j}{n}\right)$, $k, j \in \mathbb{Z}, n \in \mathbb{N}$. A systematic study of Generalized sampling operators (1) and (4) for arbitrary kernel functions φ with (2) and (3) was initiated at RWTH Aachen by P. L. Butzer and his collaborators (students) since 1977 (see, e.g., [12]-[16] and [26]).

In recent years, these operators and some of their modifications have been of great importance in the development of mathematical models for signal and image recovering, as studied by research group (RITA network) from Perugia led by Bardaro C. and Vinti G. (see, e.g., [1]-[11], [17], [18] and [29]).

The goal of this study is to find a positive solution to the approximation (or superposition) problem for operators and functionals. In other words, we will propose a generalization and extension of the theory of interpolation to operators by introducing an integral operator, called Urysohn type operators. The new operators are more flexible than the previous one and, practically, the convergence problems can be extended to the functionals and operators.

The basis used in this construction are the Fréchet and Prenter Denstity theorems together with Urysohn type operator values instead of the rational sampling values of the function.

2 Preliminaries and auxiliary results

In this section we shall introduce some notation and background material used throughout this paper.

As usual, we denote by $C(\mathbb{R})$ the Banach space of continuous and bounded functions $u : \mathbb{R} \to \mathbb{R}$ with norm

$$\|u\| = \sup\{|u(x)| : x \in \mathbb{R}\}.$$

Unlike these approaches, in the present study we will investigate these problems for operators or functionals by using Urysohn type operators. To define an Urysohn type operator and obtain some positive answers to the approximation problems, we consider the following Urysohn integral operators that were discussed by Urysohn.

$$F(t, x(\cdot)) = \int_0^1 f(t, s, x(s))ds, \quad t \in [0, 1], \quad 0 \le x(\cdot) \le 1 \tag{5}$$

with unknown kernel f: If such a representation exists, then the kernel of this integral operators $f(t, \cdot, x(\cdot))$ is called Green's functions. A well-known fact is that its properties and values depend on the properties and values of the function $x(\cdot)$ (see, e.g., [8], [9], [20]-[25], [28]).

For a constant function $x(\cdot) = a$, we set $Fa(t) = F(a)$.

Equation (5) was discussed by Urysohn in 1923-1924 in [32]-[33]. Equations of this type appear in many applications. For example, it occurs in solving problems arising in economics, engineering, and physics (see [27], [30], [34] and [35]).

Recall that, since the Dirac Delta function is actually a generalized functions (or distributions), the derivative of the Heaviside unit step function $(H(x))$ is the Dirac Delta function $(\delta(x))$ in distributional sense, namely

$$\delta(x) = \frac{dH(x)}{dx}$$

holds true.

In view of the relations between Dirac and Heaviside unit step functions, we assume that the following continuous interpolation conditions hold.

$$Fx(t) := F(t, x_i(\cdot, s)) = \int_0^1 f(t, z, x_i(z, s))dz, \qquad t \in [0, 1] \tag{6}$$

where $-\infty < x_i(z, s) < \infty$ defined as $x_i(z, s) = \frac{i}{n}H(z - s)$; $s \in [0, 1]$; and $i = \ldots -2, -1, 0, 1, 2, \ldots$.

It is clear that

$$\frac{\partial F(t, x_i(\cdot, s))}{\partial s} = \frac{\partial F\left(t, \frac{i}{n}H(\cdot - s)\right)}{\partial s} = f(t, s, 0) - f\left(t, s, \frac{i}{n}\right) \tag{7}$$

holds true, where $x_i(\cdot, s) = \frac{i}{n}H(\cdot - s)$; $s \in [0, 1]$; and $i = \ldots -2, -1, 0, 1, 2, \ldots$.

