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A projection method with smoothing transformation for second kind Volterra integral equations

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Abstract

In this paper we present a projection method for linear second kind Volterra integral equations with kernels having weak diagonal and/or boundary singularities of algebraic type. The proposed approach is based on a specific optimal interpolation process and a smoothing transformation. The convergence of the method is proved in suitable spaces of functions, equipped with the uniform norm. Several tests show the accuracy of the presented method.

1 Introduction

Let us consider the following Volterra integral equation

$$f(y)+\int_{-1}^{y} k(x,y)f(x)(y-x)^{\alpha}(1+x)^{\beta}dx = g(y), \quad y \in I \equiv [-1,1], \tag{1}$$

where k and g are given functions defined on $\Delta = \{(x,y) : -1 < x \le y \le 1\}$ and I, respectively, f is the unknown solution and $\alpha, \beta > -1$.

The case $\alpha = \beta = 0$, which is when the kernel is a smooth function, has been extensively investigated and today there are several numerical methods which are able to approximate the solution of (1), which in this case turns out to be smooth. Among them we mention the iterated collocation methods presented in [3, 15, 22, 27] and the spectral collocation methods proposed in [12, 24, 26]. However, for a complete bibliography, we refer to [4, Chapter 2] and the reference included there.

The case $\alpha \in (-1,0)$ and $\beta = 0$ is more delicate to treat, since the kernel is singular along the boundary as $y \to x$. The solution inherits the weak singularity at $x = -1$, even when the right-hand side is a smooth function (see, for instance, ([4, Chapter 6] and [20]). Nevertheless, there is a wide range of literature concerning numerical methods to approximate the solution of (1) (see, for instance, [2, 4, 5, 6, 8, 7, 14, 19]).

In particular, in [5] the authors consider the corresponding equation (1) defined in $[0,1]$ with $\alpha \in (-1,0)$ and $\beta = 0$. Their starting point is the behavior of the unknown function f near the point $x = 0$. Indeed, under the assumption that g and k have m continuous derivatives, then $f^{(m)}(y) \sim y^{1-m-\alpha}$. Taking into account that behavior, they regularize the equation and get a new one in which the unknown solution is smoother than the original one. Then, they propose a Jacobi-collocation method for the regularized equation and give a rigorous analysis of the error in spaces equipped with the uniform and the L^2 norm.

In this paper we deal with a more general case: the situation where the kernel can be singular along the diagonal as $y \to x$ and has a singularity along the side $y = -1$ as $y \to -1$. Such equations have already been investigated, essentially by means of piecewise approximations. For instance, in [13] the authors propose a piecewise polynomial collocation method on a mildly graded or uniform grid after regularizing the equation by a smoothing transformation.

First we develop a projection method based on an interpolation process with optimal Lebesgue constants. Such a process is based on the well-known “additional nodes method" [16] and allows us to prove the stability and the convergence of the method in spaces equipped with uniform norm. Once the error is stated, we introduce a smoothing transformation to improve the order of convergence. This is a typical approach which has been already applied in several contexts [11, 19, 21] and allow us to improve the smoothness properties of the solution and consequently the error. The aforesaid projection method is then applied to the regularized equation and a new convergence estimate is derived. Such an estimate furnishes an error which depends on the smoothing parameter and improves those given in [5].

We want to emphasize that in this paper we consider Volterra integral equations into Zygmund-type spaces (see (2)), which the unique solution naturally belongs to, by virtue of its pathology. Zygmund-type spaces are the right environment for studying

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functions with algebraic singularities at ± 1 and, if conducted in these functional spaces, a theoretical analysis of the method provides accurate errors of approximation. This adaptability is well known [9]. For instance, let us consider the function $f(x) = (x+1)^{5/2}$. If we look at the space of functions having p continuous derivatives C^p then $f \in C^2$ and the error of best polynomial approximation is of the order $\mathcal{O}(m^{-2})$. However, if we look at f as an element of the Zygmund-type space Z_λ, then $f \in Z_5$ and the error of best polynomial approximation goes to zero as m^{-5} (see estimate (6)).

The outline of the paper is as follows. In Section 2 we introduce some notations, functional spaces and the optimal Lagrange interpolation process we will use in the numerical method described in Section 3. In Section 4 we show by some numerical tests the accuracy of the procedure and in Section 5 we give the proofs of our main results.

2 Preliminaries

2.1 Notation

Throughout the whole paper we will denote by $\mathcal{C}$ a positive constant having different meanings in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \ldots$, and $\mathcal{C} = \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. If $A, B > 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a constant $\mathcal{C} \neq \mathcal{C}(A, B)$ such that $\mathcal{C}^{-1}B \leq A \leq \mathcal{C}B$.

Moreover, $\mathbb{P}_m$ will denote the space of the algebraic polynomials of degree at most m and for a bivariate function $k(x, y)$ the notation k_x (or k_y) will be adopted to regard k as function of the only variable y (or x).

2.2 Function Spaces

Let us denote by $C^0(A)$ the space of all continuous functions in any interval $A \subset \mathbb{R}$ equipped with the norm

$$\|f\|_A = \sup_{x\in A} |f(x)|$$

and by $C^p(A)$, $p \in \mathbb{N}$ the space of functions having the p-th continuous derivative in A. If $A = [-1, 1]$ we set $C^0 := C^0([-1, 1])$, $C^p := C^p([-1, 1])$, and

$$\|f\|_\infty := \sup_{|x|\leq 1} |f(x)|.$$

For any $f \in C^0$ and for an integer $k \geq 1$, we denote by $\Omega_\varphi^k(f, t)$ the main part of the φ-*modulus of smoothness* [9] defined as

$$\Omega_\varphi^k(f, t) = \sup_{0<\tau\leq t} \|\Delta_{\tau\varphi}^k f\|_{I_{k\tau}}, \quad I_{k\tau} = [-1 + (2k\tau)^2, 1 - (2k\tau)^2],$$

with

$$\Delta_{\tau\varphi}^k f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\left(x + \frac{\tau\varphi(x)}{2}(k - 2i)\right), \qquad \varphi(x) = \sqrt{1 - x^2}.$$

By means of $\Omega_\varphi^k(f, t)$ it is possible to define the Zygmund-type space of order $\lambda \in \mathbb{R}^+$ with $0 < \lambda < k$ as

$$Z_{\lambda,k} = \left\{ f \in C^0 : \sup_{t>0} \frac{\Omega_\varphi^k(f, t)}{t^\lambda} < \infty \right\}, \tag{2}$$

equipped with the norm

$$\|f\|_{Z_{\lambda,k}} := \|f\|_\infty + \sup_{t>0} \frac{\Omega_\varphi^k(f, t)}{t^\lambda}.$$

Denoting by $E_m(f)_\infty = \inf_{P_m\in\mathbb{P}_m} \|f - P_m\|_\infty$ the error of best polynomial approximation of a given function $f \in C^0$, the following equivalence holds true [10, Theorem 2.1]

$$\sup_{t>0} \frac{\Omega_\varphi^k(f, t)}{t^\lambda} \sim \sup_{i\geq 0} (1 + i)^\lambda E_i(f)_\infty, \tag{3}$$

where the constants in "$\sim$" depend on λ. By (3), it follows that the definition of the Zygmund spaces in (2) is independent of $k > \lambda$, enabling to set $Z_\lambda := Z_{\lambda,k}$.

When $\lambda = r$ is a positive integer, denoting by $AC(-1, 1)$ the set of all the functions which are absolutely continuous on every closed subset of $(-1, 1)$, let

$$W_r = \left\{ f \in C^0 : f^{(r-1)} \in AC(-1, 1) \quad \|f^{(r)}\varphi^r\|_\infty < \infty \right\},$$

be the Sobolev space of order r, endowed with the norm

$$\|f\|_{W_r} = \|f\|_\infty + \|f^{(r)}\varphi^r\|_\infty.$$

Let us note that

$$W_{\lfloor\lambda\rfloor+1} \subset Z_\lambda \subset W_{\lfloor\lambda\rfloor}$$

$\lfloor\lambda\rfloor$ being the smallest integer greater than or equal to $\lambda > 0$.

