Subdivision Schemes for Geometric Modelling

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Outline

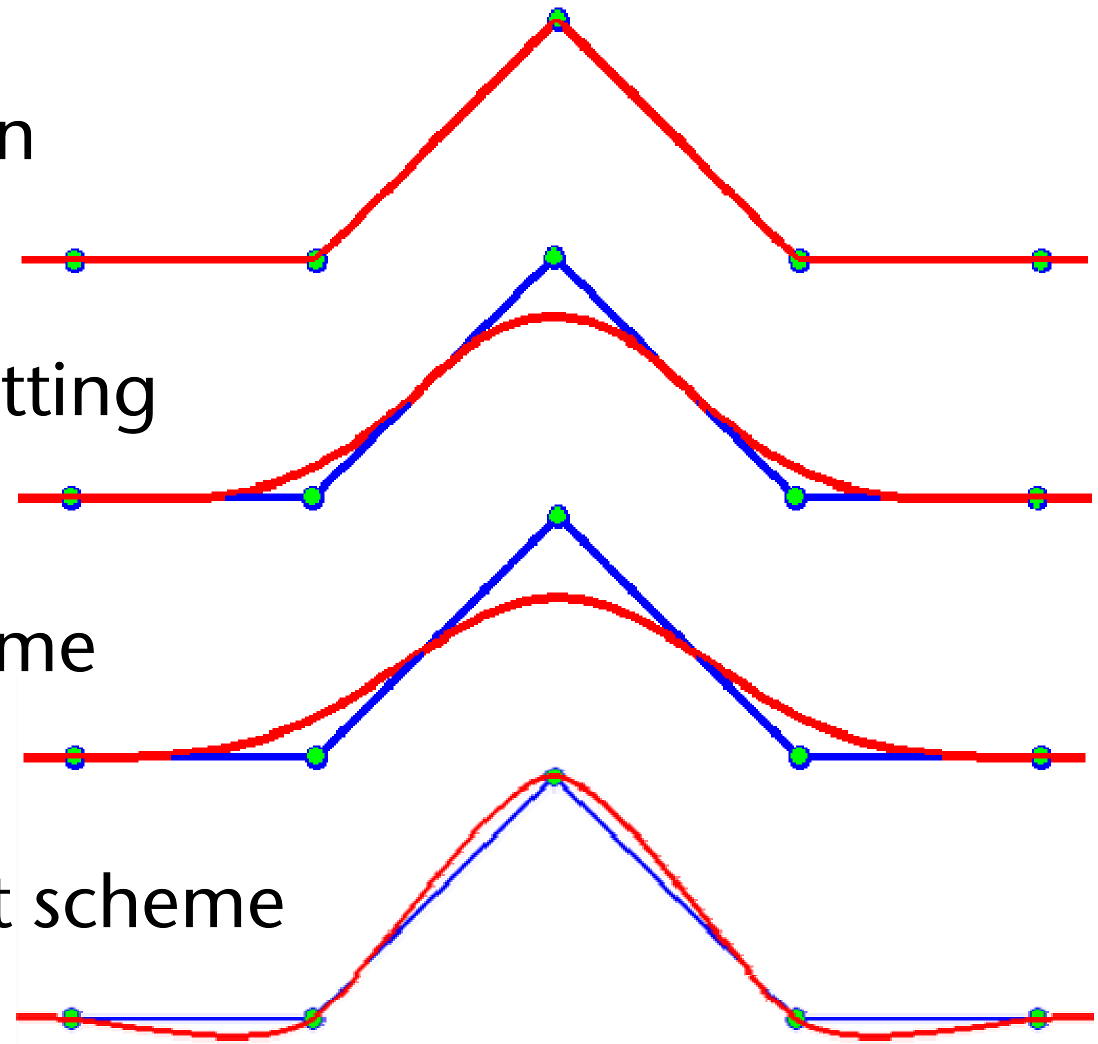
- Sep 5 – Subdivision as a linear process
  - basic concepts, notation, subdivision matrix
- Sep 6 – The Laurent polynomial formalism
  - algebraic approach, polynomial reproduction
- *Sep 7 – Smoothness analysis*
  - *Hölder regularity of limit by spectral radius method*
- Sep 8 – Subdivision surfaces
  - overview of most important schemes & properties