Denote

$$F_1\left(t, s, \frac{i}{n}\right) := f(t, s, 0) - f\left(t, s, \frac{i}{n}\right) = \frac{\partial F(t, x_i(\cdot, s))}{\partial s}. \tag{8}$$

Now, we can construct a new type Sampling operator by means of the Urysohn interpolation conditions given in (6) with unknown function $f:[0,1]^2\times\mathbb{R}\to\mathbb{R}$, which we plan to use it for the solution of the convergence problem to the Urysohn operator $Fx(t)$.

Owing to the above definitions, first of all, we introduce the sequence of one dimensional Urysohn type operators studied in this paper.

Definition 1. Let F be the Urysohn integral operator of f. Then the Urysohn type generalized sampling operator is defined as:

$$(US_nF)x(t):=(US_nF)(t;x(\cdot,s))=\int_0^1\left[\sum_{k=-\infty}^{\infty}f\left(t,s,\frac{k}{n}\right)\varphi_{k,n}(x(s))\right]ds \tag{9}$$

where $\varphi_{k,n}(x(s))=\varphi(nx(s)-k)$ is an arbitrary kernel function satisfies (2) and (3). In particular, we will put $Dom\ (USF)=\bigcap_{n\in\mathbb{N}}Dom\ (US_nF)$, where $Dom\ (US_nF)$ is the set of all bounded functions $f:[0,1]^2\times\mathbb{R}\to\mathbb{R}$ for which the operator is well defined.

Remark 1. By (5) and (7), (US_nF) can be written as;

$$(US_nF)x(t)=F(0)-\int_0^1\left[\sum_{k=-\infty}^{\infty}\frac{\partial F\left(t,\frac{k}{n}H(t-s)\right)}{\partial s}\varphi_{k,n}(x(s))\right]ds.$$

Throughout this work, we assume that.the first two central moments of the generalized sampling operators satisfy

$$m_1(\varphi)\ :=\sum_{k=-\infty}^{\infty}\varphi(u-k)(u-k)=0,$$
$$m_2(\varphi)\ :=\sum_{k=-\infty}^{\infty}\varphi(u-k)(u-k)^2=C$$

for every $u\in\mathbb{R}$ and for a given constant $C\in\mathbb{R}$.

In general, for every $u\in\mathbb{R}$ and for some $\beta>0$, we assume that the discrete absolute moment of order β are finite, i.e.,

$$M_\beta(\varphi):=\sup_{u\in\mathbb{R}}\sum_{k=-\infty}^{\infty}|\varphi(u-k)|\,|u-k|^\beta<\infty$$

(see [4] and [19]). The formula for $M_\beta(\varphi)$ in case $\beta=0$ is exactly A_φ.

3 Convergence property

3.1 One Dimensional case

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems.

Definition 2. Let $f\in C\left([0,1]^2\times\mathbb{R}\right)$ and $\delta>0$ be given. Then the partial modulus of continuity of f is given by;

$$\omega_3(\delta):=\sup_{(t,s)\in[0,1]^2}\sup_{|u_1-u_2|\le\delta}|f(t,s,u_1)-f(t,s,u_2)|. \tag{10}$$

Clearly, if $\delta=|u_1-u_2|$, one has

$$|f(t,s,u_1)-f(t,s,u_2)|\le\omega_3(|u_1-u_2|)\le\left(1+\frac{|u_1-u_2|}{\delta}\right)\omega_3(\delta).$$

We are now ready to establish one of the main results of this study:

Theorem 1. *Let $f:[0,1]^2\times\mathbb{R}\to\mathbb{R}$ be a bounded function and let F be the Urysohn integral operator of the function f. Then,*

$$\lim_{n\to\infty}(US_nF)x(t)=Fx(t),$$

holds true at each point $x(s)$ of continuity of f.