To estimate the error of best polynomial approximation, let us recall the weaker version of the Jackson Theorem [9, Theorem 8.2.1]

$$E_m(f)_\infty \le \mathcal{C}\int_0^{\frac{1}{m}} \frac{\Omega_\varphi^k(f,t)}{t}dt, \qquad \mathcal{C} \ne \mathcal{C}(m,f), \tag{4}$$

and the estimate [16, (2.5.13)]

$$\Omega_\varphi^k(f,t) \le \mathcal{C}t^k \sup_{0<h\le t} \|f^{(k)}\varphi^k\|_{I_{kh}}, \qquad \mathcal{C} \ne \mathcal{C}(t,f). \tag{5}$$

In particular, the following Favard inequality [9] holds $\forall f \in Z_\lambda$,

$$E_m(f)_\infty \le \frac{\mathcal{C}}{m^\lambda}\|f\|_{Z_\lambda}, \quad \mathcal{C} \ne \mathcal{C}(m,f). \tag{6}$$

2.3 Optimal Lagrange interpolation processes

Denoting by $v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ the Jacobi weight of parameters $\alpha,\beta > -1$, let $\{p_m(v^{\alpha,\beta})\}_{m=0}^\infty$ be the sequence of the corresponding orthonormal polynomials having positive leading coefficients and let $x_{m,1}^{\alpha,\beta} < x_{m,2}^{\alpha,\beta} < \cdots < x_{m,m}^{\alpha,\beta}$ be the zeros of the m-th polynomial $p_m(v^{\alpha,\beta})$.

For a given function $f \in C^0$, let $\mathcal{L}_m^{\alpha,\beta}(f) \in \mathbb{P}_{m-1}$ be the Lagrange polynomial interpolating f at the zeros of $p_m(v^{\alpha,\beta})$, i.e.

$$\mathcal{L}_m^{\alpha,\beta}(f,x) = \sum_{i=1}^m \ell_{m,i}^{\alpha,\beta}(x) f(x_{m,i}^{\alpha,\beta}), \qquad \ell_{m,i}^{\alpha,\beta}(x) = \frac{p_m(v^{\alpha,\beta},x)}{p_m'(v^{\alpha,\beta},x_{m,i}^{\alpha,\beta})(x-x_{m,i}^{\alpha,\beta})}.$$

Denoting by $\|\mathcal{L}_m^{\alpha,\beta}\|$ the m-th Lebesgue constant, i. e. the norm of the operator $\mathcal{L}_m^{\alpha,\beta} : C^0 \to C^0$

$$\|\mathcal{L}_m^{\alpha,\beta}\| = \sup_{\|f\|=1} \|\mathcal{L}_m^{\alpha,\beta}(f)\|_\infty,$$

it is well known (see, for instance, [16, Chapter 4]) that the sequence $\{\|\mathcal{L}_m^{\alpha,\beta}\|\}_m$ plays an essential role in the study of the convergence of the Lagrange polynomial, since

$$\|f - \mathcal{L}_m^{\alpha,\beta}(f)\|_\infty \le (1 + \|\mathcal{L}_m^{\alpha,\beta}\|)E_{m-1}(f)_\infty.$$

According to the Faber theorem, it is $\|\mathcal{L}_m^{\alpha,\beta}\| \ge \frac{1}{12}\log m$ and, in view of a classical result by Szëgo, the following behaviour arises

$$\|\mathcal{L}_m^{\alpha,\beta}\| \sim \begin{cases} \log m, & -1 < \alpha,\beta \le -\frac{1}{2} \\ m^{\max\{\alpha,\beta\}+\frac{1}{2}}, & \text{otherwise.} \end{cases}$$

This means that the Lebesgue constants of Lagrange interpolating polynomial based on the zeros of Legendre polynomials ($\alpha = \beta = 0$) or second kind Chebyschev polynomials ($\alpha = \beta = \frac{1}{2}$), diverge algebraically as $m \to \infty$. On the other hand, it is possible to modify the above interpolation processes by using the *additional nodes method* (see e.g. [16, p. 252]) to obtain corresponding Lebesgue constants behaving like $\log m$, also in the case $\alpha > -1/2$ or $\beta > -1/2$. The additional nodes method was extensively used by several authors and in different contexts, and nowadays is referred to as the "additional nodes method" (see [23], [16] and the references therein for instance [18].) To describe the modified process, let

$$y_j = -1 + j\frac{1 + x_{m,1}^{\alpha,\beta}}{1+s}, \quad j = 1,2,\ldots,s,$$
$$t_j = x_{m,1}^{\alpha,\beta} + j\frac{1 - x_{m,m}^{\alpha,\beta}}{1+r}, \quad j = 1,2,\ldots,r,$$

be the additional nodes and define the polynomials

$$A_s(x) := A_{m,s}(x) = \prod_{j=1}^s (x - y_j), \quad B_r(x) := B_{m,r}(x) = \prod_{j=1}^r (x - t_j).$$

Then, let us denote by $L_{m,r,s}^{\alpha,\beta}(f) \in \mathbb{P}_{m+r+s-1}$ the Lagrange polynomial interpolating $f \in C^0$ at the zeros of $Q_{m+r+s} := A_s\, p_m^{\alpha,\beta}\, B_r$. In the case $r = s = 0$ it is $L_{m,r,s}^{\alpha,\beta}(f) \equiv \mathcal{L}_m^{\alpha,\beta}(f)$.

Defining

$$z_i^{\alpha,\beta} := \begin{cases} y_i, & i = 1,2,\ldots,s \\ x_{m,i-s}^{\alpha,\beta}, & i = s+1,\ldots,s+m \\ t_{i-s-m}, & i = s+m+1,\ldots,s+m+r \end{cases} \tag{7}$$

and

$$l_j^{\alpha,\beta}(x) = \prod_{\substack{p=1 \\ p\ne j}}^{m+r+s} \frac{(x - z_p^{\alpha,\beta})}{(z_j^{\alpha,\beta} - z_p^{\alpha,\beta})},$$

the polynomial $L^{\alpha,\beta}_{m,r,s}(f)$ can be written in the form

$$L^{\alpha,\beta}_{m,r,s}(f,x)=\sum_{j=1}^{m+r+s} l_j^{\alpha,\beta}(x)f(z_j^{\alpha,\beta}). \tag{8}$$

Next theorem states the conditions under which the sequence $\{\|L^{\alpha,\beta}_{m,r,s}\|\}_m$ behaves like $\log m$ [16, p.254]:

Theorem 2.1. *Let $\alpha,\beta>-1$ and r,s non negative integers. Then,*

$$\sup_{\|f\|_\infty=1}\|L^{\alpha,\beta}_{m,r,s}(f)\|_\infty\sim\log m$$

if and only if the parameters α,β,r,s satisfy the relations

$$\frac{\alpha}{2}+\frac{1}{4}\le r<\frac{\alpha}{2}+\frac{5}{4},\qquad \frac{\beta}{2}+\frac{1}{4}\le s<\frac{\beta}{2}+\frac{5}{4}. \tag{9}$$

We remark that under the assumption of Theorem 2.1, for each $f\in Z_\lambda$ we have

$$\|f-L^{\alpha,\beta}_{m,r,s}(f)\|_\infty\le\mathcal{C}\frac{\log m}{m^\lambda}\|f\|_{Z_\lambda}, \tag{10}$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$. Denoting by L^1 the usual space of measurable functions in $[-1,1]$ endowed with the norm $\|f\|_1=\int_{-1}^{1}|f(x)|dx<\infty$, we can state the following result, useful in different contexts.