The functional setting

- initial data $\boldsymbol{f}^0 = (f_i^0)_{i\in\mathbb{Z}}$
- mask $\boldsymbol{a} = (a_i)_{i\in\mathbb{Z}}$
- refinement rule $f_i^{j+1} = \sum_k a_{i-2k} f_k^j$
- parameter values $(t_i^j)_{i\in\mathbb{Z},j\in\mathbb{N}}$
- piecewise linear functions F^j with $F^j(t_i^j) = f_i^j$
- limit function $S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0 = \lim_{j\to\infty} F^j$
- consider initial data $\boldsymbol{\delta}^0 = (\delta_{i,0})_{i\in\mathbb{Z}} = (\dots, 0, 1, 0, \dots)$
- ***basic limit function*** $\phi_{\boldsymbol{a}} = S_{\boldsymbol{a}}^\infty \boldsymbol{\delta}^0$
- by linearity of the scheme $S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0 = \sum_k \phi_{\boldsymbol{a}}(\cdot - k) f_k^0$

Basic limit function

- **Examples**
  - polygon subdivision
  - Chaikin's corner cutting
  - cubic B-spline scheme
  - interolating 4-point scheme



Convergence

- if the sequence $(F^j)_{j\in\mathbb{N}}$ of piecewise linear functions converges (uniformly) for any initial data, then the scheme $S_{\boldsymbol{a}}$ is called ***convergent***
  - the limit is necessarily a continuous function
- necessary conditions for $S_{\boldsymbol{a}}$ to be convergent
  - even/odd coefficients of the mask sum to 1

    $\Leftrightarrow \qquad a(z) = (1+z)b(z) \quad$ and $\quad b(1) = 1$
  - 1 is the single dominant eigenvalue of the local subdivision matrix

Convergence

- **Example**
  - scheme with mask $\boldsymbol{a} = [\,-7, 7, 16, 16, 7, -7\,]/16$
  - even/odd coefficients sum to 1
  - local subdivision matrix
    - eigenvalues: $1, \frac{7}{8}, \frac{1}{2} \pm \frac{\mathrm{i}}{4}\sqrt{10}$

$$S = \begin{pmatrix} 7 & 16 & -7 & 0 \\ -7 & 16 & 7 & 0 \\ 0 & 7 & 16 & -7 \\ 0 & -7 & 16 & 7 \end{pmatrix}$$

  - necessary conditions for convergence satisfied
  - still, the scheme does not converge
- more analysis needed!

Convergence

- **Theorem**

  the scheme S_a converges, if and only if the scheme S_b is ***contractive***, i.e. $S_b^\infty f^0 = 0$ for any initial data

  - remember: S_b is the scheme for the differences

- the scheme S_b is contractive, if

$$\max_{i\in\mathbb{Z}} |f_i^{j+1}| \le \mu \max_{i\in\mathbb{Z}} |f_i^j|, \qquad \mu < 1$$

- and that is the case if

$$||\boldsymbol{b}|| = \max\left(\sum_i |b_{2i}|, \sum_i |b_{2i+1}|\right) < 1$$

Contractivity

- **Examples**
  - general primal 3-point $\boldsymbol{a} = [\, w, ½, 1-2w, ½, w \,]$
    - difference scheme $\boldsymbol{b} = [\, w, ½-w, ½-w, w \,]$
    - $\|\boldsymbol{b}\| = |w| + |½-w| < 1 \quad$ for $\quad w \in (-¼, ¾)$
    - S_b is contractive, hence the scheme converges
  - scheme with mask $\boldsymbol{a} = [\, -7, 7, 16, 16, 7, -7 \,]/16$
    - difference scheme $\boldsymbol{b} = [\, -7, 14, 2, 14, -7 \,]/16$
    - $\|\boldsymbol{b}\| = \max(7+2+7, 14+14)/16 = 7/4 > 1$
    - S_b is not contractive, but …

Contractivity

- **Example**
  - scheme with mask $\boldsymbol{a} = [\,-1, 1, 8, 8, 1, -1\,]/8$
    - difference scheme $\boldsymbol{b} = [\,-1, 2, 6, 2, -1\,]/8$
    - $\|\boldsymbol{b}\| = \max(1+6+1, 2+2)/8 = 1$
    - S_b is not contractive, but ...
  - consider 2 steps of the scheme, i.e. the scheme $S_{\boldsymbol{b}}^2$ with symbol $b(z)\,b(z^2)$
    - mask $\boldsymbol{b}^2 = [1,-2,-8,2,7,16,32,16,7,2,-8,-2,1]/64$
    - $\|\boldsymbol{b}^2\| = \max\,(1+7+7+1,\ 2+16+2,\ 8+32+8)/64 < 1$
    - $S_{\boldsymbol{b}}^2$ is contractive, hence the scheme converges