Proof. In view of the definition of the operator (9) and (6), we have

$$\begin{aligned}(US_nF)x(t)-Fx(t) &= F(0)-\int_0^1\left[\sum_{k=-\infty}^{\infty}\frac{\partial F\left(t,\frac{k}{n}H(t-s)\right)}{\partial s}\varphi_{k,n}(x(s))\right]ds-Fx(t)\\ &= F(0)-Fx(t)-\int_0^1\left[\sum_{k=-\infty}^{\infty}\frac{\partial F\left(t,\frac{k}{n}H(t-s)\right)}{\partial s}\varphi_{k,n}(x(s))\right]ds\\ &= F(0)-Fx(t)-\int_0^1\sum_{k=-\infty}^{\infty}F_1\left(t,s,\frac{k}{n}\right)\varphi_{k,n}(x(s))\,ds.\end{aligned}$$

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Note that, by (7) and (8), one has

$$F(0)-Fx(t)=\int_0^1 f(t,s,0)ds-\int_0^1 f(t,s,x(s))ds=\int_0^1 F_1(t,s,x(s))\,ds.$$

So we obtain

$$(US_nF)x(t)-Fx(t)=\int_0^1\sum_{k=-\infty}^{\infty}\varphi_{k,n}(x(s))\left[F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right]ds.$$

Let us divide the last term into two parts as;

$$(US_nF)x(t)-Fx(t)=P_1+P_2,$$

where

$$P_1=\int_0^1\sum_{\left|\frac{k}{n}-x(s)\right|<\delta}\varphi_{k,n}(x(s))\left[F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right]ds$$

and

$$P_2=\int_0^1\sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta}\varphi_{k,n}(x(s))\left[F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right]ds.$$

Hence one has

$$|(US_nF)x(t)-Fx(t)|\leq|P_1|+|P_2|$$

together with

$$|P_1|\leq\int_0^1\sum_{\left|\frac{k}{n}-x(s)\right|<\delta}\left|\varphi_{k,n}(x(s))\right|\left|F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right|ds,$$

and

$$|P_2|\leq\int_0^1\sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta}\left|\varphi_{k,n}(x(s))\right|\left|F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right|ds.$$

Since $x(s)$ is a continuity point of f, then clearly

$$\left|F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right|<\epsilon$$

holds true when $\left|\frac{k}{n}-x(s)\right|<\delta$, and

$$\left|F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right|\leq 2\,\|f\|$$

holds true, when $\left|\frac{k}{n}-x(s)\right|\geq\delta$.

So

$$\begin{aligned}|P_1| &\leq \int_0^1\sum_{\left|\frac{k}{n}-x(s)\right|<\delta}\left|\varphi_{k,n}(x(s))\right|\left|F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right|ds\\ &\leq A_\varphi\epsilon,\end{aligned}$$

and

$$\begin{aligned}|P_2| &\leq \int_0^1\sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta}\left|\varphi_{k,n}(x(s))\right|\left|F_1(t,s,x(s))-F_1\left(t,s,\frac{k}{n}\right)\right|ds\\ &\leq 2\,\|f\|\int_0^1\sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta}\left|\varphi_{k,n}(x(s))\right|ds\\ &\leq 2\,\|f\|\,\frac{M_2(\varphi)}{\delta^2n^2}=O(n^{-2}).\end{aligned}$$

Collecting these estimates we have

$$\lim_{n\to\infty}(US_nF)x(t)=Fx(t).$$

This completes the proof.

Theorem 2. *Let F be the Urysohn integral operator of the function f. Then (US_nF) converges to F uniformly in x. That is*

$$\lim_{n\to\infty} \|(US_nF) - F\| = 0.$$

Owing to the Weierstrass criterion (for uniform convergence), the proof of this theorem is similar and a consequence of the previous theorem.

Theorem 3. *Let F be the Urysohn integral operator of the function f. Then for every $\varepsilon > 0$*

$$|(US_nF)x(t) - Fx(t)| \le 2\omega_3(f;\delta)$$

holds true at each point $x(s)$ of continuity of f, where $\delta = \sqrt{M_2(\varphi)/n^2}$.