Theorem 2.2. *Let $r,s\in\mathbb{N}$, $\alpha,\beta>-1$ and $u=v^{\gamma,\delta}$ with $\gamma,\delta>-1$. If*

$$\frac{uv^{r,s}}{\sqrt{v^{\alpha,\beta}\varphi}}\in L^1,\qquad \frac{\sqrt{v^{\alpha,\beta}\varphi}}{v^{r,s}}\in L^1, \tag{11}$$

then for any $f\in C^0$ we have

$$\|L^{\alpha,\beta}_{m,r,s}(f)u\|_1\le\mathcal{C}\|f\|_\infty$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$.

Let us note that in the case $r=s=0$ the previous theorem was proved in [17].

3 Main results

Let us consider equation (1). By the change of variable

$$x=\gamma(t,y)=\frac{1+t}{2}y+\frac{t-1}{2},$$

the interval $[-1,y]$ is mapped into $[-1,1]$, so that equation (1) can be rewritten as follows

$$f(y)+\mu\int_{-1}^{1}\hat{k}(t,y)f(\gamma(t,y))v^{\alpha,\beta}(t)dt=g(y), \tag{12}$$

where $\mu=2^{-(\alpha+\beta+1)}$ and

$$\hat{k}(t,y)=(1+y)^{\alpha+\beta+1}k(\gamma(t,y),y). \tag{13}$$

Setting

$$(\mathcal{K}f)(y)=\mu\int_{-1}^{1}\hat{k}(t,y)f(\gamma(t,y))v^{\alpha,\beta}(t)dt, \tag{14}$$

next proposition states conditions under which the operator $\mathcal{K}$ is compact in some subspaces of C^0.

Proposition 3.1. *Assuming that*

$$\sup_{|x|\le1}\|k_x v^{0,\alpha+\beta+1}\|_{Z_\lambda}<\infty, \tag{15}$$

we have

$$\|\mathcal{K}f\|_\infty\le\mathcal{C}\|f\|_\infty,\quad\forall f\in C^0,\quad\mathcal{C}\neq\mathcal{C}(f) \tag{16}$$

and

$$\lim_m\left(\sup_{f\in Z_\lambda}E_m(\mathcal{K}f)_\infty\right)=0. \tag{17}$$

By (17) it follows that $\mathcal{K}:Z_\lambda\to C^0$ is a compact operator (see, for instance, [25, p.93]). Therefore, by the Fredholm Alternative Theorem, we can conclude that for any given function $g\in Z_\lambda$, the equation (1) admits a unique solution $f^*\in Z_\lambda$. Consequently, we can state the following existence and uniqueness theorem.

Theorem 3.2. *Equation (1) admits a unique solution $f^*\in Z_\lambda$, for any given function $g\in Z_\lambda$*

3.1 The collocation method

In this section we propose a collocation method obtained by projecting the equation (12) on the finite dimensional space $\mathbb{P}_{m+r+s-1}$ by means of the Lagrange operator $L_{m,r,s}^{\alpha,\beta}$ defined in (8).

To this end we define the polynomial sequences $\{g_m\}_m$ and $\{\mathcal{K}_m f\}_m$ as

$$g_m = L_{m,r,s}^{\alpha,\beta}(g) \tag{18}$$

and

$$(\mathcal{K}_m f) = L_{m,r,s}^{\alpha,\beta}(\mathcal{K}_m^* f), \tag{19}$$

where

$$(\mathcal{K}_m^* f)(y) = \mu \int_{-1}^{1} \mathcal{L}_m^{\alpha,\beta}\left(\left[L_{m,r,s}^{\alpha,\beta}(\hat{k}_y) f(\gamma(\cdot,y))\right], x\right) v^{\alpha,\beta}(x)dx \tag{20}$$

$$= \mu \sum_{\nu=1}^{m} \lambda_{m,\nu}^{\alpha,\beta} L_{m,r,s}^{\alpha,\beta}(\hat{k}_y, x_{m,\nu}^{\alpha,\beta}) f(\gamma(x_{m,\nu}^{\alpha,\beta}, y)) \tag{21}$$

$$= \mu \sum_{\nu=1}^{m} \lambda_{m,\nu}^{\alpha,\beta} \hat{k}(x_{m,\nu}^{\alpha,\beta}, y) f(\gamma(x_{m,\nu}^{\alpha,\beta}, y)), \tag{22}$$

$\{\lambda_{m,\nu}^{\alpha,\beta}\}_{\nu=1}^m$ being the Christoffel numbers with respect to $v^{\alpha,\beta}$.

Let us note that the equality (21) holds, since the m-th Gauss rule we applied to the integral in (20), is exact for polynomials in $\mathbb{P}_{2m-1}$. Moreover, let us also point out that setting $\{z_i := z_i^{\alpha,\beta}\}_{i=1}^{m+r+s}$ with $z_i^{\alpha,\beta}$ given in (7), and $\{\gamma_i(x) = \gamma(x, z_i)\}_{i=1}^{m+r+s}$, one has

$$(\mathcal{K}_m^* f)(z_i) = \mu \sum_{\nu=1}^{m} \lambda_{m,\nu}^{\alpha,\beta} \hat{k}(x_{m,\nu}^{\alpha,\beta}, z_i) f(\gamma_i(x_{m,\nu}^{\alpha,\beta})).$$

Now let us consider the following finite dimensional equation

$$f_m(y) + \mu \sum_{i=1}^{m+r+s} l_i^{\alpha,\beta}(y) \sum_{\nu=1}^{m} \lambda_{m,\nu}^{\alpha,\beta} \hat{k}(x_{m,\nu}^{\alpha,\beta}, z_i) f_m(\gamma_i(x_{m,\nu}^{\alpha,\beta})) = g_m(y), \tag{23}$$

where

$$f_m(y) = \sum_{k=1}^{m+r+s} l_k^{\alpha,\beta}(y) f_m(z_k) \in \mathbb{P}_{m+r+s-1}. \tag{24}$$

By collocating (23) at $\{z_k\}_{k=1}^{m+r+s}$, using (18) and

$$f_m(\gamma_i(y)) = \sum_{k=1}^{m+r+s} l_k^{\alpha,\beta}(\gamma_i(y)) f_m(z_k) \tag{25}$$

we get the following square linear system of order $m+r+s$

$$\sum_{j=1}^{m+r+s}\left[\delta_{kj} + \mu \sum_{\nu=1}^{m} \lambda_{m,\nu}^{\alpha,\beta} \hat{k}(x_{m,\nu}^{\alpha,\beta}, z_k) l_j^{\alpha,\beta}(\gamma_k(x_{m,\nu}^{\alpha,\beta}))\right] c_j = g(z_k), \qquad k = 1, \ldots, m+r+s \tag{26}$$

where $c_j = f_m(z_j)$.

We point out that the polynomial given in (24) interpolates the unknown function f_m, whereas the polynomial given in (25) approximates (not interpolates) $f_m(\gamma_i(y))$.

Denoting by I the identity matrix of order $m+r+s$, setting

$$\mathbf{c} = [c_1, \ldots, c_{m+r+s}]^T, \quad \mathbf{g} = [g(z_1), \ldots, g(z_{m+r+s})]^T$$

and denoting by A the matrix of order $m+r+s$ whose entries are

$$A(k,j) = \mu \sum_{\nu=1}^{m} \lambda_{m,\nu}^{\alpha,\beta} \hat{k}(x_{m,\nu}^{\alpha,\beta}, z_k) l_j^{\alpha,\beta}(\gamma_k(x_{m,\nu}^{\alpha,\beta})), \qquad 1 \le k, j \le m+r+s,$$

the system (26) can be rewritten in a compact form, as follows

$$(I + A)\mathbf{c} = \mathbf{g}.$$

Once the previous system is solved, we can find the solution of the initial equation (1) according to (24).