Convergence

- **Theorem**

  the scheme $S_{\boldsymbol{a}}$ converges, if and only if the scheme $S_{\boldsymbol{b}}$ is ***contractive***

- the scheme $S_{\boldsymbol{b}}$ is contractive, if $\|\boldsymbol{b}^\ell\| < 1$ for some $\ell > 0$, with

$$\|\boldsymbol{b}^\ell\| = \max\left\{\sum_i |b^\ell_{k-2^\ell i}| : 0 \le k < 2^\ell\right\}$$

where b^ℓ_i are the coefficients of the scheme $S^\ell_{\boldsymbol{b}}$ with symbol $b^\ell(z) = b(z)\,b(z^2)\cdots b(z^{2^{\ell-1}})$

Smoothness

- **Theorem**

  if the scheme $S_{\boldsymbol{b}}$ converges, then the limit curves of the scheme $S_{\boldsymbol{a}}$ with symbol

$$a(z) = \left(\frac{1+z}{2}\right)^m b(z)$$

  are C^m-continuous

- $S_{\boldsymbol{b}}$ is the scheme for the m-th *divided differences* and

$$(S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0)^{(m)} = S_{\boldsymbol{b}}^\infty(\Delta^m \boldsymbol{f}^0)$$

- **Example**
  - 4-point scheme
    - symbol: $a(z) = \frac{1+z}{2} b(z), \quad b(z) = \frac{-1+4z-z^2}{8z^3}(1+z)^3$
    - mask of S_b : $\boldsymbol{b} = [\,-1, 1, 8, 8, 1, -1\,]/8$
    - this scheme converges (see above)
    - the limit curves of the 4-point scheme are C^1-continuous
  - to check C^2-continuity, consider $a(z) = \frac{(1+z)^3}{4} c(z)$
    - but $\boldsymbol{c} = [\,-1, 3, 3, -1]/4 \;\Rightarrow\; \|\boldsymbol{c}\| = 1$

      and $\boldsymbol{c}^2 = [\,1, -3, -6, 10, 6, 6, 10, -6, -3, 1]/16 \;\Rightarrow\; \|\boldsymbol{c}^2\| = 1$
    - likewise for $\boldsymbol{c}^\ell \;\Rightarrow\; S_{\boldsymbol{c}}$ not contractive $\;\Rightarrow\;$ no C^2-continuity

Hölder regularity

- **Definition**

  a function ϕ is called ***Hölder regular of order $n+\alpha$*** ($n \in \mathbb{N}$, $0 < \alpha \leq 1$), if it is n times continuously differentiable and $\phi^{(n)}$ is Lipschitz of order α, i.e.

  $$|\phi^{(n)}(x+h) - \phi^{(n)}(x)| \leq c\,|h|^{\alpha}$$

  for all x and h and some constant c

- remember: a function that is Lipschitz of order 1 is not necessarily differentiable
- Hölder regularity of order $n+1$ is weaker than being $n+1$ times differentiable

Lower bound on Hölder regularity

- **Theorem**

  the scheme S_a with symbol $a(z) = \left(\frac{1+z}{2}\right)^m b(z)$

  generates limit curves with Hölder regularity

  $r \geq m - \log_2(\|b^\ell\|)/\ell$ for any ℓ

- **Examples**
  - 4-point scheme $a(z) = \left(\frac{1+z}{2}\right)^4 \frac{-1+4z-z^2}{z^2}$
    - $m=4$, $b=[-1,4,-1] \Rightarrow r \geq 4 - \log_2(4) = 2$
  - cubic B-spline scheme $a(z) = \left(\frac{1+z}{2}\right)^4 \frac{2}{z^2}$
    - $m=4$, $b=[2] \Rightarrow r \geq 4 - \log_2(2^\ell)/\ell = 3$