Proof. Clearly one has

$$\begin{aligned} |(US_nF)x(t) - Fx(t)| &\le \int_0^1 \sum_{k=-\infty}^{\infty} \left|\varphi_{k,n}(x(s))\right| \left|F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right| ds \\ &:= I_{n,1}(x), \end{aligned} \tag{11}$$

say. Taking into account the modulus of continuity, we can write (11) as follows;

$$I_{n,1}(x) \le \int_0^1 \sum_{k=-\infty}^{\infty} \left|\varphi_{k,n}(x(s))\right| \omega_3\left(\left|x(s) - \frac{k}{n}\right|\right) ds.$$

We have

$$\begin{aligned} I_{n,1}(x) &\le \int_0^1 \sum_{k=-\infty}^{\infty} \left|\varphi_{k,n}(x(s))\right| \omega_3\left(\left|x(s) - \frac{k}{n}\right|\right) ds \\ &\le \omega_3(\delta)\left\{1 + \delta^{-1}\int_0^1 \sum_{k=-\infty}^{\infty} \left|\frac{k}{n} - x(s)\right| \left|\varphi_{k,n}(x(s))\right| ds\right\} \\ &\le \omega_3(\delta)\left\{1 + \delta^{-2}\int_0^1 \sum_{k=-\infty}^{\infty} \left(\frac{k}{n} - x(s)\right)^2 \left|\varphi_{k,n}(x(s))\right| ds\right\} \\ &\le \omega_3(\delta)\left\{1 + \frac{M_2(\varphi)}{\delta^2 n^2}\right\}. \end{aligned}$$

If we choose

$$\delta = \sqrt{\frac{M_2(\varphi)}{n^2}},$$

then one can obtain the desired estimate, namely,

$$|(US_nF)x(t) - Fx(t)| \le 2\omega_3(\delta).$$

Thus the proof is now complete.

3.2 Two Dimensional case

In view of the one dimensional case, we obtain the following convergence results related to the two dimensional generalized sampling operators.

As to the two dimensional case, we assume that the two dimensional continuous interpolation conditions hold:

$$F\left(x_i(t), y_j(t)\right) = \int_0^1 \int_0^1 f(t,s,z,x_i(s),y_j(z))\,ds\,dz, \qquad t \in [0,1] \tag{12}$$

where

$$\begin{aligned} x_i(s) &= \frac{i}{n} H(s-\xi);\, \xi \in [0;1], \\ y_j(z) &= \frac{j}{n} H(z-\varsigma);\, \varsigma \in [0;1] \end{aligned} \tag{13}$$

and $i, j = \ldots -2, -1, 0, 1, 2, \ldots$.

Taking into account (12) and (13), by a straightforward calculation the stated identities follow.

$$\begin{aligned} & F\left(\frac{i}{n}H(s-\xi), \frac{j}{n}H(z-\varsigma)\right) \\ = & \int_{\varsigma}^{1}\int_{\xi}^{1} f(t,s,z,\frac{i}{n},\frac{j}{n})dsdz + \int_{0}^{\varsigma}\int_{\xi}^{1} f(t,s,z,\frac{i}{n},0)dsdz \\ & + \int_{0}^{\varsigma}\int_{0}^{\xi} f(t,s,z,0,0)dsdz + \int_{\varsigma}^{1}\int_{0}^{\xi} f(t,s,z,0,\frac{j}{n})dsdz \end{aligned} \tag{14}$$

and hence, one has

$$\begin{aligned} \frac{\partial^2 F\left(\frac{i}{n}H(s-\xi), \frac{j}{n}H(z-\varsigma)\right)}{\partial\xi\partial\varsigma} = & f(t,\xi,\varsigma,\frac{i}{n},\frac{j}{n}) - f(t,\xi,\varsigma,\frac{i}{n},0) \\ & + f(t,\xi,\varsigma,0,0) - f(t,\xi,\varsigma,0,\frac{j}{n}). \end{aligned}$$

Say

$$F_1\left(t,\xi,\varsigma,\frac{i}{n},\frac{j}{n}\right) := \frac{\partial^2 F\left(\frac{i}{n}H(s-\xi), \frac{j}{n}H(z-\varsigma)\right)}{\partial\xi\partial\varsigma}. \tag{15}$$