Next theorem contains useful properties of the sequences $\{\mathcal{K}_m\}_m$ and $\{\mathcal{K}_m^*\}$ which will be essential for studying the stability and the convergence of the described method.

Theorem 3.3. *Let $\mathcal{K}$ and $\mathcal{K}_m^*$ be the operators defined in (14) and (20), respectively. If conditions (9) are satisfied, and for some real $\lambda > 0$ it is*

$$\sup_{|y|\le 1} \|k_y v^{0,\alpha+\beta+1}\|_{Z_\lambda} < \infty, \tag{27}$$

then

$$\|\mathcal{K} - \mathcal{K}_m^*\|_{Z_\lambda \to C^0} \le \mathcal{C} \frac{\log m}{m^\lambda}$$

where $\mathcal{C} \ne \mathcal{C}(m)$.

Theorem 3.4. *Let $\mathcal{K}_m^*$ and $\mathcal{K}_m$ be the operators defined in (22) and (19), respectively. If the assumptions of Theorem 3.3 are satisfied, then*

$$\|(\mathcal{K}_m - \mathcal{K}_m^*)\|_{Z_\lambda \to C^0} \le \mathcal{C} \frac{\log m}{m^\lambda}$$

where $\mathcal{C} \ne \mathcal{C}(m)$.

An immediate consequence of the previous two results is the following theorem.

Theorem 3.5. *Under the assumptions of Theorem 3.3 we have*

$$\|(\mathcal{K} - \mathcal{K}_m)\|_{Z_\lambda \to C^0} \le \mathcal{C} \frac{\log m}{m^\lambda}$$

where $\mathcal{C} \ne \mathcal{C}(m)$.

About the convergence of the method, we can prove the following.

Theorem 3.6. *Let us assume that the conditions of Theorem 3.3 are satisfied and that the right-hand side $g \in Z_\lambda$. Then, for sufficiently large m (say $m > m_0$), equation (23) has a unique solution $f_m^* \in \mathbb{P}_{m+r+s-1}$. Moreover, denoting by f^* the unique solution of equation (1), the following estimate holds true*

$$\|f^* - f_m^*\|_\infty = \mathcal{O}\left(\frac{\log m}{m^\lambda}\right), \tag{28}$$

where the constants in "$\mathcal{O}$" are independent of m.

Example 3.1. Let us test the proposed method on the following equation

$$f(y) + \int_{-1}^{y} e^{xy} \sin(\sqrt{1+x}) f(x) \sqrt{1+x}\, dx = e^{(1+y)^{\frac{1}{3}}}. \tag{29}$$

Since the exact solution is not available, we assume as exact those values of f_m obtained with $m = 512$ computing the absolute errors

$$\varepsilon_m^{512}(f) := \max_y |f_m(y) - f_{512}(y)|,$$

as well as the condition numbers $\mathrm{cond}(I + A)$ in infinity norm of the matrix $I + A$ of system (26). Following the procedure described at the beginning of this section, we get equation (12) where $\hat{k} \in Z_1$ and $g \in Z_{2/3}$. Thus, according to (28) we expect an error of the order at least $\mathcal{O}\left(\frac{\log m}{m^{2/3}}\right)$.

m	$\varepsilon_m^{512}(f)$	$\mathrm{cond}(I+A)$
4	1.14e-02	8.89e+00
8	8.30e-03	8.02e+00
16	4.41e-03	7.29e+00
32	1.47e-03	7.07e+00
64	4.32e-04	7.01e+00
256	4.63e-05	7.00e+00

Table 1: Numerical results for Equation (29)

The numerical results (Table 1) confirm the rate of convergence given in (28) and the well conditioning of the linear system. However, in virtue of the low smoothness of the kernel and right hand side, the order of convergence is slow. Hence, in the next section we introduce a numerical procedure which aims at regularizing the solution of the equation and consequently at improving the rate of convergence.

3.2 A regularized procedure

As already mentioned, the unknown solution of equation (1) is typically non-smooth at $y=-1$ where its derivative becomes unbounded (see, for instance, [13, 19]). Then, following a well known approach [11, 19, 21], in order to eliminate such a singularity, we introduce a change of variable in equation (1).

Specifically, we will consider the following "smoothing" transformation which has been widely adopted in the numerical methods for solving Volterra and Fredholm integral equations [11, 19, 21]

$$\phi_q(z)=2^{1-q}(1+z)^q-1, \qquad q\in\mathbb{N}. \tag{30}$$

Let us remark that $\phi_q'(z)\geq 0$ on $[-1,1]$ and that the inverse function is explicitly known

$$\phi_q^{-1}(z)=2^{1-\frac{1}{q}}(1+z)^{\frac{1}{q}}-1. \tag{31}$$

Setting $x=\phi_q(t)$ and $y=\phi_q(z)$ in (1), we get the following equation

$$\hat{f}(z)+\rho\int_{-1}^{z}\hat{f}(t)\tilde{k}(t,z)(z-t)^{\alpha}(1+t)^{\xi}dt=\hat{g}(z) \tag{32}$$

where $\rho=q2^{(1-q)(\beta+\alpha+1)}$, $\xi=(\beta+1)q-1$, $\hat{f}(z)=f(\phi_q(z))$ is the new unknown function,

$$\tilde{k}(t,z)=k(\phi_q(t),\phi_q(z))\left[\sum_{j=0}^{q-1}(1+t)^j(1+z)^{q-1-j}\right]^{\alpha} \tag{33}$$

and

$$\hat{g}(z)=g(\phi_q(z)). \tag{34}$$

We remark that the function $\sum_{j=0}^{q-1}(1+t)^j(1+z)^{q-1-j}$ in the new kernel $\tilde{k}$ does not vanish in $[-1,\,1]$. Moreover, for any smoothing parameter $q\in\mathbb{N}$ it results $\xi>-1$.

By mapping the interval $[-1,\,y]$ into $[-1,\,1]$, through the change of variable

$$t=\gamma(x,z)=\frac{1+z}{2}x+\frac{z-1}{2}, \tag{35}$$

(32) can be rewritten as

$$\hat{f}(z)+\varrho\int_{-1}^{1}h(x,z)\hat{f}(\gamma(x,z))v^{\alpha,\eta}(x)dx=\hat{g}(z) \tag{36}$$

where $\varrho=q\,2^{-q(2\beta+\alpha+2)+(\beta+1)}$, $\eta=\xi-\lfloor\xi\rfloor$, and

$$h(x,z)=(1+z)^{(\beta+1)q+\alpha}(1+x)^{\lfloor\xi\rfloor}\tilde{k}(\gamma(x,z),z).$$

Taking into account (33), (30) and (35), the kernel h can be also rewritten as

$$h(x,z)=(1+z)^{q(1+\beta+\alpha)}(1+x)^{\lfloor\xi\rfloor}k(\phi_q(\gamma(x,z)),\phi_q(z))\left[\sum_{j=0}^{q-1}\left(\frac{1+x}{2}\right)^j\right]^{\alpha}. \tag{37}$$

Next proposition states that the new known functions are smoother then the original ones.

Proposition 3.7. *The following statements hold true:*

1. *Let g be the right-hand side of the original equation (1) and let us assume that*

$$g(x)=g_1(x)g_2(v^{0,\delta}(x)), \qquad \delta>0, \quad g_1,g_2\in C^s.$$

Then $g\in Z_{2\delta}$ with $2\delta\leq s$ and the function $\hat{g}$ defined in (34) belongs to $Z_{2q\delta}$ with $2q\delta\leq s$.

2. *Let k be the kernel function of the original equation (1). Assuming that*

$$k(x,y)=k_1(x,y)k_2(v^{0,\delta}(x),v^{0,\delta}(y)), \quad \delta>0, \quad k_1,k_2\in C^s\times C^s,$$

then $k_x,k_y\in Z_{2\delta}$. Moreover, under the assumption $\alpha+\beta+1\geq 0$, the function h defined in (37) is such that

$$h_z\in Z_{2q\delta}, \quad 2q\delta<s$$
$$h_x\in Z_{\zeta}, \quad \zeta=\min\{[2q(1+\alpha+\beta)],2q\delta\}.$$

At this point in order to approximate the solution of equation (36) we apply the collocation method described in the previous paragraph.