Lower bound on Hölder regularity

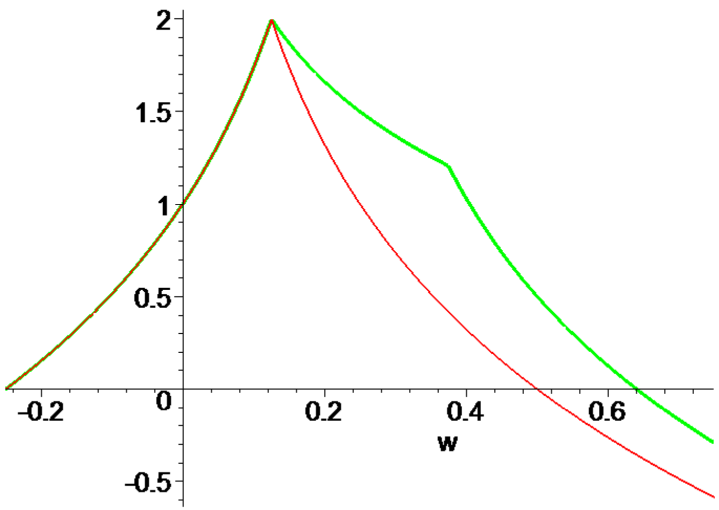
- **Example**
  - general primal 3-point

$$a(z) = \left(\frac{1+z}{2}\right)^2 \frac{4w+(2-8w)z+4wz^2}{z^2}$$

  - $m=2,\ \ell=1,\ \boldsymbol{b}=[4w, 2-8w, 4w] \ \Rightarrow\ r \geq 2 - \log_2(\|\boldsymbol{b}\|)$
  - $m=2,\ \ell=2,$

    $\boldsymbol{b}^2=[16w^2, 8w(1-4w),$
    $\quad 8w(1-2w),\ 4(1-4w)^2, \ldots]$

    $\Rightarrow\ r \geq 2 - \log_2(\|\boldsymbol{b}^2\|)/2$
  - larger ℓ, ...



Support

- suppose the mask $\boldsymbol{a}$ is ***supported*** on $[0, N]$, i.e. $a_i = 0$ for $i < 0$ and $i > N$
  - all masks are of this kind after an index shift
- refine the initial data $\boldsymbol{f}^0 = (\dots, 0, \underline{1}, 0, \dots)$
  - remember: $f_i^{j+1} = \sum_k a_{i-2k} f_k^j$
  - hence, $\boldsymbol{f}^1$ is supported on $[0, N]$
  - likewise, $\boldsymbol{f}^j$ is supported on $[0, (2^j - 1)N]$
- assume primal parameterization $t_i^j = i/2^j$
  - the support of the basic limit function $\phi_{\boldsymbol{a}}$ is $[0, N]$

Support

- for arbitrary initial data $\boldsymbol{f}^0$, the limit function is

$$S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}^0 = \sum_k \phi_{\boldsymbol{a}}(\cdot - k) f_k^0$$

- assume the support of $\phi_{\boldsymbol{a}}$ is $[0, N]$, then the values of the limit function $S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}^0$
  - on [0,1] are determined by the N control points
    $$f^0_{-N+1}, f^0_{-N+2}, \ldots, f^0_0$$
  - on [0, ½] are determined by $f^1_{-N+1}, f^1_{-N+2}, \ldots, f^1_0$
  - on [½, 1] are determined by $f^1_{-N+2}, f^1_{-N+3}, \ldots, f^1_1$

Example

- cubic B-spline scheme with $N=4$

$$\begin{pmatrix} \vdots \\ f^1_{-3} \\ f^1_{-2} \\ f^1_{-1} \\ f^1_{0} \\ f^1_{1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & \\ & \frac{1}{8} & \frac{6}{8} & \frac{1}{8} & & \\ & & \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{8} & \frac{6}{8} & \frac{1}{8} & \\ & & & \frac{1}{2} & \frac{1}{2} & \\ & & & & & \ddots \end{pmatrix} \cdot \begin{pmatrix} \vdots \\ f^0_{-3} \\ f^0_{-2} \\ f^0_{-1} \\ f^0_{0} \\ f^0_{1} \\ \vdots \end{pmatrix}$$

- $u^0 = (f^0_{-3}, \ldots, f^0_0)$ determines $S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0$ on [0,1]
- $A_0 u^0$ determines the values on [0,½]
- $A_1 u^0$ determines the values on [½,1]

Another local limit analysis

- suppose the mask $\boldsymbol{a}$ of a convergent scheme is *supported* on $[0, N]$
  - consider the local $N \times N$ subdivision matrices A_0, A_1
  - take any $x \in [0,1]$ with binary representation

$$x = 0.i_1 i_2 i_3 i_4 \ldots \qquad \text{with} \quad i_k \in \{0,1\}$$

  - then the limit value $S_{\boldsymbol{a}}^{\infty} \boldsymbol{f}^0(x)$ is given (N times) by

$$\ldots A_{i_4} A_{i_3} A_{i_2} A_{i_1} u^0$$

with $u^0 = (f^0_{-N+1}, \ldots, f^0_0)$

Joint spectral radius

- **Definition**

  the *joint spectral radius* of two matrices A_0, A_1 is

$$\rho(A_0, A_1) = \limsup_{k\to\infty}\left(\max\left\{\|A_{i_k}\cdots A_{i_2}A_{i_1}\|_\infty^{1/k} : i_k \in \{0,1\}\right\}\right)$$