Definition 3. Given a continuous and bounded function f and an arbitrary kernel $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies (2) and (3). We consider a sequence $USTF = (UST_n F)$ of operators, called Urysohn type two dimensional Generalized Sampling operators defined on $\mathbb{R}^2$, having the form:

$$\begin{aligned} & (UST_n F)(x(t), y(t)) \\ = & \int_0^1\int_0^1 \left[\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} f\left(t,s,z,\frac{k}{n},\frac{j}{n}\right)\varphi_{k,n}(x(s))\,\varphi_{j,n}(x(s))\right]dsdz, \end{aligned}$$

acting on bounded functions f on $[0,1]^3\, x\mathbb{R}^2$. We will put $Dom\ (USTF) = \bigcap_{n\in\mathbb{N}} Dom\ (UST_n F)$ as the set of all functions f for which the operator is well defined.

Theorem 4. *Let F be the Urysohn integral operator of f, with $(x, y) \in R^2$. Then*

$$\lim_{n\to\infty}(UST_n F)(x(t), y(t)) = F(x(t), y(t))$$

at each point $x(s)$ and $y(z)$ of continuity of f.

Proof. In view of the definition of the operator (9) and (6), we have

$$\begin{aligned} & (UST_n F)(x(t), y(t)) - F(x(t), y(t)) \\ = & \int_0^1\int_0^1 \left[\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\left[\begin{array}{c} F_1(t,\xi,\varsigma,x(s),y(z)) - \\ -F_1\left(t,\xi,\varsigma,\frac{k}{n},\frac{j}{n}\right)\end{array}\right]\varphi_{k,n}(x(s))\,\varphi_{j,n}(x(s))\right]dsdz \end{aligned}$$

Let us divide the last term into four parts as;

$$|(UST_n F)(x(t), y(t)) - F(x(t), y(t))| \le P_1 + P_2 + P_3 + P_4,$$

where

$$\begin{aligned} P_1 &= \int_0^1\int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta_1}\ \sum_{\left|\frac{j}{n}-y(z)\right|<\delta_2} |\varphi_{k,n}(x(s))\,\varphi_{j,n}(x(s))| \left|F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{j}{n}\right)\right| dsdz, \\ P_2 &= \int_0^1\int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta_1}\ \sum_{\left|\frac{j}{n}-y(z)\right|\ge\delta_2} |\varphi_{k,n}(x(s))\,\varphi_{j,n}(x(s))| \left|F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{j}{n}\right)\right| dsdz, \\ P_3 &= \int_0^1\int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\ge\delta_1}\ \sum_{\left|\frac{j}{n}-y(z)\right|<\delta_2} |\varphi_{k,n}(x(s))\,\varphi_{j,n}(x(s))| \left|F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{j}{n}\right)\right| dsdz, \end{aligned}$$

and

$$P_4=\int_0^1\int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta_1}\ \sum_{\left|\frac{j}{n}-y(z)\right|\geq\delta_2} \left|\varphi_{k,n}(x(s))\,\varphi_{j,n}(x(s))\right|\left|F_1(t,s,z,x(s),y(z))-F_1\left(t,s,z,\frac{k}{n},\frac{j}{n}\right)\right| dsdz.$$

Since x, y are continuity points of f, then there exist $\delta_1, \delta_2 > 0$ such that for every $\epsilon > 0$

$$\left|F_1(t,s,z,x(s),y(z))-F_1\left(t,s,z,\frac{k}{n},\frac{j}{n}\right)\right|<\epsilon$$

holds true when $\left|\frac{k}{n}-x(s)\right|<\delta_1$ and $\left|\frac{j}{n}-y(z)\right|<\delta_2$. So one can easily obtain

$$P_1 < A_\varphi^2\epsilon.$$

As to the other terms

$$\left|F_1(t,s,z,x(s),y(z))-F_1\left(t,s,z,\frac{k}{n},\frac{j}{n}\right)\right|\leq 2M$$

holds true for some $M > 0$, when $\left|\frac{k}{n}-x(s)\right|\geq\delta_1$ or $\left|\frac{j}{n}-y(z)\right|\geq\delta_2$.