In a nutschell, we project equation (36) on the finite dimensional space $\mathbb{P}_{m+r+s-1}$ by means of the Lagrange operator $L^{\alpha,\eta}_{m,r,s}$ where

$$\frac{\alpha}{2}+\frac{1}{4}\le r<\frac{\alpha}{2}+\frac{5}{4}, \qquad \frac{\eta}{2}+\frac{1}{4}\le s<\frac{\eta}{2}+\frac{5}{4}. \tag{38}$$

Then, by proceeding as done in Section 3.1 we introduce the sequences

$$\hat{g}_m = L^{\alpha,\eta}_{m,r,s}(\hat{g})$$

and

$$(\hat{\mathcal{K}}_m\hat{f}) = L^{\alpha,\eta}_{m,r,s}(\hat{\mathcal{K}}^*_m\hat{f}) \tag{39}$$

where

$$(\mathcal{K}^*_m f)(z) = \varrho\int_{-1}^{1}\mathcal{L}^{\alpha,\eta}_m\left(\left[L^{\alpha,\eta}_{m,r,s}(h_z)\hat{f}(\gamma(\cdot,z))\right],x\right)\,v^{\alpha,\eta}(x)dx \tag{40}$$

and we consider the following finite dimensional equation

$$\hat{f}_m(z)+\varrho\sum_{i=1}^{m+r+s} l_i^{\alpha,\eta}(z)\sum_{\nu=1}^{m}\lambda^{\alpha,\eta}_{m,\nu}\,h(x^{\alpha,\eta}_{m,\nu},z_i)\hat{f}_m(\gamma_i(x^{\alpha,\eta}_{m,\nu})) = \hat{g}_m(z), \tag{41}$$

where here $\{z_k := z_k^{\alpha,\eta}\}_{k=1}^{m+r+s}$ and

$$\hat{f}_m(z) = \sum_{j=1}^{m+r+s} l_j^{\alpha,\eta}(z)\hat{f}_m(z_j). \tag{42}$$

Hence, by collocating at $\{z_k\}_{k=1}^{m+r+s}$, we get the following linear system

$$\sum_{j=1}^{m+r+s}\left[\delta_{kj}+\varrho\sum_{\nu=1}^{m}\lambda^{\alpha,\eta}_{m,\nu}\,h(x^{\alpha,\eta}_{m,\nu},z_k)\,l_j^{\alpha,\eta}(\gamma_k(x^{\alpha,\eta}_{m,\nu}))\right]\hat{c}_j = \hat{g}(z_k), \qquad k=1,\dots,m+r+s \tag{43}$$

where $\hat{c}_j = \hat{f}_m(z_j)$. In a matrix form, setting

$$\mathbf{c} = [\hat{c}_1,\dots,\hat{c}_{m+r+s}]^T, \quad \mathbf{g} = [\hat{g}(z_1),\dots,\hat{g}(z_{m+r+s})]^T$$

and denoting by $\hat{A}$ the matrix

$$\hat{A}(k,j) = \varrho\sum_{\nu=1}^{m}\lambda^{\alpha,\beta}_{m,\nu}\,h(x^{\alpha,\beta}_{m,\nu},z_k)\,l_j^{\alpha,\beta}(\gamma_k(x^{\alpha,\beta}_{m,\nu})), \qquad 1\le k,j\le m+r+s,$$

the linear system (43) can be written as

$$(I+\hat{A})\mathbf{c} = \mathbf{g}.$$

Once such a system is solved, we can find the solution of the regularized equation (36) according to (42). Moreover, by using the inverse function (31), we can directly recover the unique solution f_m of the initial equation (1) being

$$f_m(y) = \sum_{j=1}^{m+r+s}\ell_j^{\alpha,\eta}(\phi_q^{-1}(x))\hat{c}_j. \tag{44}$$

Next theorem states the convergence of the collocation method applied to the regularized equation.

Theorem 3.8. *Let us assume that $\alpha+\beta+1\ge 0$, conditions (38) are satisfied, and the assumptions of Proposition 3.7 are fulfilled. Then, for sufficiently large m (say $m>m_0$), equation (41) has a unique solution $\hat{f}^*_m\in\mathbb{P}_{m+r+s-1}$. Moreover, denoting by $\hat{f}^*$ the unique solution of equation (32), the following estimate holds true*

$$\|\hat{f}^*-\hat{f}^*_m\|_\infty = \mathcal{O}\left(\frac{\log m}{m^\zeta}\right), \quad \zeta=\min\{2q(\alpha+\beta+1),2q\delta\} \tag{45}$$

where the constants in "$\mathcal{O}$" are independent of m.

We want to remark that our regularization technique provides more accurate results with respect to others available in the literature (see, for instance [5]). Consider indeed the following equation [6, Example 5.1] which has been examined also in Section 4 (see Example 4.2)

$$f(y)+\int_{-1}^{y} f(x)(y-x)^{-0.35}dx = (1+y)^{3.6}+(1+y)^{4.25}B(4.6,0.65)$$

where B$(\cdot,\cdot)$ is the Beta function. The exact solution is $f^*(y) = (1+y)^{3.6} \in C^3$. If we apply the procedure given in [5, Theorem 4.1] we get an error of the order $\mathcal{O}\left(m^{-2.5}\log m\right)$, since $g \in C^3$ and $k \equiv 1$. However, according to (28) if we apply our method the error behaves like $\mathcal{O}\left(m^{-q\,1.3}\log m\right)$ since $g \in Z_{7.2}$ and (27) is satisfied with $\lambda = 1.3$. Hence, if for instance $q = 2$, we get a slightly better error but if we take $q = 3$ we get a high order of convergence. This is one of the advantages of our method, i.e. the order of convergence depends on the smoothing parameter q and then we can take q as large as we want in order to get an accurate approximation .

Let us also remark that we have a good error due to the regularizing technique and the use of an optimal interpolation process, but also due to the choice of the Zygmund-type spaces in which to look for the solution. Indeed, if we had considered the spaces C^p then, in the above example, we would have had an error of the order $\mathcal{O}\left(m^{-\lfloor q\,0.65\rfloor}\log m\right)$.

4 Numerical Tests

In this section we show the effectiveness of the proposed numerical method, by means of some numerical examples.

For each test, we solve system (43), we perform the solution $\hat{f}_m$ given by (42) of the regularized equation, and then we construct the solution f of the considered equation according to (44).

In the second example the exact solution f^* is available and then we compute the absolute errors

$$\epsilon_m(f) := \max_y |f_m(y) - f^*(y)|.$$

In the reminder ones, since the exact solution is not available, we assume as exact the one obtained for a fixed value of M that we will specify in each test and we compute the absolute errors

$$\epsilon_m^M(f) = \max_y |f_m(y) - f_M(y)|.$$

Example 4.1. As a first test, let us consider again equation (29) examined in Example 3.1 and let us apply the regularized procedure. The new kernel and right-hand side belong to the spaces Z_q and $Z_{\frac{2q}{3}}$, respectively. Hence, according to Theorem 3.8, the order of convergence of the method is $\mathcal{O}\left(\frac{\log m}{m^{\frac{2q}{3}}}\right)$. The numerical results given in Table 2 confirm the theoretical error and also show that the condition number in infinity norm cond$(I+\hat{A})$ of the regularized system is uniformly bounded with respect to m for each q.