  - is bounded by the spectral radii and the norms of A_0 and A_1

$$\max\{\rho(A_0), \rho(A_1)\} \leq \rho(A_0, A_1) \leq \max\{\|A_0\|_\infty, \|A_1\|_\infty\}$$

  - does not dependent on the chosen matrix norm
  - is usually very hard to determine exactly

Hölder regularity

- **Theorem**

  the scheme S_a with symbol $a(z) = \left(\frac{1+z}{2}\right)^m b(z)$ generates limit curves with Hölder regularity $r = m - \log_2(\mu)$, where μ is the joint spectral radius of the local matrices B_0, B_1 from the scheme S_b

  - in practice, the lower and upper bounds on μ are used to get upper and lower bounds on r
  - the lower bound then is the same as before, because $\|\boldsymbol{b}^k\| = \max\left\{\|B_{i_k} \cdots B_{i_2} B_{i_1}\|_\infty : i_k \in \{0,1\}\right\}$

Hölder regularity

- **Example**
  - cubic B-spline scheme $a(z) = \left(\frac{1+z}{2}\right)^4 \frac{2}{z^2}$
    - $\boldsymbol{b} = [2] \;\Rightarrow\; B_0 = B_1 = (2) \;\Rightarrow\; \mu = 2 \;\Rightarrow\; r = 4 - \log_2(2) = 3$
    - scheme gives C^2 limit curves, whose second derivatives are Lipschitz of order 1; sometimes called $C^{3-\epsilon}$
  - 4-point scheme $a(z) = \left(\frac{1+z}{2}\right)^4 \frac{-1+4z-z^2}{z^2}$
    - $\boldsymbol{b} = [-1,4,-1] \;\Rightarrow\; B_0 = \begin{pmatrix} 4 & \\ -1 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & -1 \\ & 4 \end{pmatrix}$
    - $\|B_0\| = \|B_1\| = \rho(B_0) = \rho(B_2) = 4 = \mu \;\Rightarrow\; r = 4 - \log_2(4) = 2$
    - scheme gives $C^{3-\epsilon}$ limit curves

Hölder regularity

Example

- dual 4-point scheme
  - evaluate local cubic interpolant in a dual fashion
  - $\boldsymbol{a} = [-5, -7, 35, 105, 105, 35, -7, -5]/128$
  - divide $m = 5$ times by $(1+z)/2 \quad \Rightarrow \quad \boldsymbol{b} = [-5, 18, -5]/4$

$$B_0 = \begin{pmatrix} \frac{9}{2} & \\ -\frac{5}{4} & -\frac{5}{4} \end{pmatrix}, \quad B_1 = \begin{pmatrix} -\frac{5}{4} & -\frac{5}{4} \\ & \frac{9}{2} \end{pmatrix}$$

  - $\|B_0\| = \|B_1\| = \rho(B_0) = \rho(B_1) = 4.5 = \mu \quad \Rightarrow \quad r = 5 - \log_2(4.5)$
  - scheme gives $C^{2.83}$ limit curves

Summary

- basic limit function $\phi_{\boldsymbol{a}}$ and support size
  - if all but N+1 consecutive mask coefficients are zero, then N is the support size of the mask and the basic limit function
- a scheme $S_{\boldsymbol{a}}$ converges if the difference scheme $S_{\boldsymbol{b}}$ is contractive
  - the norm of $\boldsymbol{b}$ or the ℓ-iterated scheme $\boldsymbol{b}^{\ell}$ is less than one
- a scheme is C^m-continuous, if the scheme for the m-th divided differences converges

Summary

- the norm of $\boldsymbol{b}^\ell$ leads to a lower bound on the Hölder regularity of the limit functions
- lower and upper bound are given by joint spectral radius analysis
  - given a scheme $S_{\boldsymbol{a}}$, divide $a(z)$ by as many factors $(1+z)/2$ as possible, say m such factors
  - for the remaining scheme $S_{\boldsymbol{b}}$ with support size N, consider the local $N \times N$ subdivision matrices B_0, B_1
  - determine the joint spectral radius $\mu = \rho(B_0, B_1)$
  - Hölder regularity of limit curves is $r = m - \log_2(\mu)$