$$\begin{aligned}P_2 &\leq 2M\int_0^1\int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta_1}\ \sum_{\left|\frac{j}{n}-y(z)\right|\geq\delta_2} \left|\varphi_{k,n}(x(s))\,\varphi_{j,n}(x(s))\right| dsdz \\ &\leq 2M\frac{A_\varphi M_2(\varphi)}{\delta_2^2 n^2}=O(n^{-2}).\end{aligned}$$

Similarly one has

$$\begin{aligned}P_3 &\leq 2M\frac{A_\varphi M_2(\varphi)}{\delta_1^2 n^2}=O(n^{-2}), \\ P_4 &\leq 2M\frac{M_2^2(\varphi)}{\delta_1^2\delta_2^2 n^4}=O(n^{-4})\end{aligned}$$

Collecting these estimates we have

$$\lim_{n\to\infty}(UST_nF)(x(t),y(t))=F(x(t),y(t)).$$

This completes the proof.

Theorem 5. *Let F be the Urysohn integral operator with $(x,y)\in R^2$. Then (UST_nF) converges to F uniformly in x, y. That is*

$$\lim_{n\to\infty}\|(UST_nF)-F\|=0.$$

As in the proof of Theorem 2, owing to the Weierstrass criterion (for uniform convergence), the proof of the Theorem 5 is similar and a consequence of the Theorem 4.

4 Practical examples, Graphical representations

Now, we will give some graphical examples for these approaches, namely convergence to functionals or operators by means of Urysohn type WKS sampling operators generated by the kernel functions of the type $\varphi(x) := \sin c(x)$.

We note that in the Figures, the graph with the red line belongs to the original operator, the graph with the green line belongs to the operators with small values of n, and finally the graph consisting of blue line to the operators with bigger values of n.

Example 1. Let us consider the operator $Fx(t)=\int_0^1 x^3(t)dt$, and we take its corresponding Urysohn type WKS sampling operator $(US_nF)x(t)$, then one has for k differs from -100 to 100, $n = 2$ and for $n = 30$.

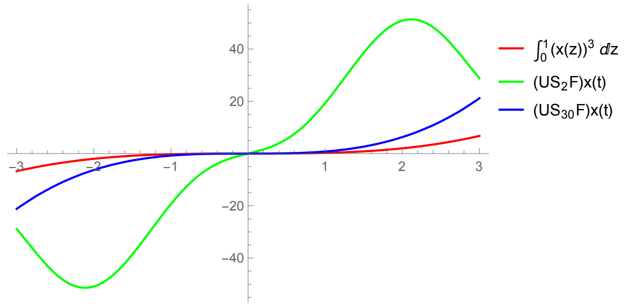

Figure 1: Approximation of $Fx(t)=\int\limits_0^1 x^3(t)dt$ by Urysohn type Sampling operator, for $n=2$ and $n=30$.

Example 2. Let us consider the operator $Fx(t)=\int\limits_0^1 x^3(t)dt$, and we take its corresponding Urysohn type WKS sampling operator $(US_nF)x\,(t)$, then one has for k differs from -1000 to 1000, $n=20$ and for $n=80$.

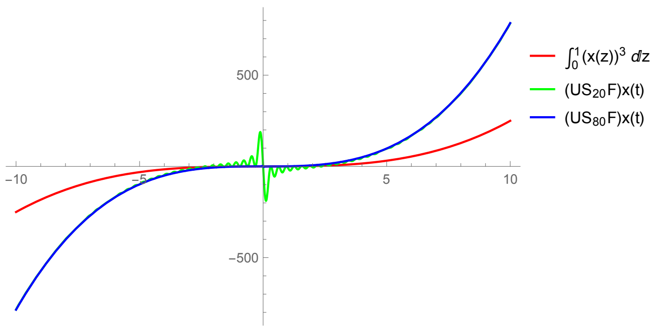

Figure 2: Approximation of $Fx(t)=\int\limits_0^1 x^3(t)dt$ by Urysohn type Sampling operator, for $n=20$ and $n=80$.