q	m	$\epsilon_m^{512}(f)$	cond$(I+\hat{A})$	q	m	$\epsilon_m^{512}(f)$	cond$(I+\hat{A})$
2	4	1.29e-02	8.87e+00	3	4	4.94e-02	8.40e+00
	8	1.31e-03	8.72e+00		8	5.59e-03	8.89e+00
	16	9.43e-04	7.60e+00		16	3.14e-05	7.86e+00
	32	2.91e-04	7.15e+00		32	5.64e-10	7.23e+00
	64	8.52e-05	7.03e+00		64	2.62e-10	7.05e+00
	256	6.69e-06	7.00e+00		256	2.50e-10	7.00e+00

Table 2: Numerical results for Example 4.1

Example 4.2. Let us consider the following Volterra integral equation [6, Example 5.1]

$$f(y) + \int_{-1}^{y} f(x)(y-x)^{-0.35}dx = (1+y)^{3.6} + (1+y)^{4.25}B(4.6, 0.65)$$

where B$(\cdot,\cdot)$ is the Beta function. The given equation admits as unique solution the function $f^*(y) = (1+y)^{3.6}$. Table 3 shows the absolute errors we get. As already mentioned at the end of Section 3, if we do not regularize, according to (28), we get an error of the order $\mathcal{O}\left(m^{-1.3}\log m\right)$ whereas if, for instance, we take $q = 2$ we get $\mathcal{O}\left(m^{-2.6}\log m\right)$.

q	m	$\epsilon_m(f)$	cond$(I+\hat{A})$	q	m	$\epsilon_m(f)$	cond$(I+\hat{A})$
1	4	7.67e-04	9.70e+00	2	4	5.67e-02	1.07e+01
	8	9.04e-06	8.33e+00		8	3.76e-07	9.28e+00
	16	8.35e-08	7.28e+00		16	1.32e-11	7.80e+00
	32	6.38e-10	6.71e+00		32	9.77e-15	6.97e+00
	64	4.01e-12	6.46e+00				
	256	3.91e-14	6.26e+00				

Table 3: Numerical results for Example 4.2

Example 4.3. Let us apply our method to the following equation

$$f(y)+\int_{-1}^{y}\frac{x}{2+y^2}f(x)(y-x)^{-\frac{3}{4}}\,dx=e^{|y|^{\frac{9}{4}}}\cos y.$$

If we look at the regularized equation, then in this case the kernel h is a smooth function with respect to the variable x whereas $h_x(z)\sim(1+z)^{q/4}\in Z_{q/2}$. The right-hand side belongs to $Z_{9/4}$. Then, for $q\geq 3$ the smoothness of the right-hand side determines the order of convergence which is in this case $\mathcal{O}\left(m^{-9/4}\log m\right)$. Table 4 contains the errors we get as well as the condition numbers of the linear system. This is an example in which our regularization strategy aiming at eliminate the singularity at $y=-1$ does not produce a high order of convergence. This is because of the presence of a right-hand side which has an internal singularity. On the other hand also in this case, our method furnishes a better estimate than those provided in [5] which would be equal to $\mathcal{O}\left(m^{-1.5}\log m\right)$.

q	m	$\varepsilon_m^{600}(f)$	cond$(I+\hat{A})$	q	m	$\varepsilon_m^{600}(f)$	cond$(I+\hat{A})$
3	8	1.79e-01	1.39e+01	4	8	3.17e-01	1.51e+01
	16	4.92e-03	1.36e+01		16	1.07e-02	1.37e+01
	32	4.99e-04	1.33e+01		32	4.63e-04	1.35e+01
	64	1.31e-04	1.32e+01		64	6.78e-05	1.31e+01
	256	4.71e-06	1.28e+01		256	1.69e-06	1.28e+01
	512	1.38e-06	1.26e+01		512	2.84e-07	1.27e+01

Table 4: Numerical results for Example 4.3

Example 4.4. Let us test our method to the following equation

$$f(y)+\int_{-1}^{y}|\sin(x+y)|^{\frac{8}{3}}f(x)(y-x)^{\frac{1}{5}}(1+x)^{\frac{2}{5}}\,dx=\arctan(1+y^2)$$

In this case the right-hand side of the regularized equation is smooth and the kernel belongs to $Z_{8/3}$ for each $q\in\mathbb{N}$. In Table 5 we report the results we get for $q=3$. If $q\geq 3$ we have errors of the same order in view of the smoothness of the kernel.

q	m	$\varepsilon_m^{700}(f)$	cond$(I+\hat{A})$
3	4	1.80e-01	3.62e+00
	8	2.66e-02	3.65e+00
	16	2.39e-03	3.73e+00
	32	9.68e-05	3.77e+00
	64	4.45e-06	3.79e+00
	256	2.81e-08	3.79e+00
	512	4.29e-10	3.79e+00

Table 5: Numerical results for Example 4.4

5 Proofs

Proof of Theorem 2.2

Setting $\mathcal{A}_m=\{x:|x|\leq 1-\frac{c}{m^2}\}$ for any fixed $c>0$, by Remez inequality (see for instance [16, p. 297]) we have

$$\|L_{m,r,s}^{\alpha,\beta}(f)u\|_1\leq\mathcal{C}\int_{\mathcal{A}_m}|L_{m,r,s}^{\alpha,\beta}(f,x)|u(x)dx.$$

Recalling that an expression of the polynomial $L_{m,r,s}^{\alpha,\beta}$ is

$$\begin{aligned}L_{m,r,s}^{\alpha,\beta}(f,x)=A_s(x)B_r(x)\mathcal{L}_m^{\alpha,\beta}\left(\frac{f}{A_sB_r},x\right)\quad&+\quad A_s(x)p_m(v^{\alpha,\beta},x)\sum_{k=1}^{r}\frac{f(t_k)}{A_s(t_k)p_m(v^{\alpha,\beta},t_k)}\prod_{i=1,i\neq k}^{r}\frac{x-t_i}{t_k-t_i}\\&+\quad B_r(x)p_m(v^{\alpha,\beta},x)\sum_{k=1}^{s}\frac{f(y_k)}{B_r(y_k)p_m(v^{\alpha,\beta},y_k)}\prod_{i=1,i\neq k}^{s}\frac{x-y_i}{y_k-y_i},\end{aligned}$$

we have

$$\begin{aligned} \|L^{\alpha,\beta}_{m,r,s}(f)u\|_1 &\leq \mathcal{C}\int_{\mathcal{A}_m}\left|A_s(x)B_r(x)\mathcal{L}^{\alpha,\beta}_m\left(\frac{f}{A_sB_r},x\right)\right|u(x)dx \\ &+ \int_{\mathcal{A}_m}\left|A_s(x)p_m(v^{\alpha,\beta},x)\sum_{k=1}^{r}\frac{f(t_k)}{A_s(t_k)p_m(v^{\alpha,\beta},t_k)}\prod_{i=1,i\neq k}^{r}\frac{x-t_i}{t_k-t_i}\right|u(x)dx \\ &+ \int_{\mathcal{A}_m}\left|B_r(x)p_m(v^{\alpha,\beta},x)\sum_{k=1}^{s}\frac{f(y_k)}{B_r(y_k)p_m(v^{\alpha,\beta},y_k)}\prod_{i=1,i\neq k}^{s}\frac{x-y_i}{y_k-y_i}\right|u(x)dx \\ &:= J_1+J_2+J_3. \end{aligned} \tag{46}$$

By [16, (4.2.24)-(4.2.25)] immediately it follows

$$J_2+J_3\leq \mathcal{C}\|f\|_\infty. \tag{47}$$

To estimate J_1 we use $(A_sB_r)(x^{\alpha,\beta}_{m,k})\sim v^{r,s}(x^{\alpha,\beta}_{m,k})$ and $(A_sB_r)(x)\sim v^{r,s}(x)$, $x\in\mathcal{A}_m$, by which we deduce

$$J_1\leq \left\|\mathcal{L}^{\alpha,\beta}_m\left(fv^{-r,-s}\right)uv^{r,s}\right\|_1$$

and by Nevai's Theorem [17, Th.1], under the assumptions (11), we get

$$J_1\leq \mathcal{C}\|f\|_\infty.$$

Hence, the thesis follows by combining last inequality with (47) and (46).□

Proof of Proposition 3.1 Since for any $f\in C^0$,

$$|(\mathcal{K}f)(y)|\leq \mathcal{C}\|f\|_\infty(1+y)^{\alpha+\beta+1}\int_{-1}^{1}|k(\gamma(x,y),y)|v^{\alpha,\beta}(x)dx\leq \mathcal{C}\|f\|_\infty,$$

under the assumption (15), the thesis (16) follows.

Let us prove that for any $f\in Z_\lambda$,

$$\Omega^r_\varphi(\mathcal{K}f,t)\leq \mathcal{C}t^\lambda\|f\|_{Z_\lambda},\quad 0<\lambda\leq r. \tag{48}$$

By (14) for $0<h\leq t$, $y\in I_{rh}=[-1+(2rh)^2,1-(2rh)^2]$ we have

$$\begin{aligned} \Delta^r_{h\varphi}(\mathcal{K}f)(y) &= \mu\int_{-1}^{1}\Delta^r_{h\varphi(y)}\left(f(\gamma(x,y))\hat{k}(\gamma(x,y),y)\right)v^{\alpha,\beta}(x)dx \\ &\leq \mathcal{C}\int_{-1}^{1}\Omega^r_\varphi(\hat{k}_xf_x,t)v^{\alpha,\beta}(x)dx \end{aligned}$$

and consequently, by using [16, (2.5.18)]

$$\Omega^k_\varphi(f,t)\leq \mathcal{C}t^k\sum_{i=0}^{\lfloor\frac{1}{t}\rfloor}(1+i)^{k-1}E_i(f)_\infty,\qquad \mathcal{C}\neq\mathcal{C}(t,f),$$

we have

$$\begin{aligned} \Omega^r_\varphi(\mathcal{K}f,t) &\leq \mathcal{C}\sup_{|x|\leq 1}\Omega^r_\varphi(\hat{k}_xf_x,t) \\ &\leq \mathcal{C}t^r\sum_{i=0}^{\lfloor\frac{1}{t}\rfloor}(1+i)^{r-1}\sup_{|x|\leq1}E_i(\hat{k}_xf_x)_\infty \\ &\leq \mathcal{C}t^r\left[\sup_{|x|\leq1}\|\hat{k}_xf_x\|_\infty+\sum_{i=1}^{\lfloor\frac{1}{t}\rfloor}(1+i)^{k-1}\sup_{|x|\leq1}E_i(\hat{k}_xf_x)_\infty\right]. \end{aligned}$$

Thus, recalling that for any $f_1,f_2\in C^0$ we have [16, p. 384]

$$E_{2m}(f_1f_2)_\infty\leq \mathcal{C}\left(\|f_1\|_\infty E_m(f_2)_\infty+2E_m(f_1)_\infty\|f_2\|_\infty\right), \tag{49}$$

by the assumption (15) and taking (13) into account we have

$$\Omega^r_\varphi(\mathcal{K}f,t)\leq \mathcal{C}\left[t^r+\sum_{i=1}^{\lfloor\frac{1}{t}\rfloor}i^{r-\lambda-1}\right]\|f\|_{Z_\lambda}\leq \mathcal{C}t^\lambda\|f\|_{Z_\lambda}$$

from which (48) follows.

By applying (4) we have

$$E_m(\mathcal{K}f)_\infty \le \mathcal{C}\int_0^{1/m}\frac{\Omega_\varphi^r(\mathcal{K}f,t)}{t}dt \le \frac{\mathcal{C}}{m^\lambda}\|f\|_{Z_\lambda},$$

and therefore (17) i.e.

$$\lim_m\left(\sup_{\|f\|_{Z_\lambda}=1} E_m(\mathcal{K}f)_\infty\right) = 0. \quad \square$$

Proof of Theorem 3.3 Setting

$$(\widetilde{\mathcal{K}}_m f)(y) = \mu\int_{-1}^1 L_{m,r,s}^{\alpha,\beta}(\hat{k}_y, x) f(\gamma(x,y))\, v^{\alpha,\beta}(x)dx,$$

we can write

$$|(Kf)(y) - K_m^* f)(y)| \le |(Kf)(y) - (\widetilde{\mathcal{K}}_m f)(y)| + |(\widetilde{\mathcal{K}}_m f)(y) - K_m^* f)(y)| =: I_1(y) + I_2(y). \tag{50}$$

Let us first consider $I_1(y)$. By (14) we have

$$I_1(y) \quad \le \quad \mu\int_{-1}^1 |f(\gamma(x,y))|\,|\hat{k}_y(x) - L_{m,r,s}^{\alpha,\beta}(\hat{k}_y, x)|v^{\alpha,\beta}(x)dx \le \mu\|f\|_\infty \max_{|y|\le 1}\|[\hat{k}_y - L_{m,r,s}^{\alpha,\beta}(\hat{k}_y)]v^{\alpha,\beta}\|_1.$$

By Theorem 2.1, in virtue of the assumption (9),

$$I_1(y) \le \mathcal{C}\|f\|_\infty \max_{|y|\le 1} E_{m+r+s-1}(\hat{k}_y)_\infty,$$

and using the weak-Jackson estimate (4), we get

$$\|I_1\|_\infty \le \frac{\mathcal{C}}{m^\lambda}\|f\|_\infty \sup_{|y|\le 1}\|\hat{k}_y\|_{Z_\lambda} \le \frac{\mathcal{C}}{m^\lambda}\|f\|_\infty \tag{51}$$

where the last bound follows taking into account the assumption (27).

Consider now $I_2(y)$. By applying (49), we can write

$$I_2(y) \le \mathcal{C}E_{2m-1}(L_{m,r,s}^{\alpha,\beta}(\hat{k}_y) f(\gamma(y)))_\infty \le \mathcal{C}\left[\|L_{m,r,s}^{\alpha,\beta}(\hat{k}_y)\|_\infty\, E_m(f(\gamma(y)))_\infty + \|f(\gamma(y)))\|_\infty E_m(L_{m,r,s}^{\alpha,\beta}(\hat{k}_y))_\infty\right]$$

from which by (10) and (15) we deduce

$$\|I_2\|_\infty \le \mathcal{C}\frac{\log m}{m^\lambda}\|f\|_{Z_\lambda}\sup_{|y|\le 1}\|\hat{k}_y\|_{Z_\lambda} \le \mathcal{C}\frac{\log m}{m^\lambda}\|f\|_{Z_\lambda}. \tag{52}$$

Therefore, by replacing (51) and (52) in (50) we have

$$\|[\mathcal{K} - \mathcal{K}_m^*]f\|_\infty \quad \le \quad \mathcal{C}\frac{\log m}{m^\lambda}\|f\|_{Z_\lambda}.$$

and the thesis immediately follows.□

Proof of Theorem 3.4 In virtue of Theorem 9, under the assumptions (9) and taking (4) into account we have

$$\begin{aligned}\|(\mathcal{K}_m^* - \mathcal{K}_m)f\|_\infty &\le \mathcal{C}\log m\, E_{m+r+s-1}(\mathcal{K}_m^* f)_\infty \\ &\le \mathcal{C}\log m \int_0^{\frac{1}{m}}\frac{\Omega_\varphi^r(\mathcal{K}_m^* f, t)_\infty}{t}dt.\end{aligned} \tag{53}$$

Let us prove that for any $f \in Z_\lambda$ with $0 < \lambda \le r$

$$\Omega_\varphi^r(\mathcal{K}_m^* f, t) \le \mathcal{C}t^\lambda\|f\|_{Z_\lambda}. \tag{54}$$