Example 3. Let us consider the operator $Fx(t)=\int\limits_0^1 x^3(t)dt$, and we take its corresponding Urysohn type WKS sampling operator $(US_nF)x\,(t)$, then one has for k differs from -1000 to 1000, $n=20$ and for $n=80$.

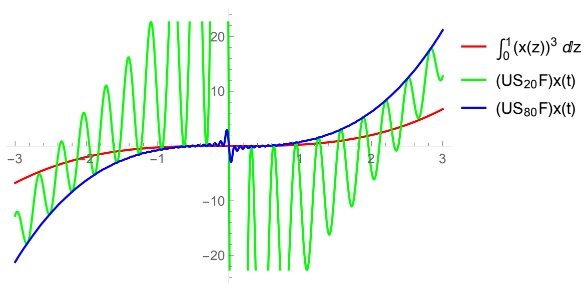

Figure 3: Approximation to $Fx(t)=\int\limits_0^1 x^3(t)dt$ by Urysohn type Sampling operator, for $n=20$ and $n=80$.

Example 4. Let us consider the operator $Fx(t)=\int\limits_0^1 \sin\left[x^3(t)+1\right]dt$, and we take its corresponding Urysohn type WKS sampling operator $(US_nF)x\,(t)$, then one has for k differs from -100 to 100, $n=2$ and for $n=10$.

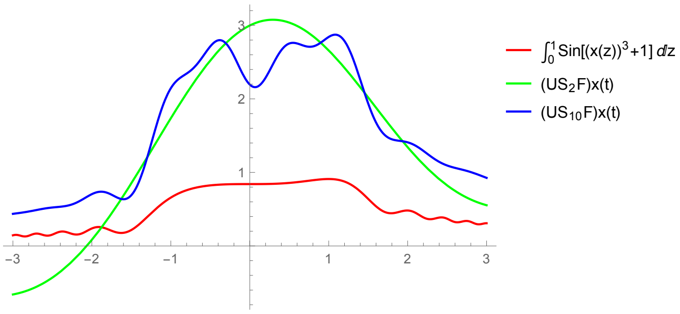

Figure 4: Approximation of $Fx(t) = \int_0^1 \sin\left[x^3(t)+1\right]dt$ by Urysohn type sampling operator, for $n = 2$ and $n = 10$.

Example 5. Let us consider the operator $Fx(t) = \int_0^1 \sin\left[x^3(t)+1\right]dt$, and we take its corresponding Urysohn type WKS Sampling operator $(US_nF)x(t)$, then one has for k differs from -1000 to 1000, $n = 2$, $n = 100$.

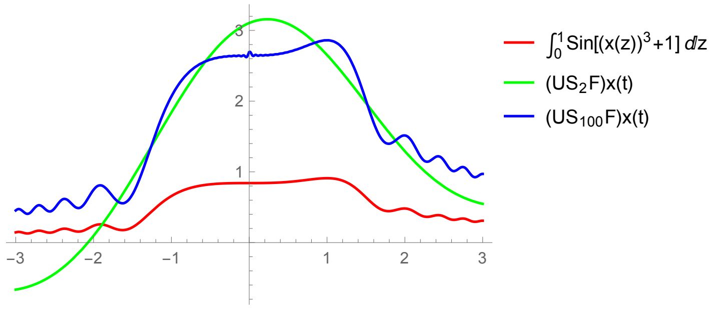

Figure 5: Approximation to $Fx(t) = \int_0^1 \sin\left[x^3(t)+1\right]dt$ by $(US_nF)x(t)$ for $n = 2$, $n = 100$.

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