For $0 < h \le t$, $y \in I_{rh} = [-1 + (2rh)^2, 1 - (2rh)^2]$ we have

$$\begin{aligned}|\Delta_{h\varphi}^r(\mathcal{K}_m^* f)(y)| \quad &\le \quad \mathcal{C}\sum_{\nu=1}^m \lambda_{m,\nu}^{\alpha,\beta}\Delta_{h\varphi(y)}^r\left(f(\gamma(x_{m,\nu}^{\alpha,\beta}, y))\hat{k}(\gamma(x_{m,\nu}^{\alpha,\beta}, y)\right) \\ &\le \quad \mathcal{C}\sum_{\nu=1}^m \lambda_{m,\nu}^{\alpha,\beta}\Omega_\varphi^r\left(\hat{k}(x_{m,\nu}^{\alpha,\beta})f(\gamma(x_{m,\nu}^{\alpha,\beta})), t\right)\end{aligned}$$

and therefore
$$\Omega_\varphi^r(\mathcal{K}_m^* f,t) \le \mathcal{C} \sup_{1\le\nu\le m} \Omega_\varphi^r\left(\bar{k}(x_{m,\nu}^{\alpha,\beta})f(x_{m,\nu}^{\alpha,\beta}),t)\right)\int_{-1}^1 v^{\alpha,\beta}(x)dx.$$
Thus following the same arguments used in the proof of relation (48), we can conclude
$$\Omega_\varphi^r(\mathcal{K}_m^* f,t) \le \mathcal{C}\, t^\lambda \, \|f\|_{Z_\lambda},$$
and then (54). By combining (53) and (54) we get
$$\|(\mathcal{K}_m^* - \mathcal{K}_m)f\|_\infty \le \mathcal{C}\frac{\log m}{m^\lambda}\,\|f\|_{Z_\lambda}$$
by which the thesis follows. □

Proof of Theorem 3.6 The first part of the theorem follows by [1, p. 55]. Moreover, by (12) and (23), setting $\widetilde{f}(x,y) = f^*(\gamma(x,y))$, we have
$$\|f^* - f_m^*\|_\infty \le \mathcal{C}\left[\|g-g_m\|_\infty + \|(\mathcal{K}-\mathcal{K}_m)\widetilde{f}\|_\infty\right]$$
from which by (10) and Theorem 3.5, we can write
$$\|f^* - f_m^*\|_\infty \le \mathcal{C}\frac{\log m}{m^\lambda}\left[\|g\|_{Z_\lambda} + \|f^*\|_{Z_\lambda}\right]$$
i.e estimate (28). □

Proof of Proposition 3.7 By the assumption $g(x) = g_1(x)g_2((1+x)^\delta)$ with $g_1, g_2 \in C^s$. For each $r \le s$, by using the Faa di Bruno's formula, we can write
$$[g_2((1+x)^\delta)]^{(r)} = r!\sum_{k=0}^r g_2^{(k)}((1+x)^\delta)\sum_{h_1+\ldots+h_k=r}\left(\frac{(1+x)^{\delta-h_1}}{h_1!}\right)\cdots\left(\frac{(1+x)^{\delta-h_k}}{h_k!}\right).$$
Thus
$$|[g_2((1+x)^\delta)]^{(r)}\varphi^r(x)| \le \mathcal{C}\sum_{k=0}^r |g_2^{(k)}((1+x)^\delta)|\,(1+x)^{\delta-\frac{r}{2}},$$
and then, for $2\delta < r \le s$ we get
$$\|[g_2((1+\phi_q)^\delta]^{(r)}\varphi^r\|_{I_{rh}} \le \mathcal{C}h^{2\delta-r}\|g_2^{(r)}\|_\infty.$$
Consequently, by (5) we can deduce that $g_2 \in Z_{2\delta}$ and therefore $g \in Z_{2\delta}$. Let us now consider $\hat{g}$. By its definition
$$\hat{g}(x) = g_1(\phi_q(x))\,g_2((1+\phi_q(x))^\delta)$$
where $g_1(\phi_q(x))$ still belongs to C^s. Considering that $g_2((1+\phi_q(x))^\delta) = g_2(2^{1-q}(1+x)^{q\delta})$ by proceeding as done before, with $q\delta$ instead of δ, we get $g_2((1+\phi_q(x))^\delta) \in Z_{2q\delta}$. Therefore $\hat{g} \in Z_{2q\delta}$.

Let us now prove the point 2. In virtue of the assumption on the kernel k, by proceeding as done for g, we can deduce that $k_x, k_y \in Z_{2\delta}$. Moreover in view of (37) we can write
$$h_z(x) = (1+x)^{\lfloor\xi\rfloor}\left[\sum_{j=0}^{q-1}\frac{(1+x)^j}{2}\right]^\alpha k_1(\phi_q(\gamma_z(x)),\phi_q(z))\,k_2\left((1+\phi_q(\gamma_z(x)))^\delta,\phi_q(z)\right).$$
The first two functions belong to C^∞, the third one belongs to C^s and so the smoothness of the $h_z(x)$ depends on the last function. About the latter, noting that
$$k_2\left((1+\phi_q(\gamma_z(x)))^\delta,\phi_q(z)\right) = k_2(2^{\delta(1-2q)}(1+z)^{q\delta}(1+x)^{q\delta},\phi_q(z))$$
by applying Faa di Bruno's formula and using (5), we can assert that it belongs to $Z_{2q\delta}$. Consequently, $h_z(x) \in Z_{2q\delta}$ too. Concerning the last point, we note that
$$h_x(z) = (1+z)^{q(\alpha+\beta+1)}\,k_1(\phi_q(\gamma_z(x)),\phi_q(z))\,k_2\left((1+\phi_q(\gamma_z(x)))^\delta,\phi_q(z)\right).$$
Again the last two functions belong to $Z_{2q\delta}$, whereas the first one belongs to $Z_{2q(1+\alpha+\beta)}$. Hence, $h_x(z) \in Z_\zeta$ with $\zeta = \min\{2q(\alpha+\beta+1), 2q\delta\}$.□

Proof of Theorem 3.8 First, let us denote by $\hat{\mathcal{K}}$ the integral operator appearing in (36), namely
$$(\hat{\mathcal{K}}f)(z) = \varrho\int_{-1}^1 h(x,z)\,\hat{f}(\gamma(x,z))\,v^{\alpha,\eta}(x)dx.$$
By a well-known result [1, p. 55] if we prove that
$$\lim_m \|\hat{\mathcal{K}} - \hat{\mathcal{K}}_m\| = 0 \tag{55}$$
where $\hat{\mathcal{K}}_m$ is given in (39), then for sufficiently large m equation (41) admits a unique solution. In order to show (55) we write
$$\|\hat{\mathcal{K}} - \hat{\mathcal{K}}_m\| \le \|\hat{\mathcal{K}} - \hat{\mathcal{K}}_m^*\| + \|\hat{\mathcal{K}}_m^* - \hat{\mathcal{K}}_m\|$$
where $\hat{\mathcal{K}}_m^*$ is defined in (40). Now, the first term can be estimated as done in the proof of Theorem 3.3 whereas the second one can be estimated by following the proof of Theorem 3.4. Moreover, taking into account Proposition 3.7 and by proceeding as in the proof of (28), we find (45).□

6 Conclusion

In this paper, we proposed a projection method for linear Volterra integral equations having kernels that can be singular along the diagonal and/or at the side $y = -1$. These are pathological cases since the solution inherits the singularity at $y = -1$.

We proved the convergence and the stability of the method providing also an error estimate. Such estimate is further improved by introducing a smoothing transformation in order to regularize the unknown solution.

In the light of the analysis carried out, we get accurate error estimates. Such accuracy is due to a combination of different approximation tools, among which, the selection of spaces which are appropriate to treat functions with singularities, the use of an optimal interpolation process and the introduction of a smoothing technique.

The numerical experiments confirmed the theoretical results showing the stability of the method, the well conditioning of the linear systems we have to solve and the dependence of the order of convergence on the smoothing parameter.

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