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Scattered data interpolation by Shepard's like methods: classical results and recent advances

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Abstract

Interpolation problems arise in many areas where there is a need to construct a continuous surface from irregularly spaced data points. This problem has a number of solutions and, among them, the choice of interpolation technique depends on the distribution of points in the data set, the application domain, the approximating function or the method that is prevalent in the discipline. We discuss on Shepard's interpolation method and some of its variations, which have been proposed in order to increase the accuracy of approximation of the original method, to improve its efficiency or even to solve specific interpolation problems.

1 Introduction

Scattered data approximation deals with the problem of reconstructing an unknown function from given scattered data, i.e. data which consist of a set of points and corresponding values, where the points have no structure or order between their relative locations. It is a fast growing research area due to its many applications such as, for instance, terrain modeling, surface reconstruction, image restoration and inpainting, surface deformation, the numerical solution of partial differential equations. The choice of the interpolation technique, among the number of solutions to the scattered data interpolation problem, depends on the distribution of points in the data set, application domain, approximating function, or the method that is prevalent in the discipline [53, 39, 14].

The most famous operator for scattered data interpolation is the Shepard operator, introduced by Donald Shepard in 1968 [50]. This operator is based on a weighted average of values at the data points. There are several variations of the original Shepard operator which have been proposed in order to increase the accuracy of approximation of the original method, to improve its efficiency or even to solve specific interpolation problems.

The aim of this paper is to give a survey on the current state of the art in Shepard's like interpolation, which covers all classic references as well as the most recent advances. In Section 2 we focus on the analysis of classical and modified Shepard's operator, in Section 3 we describe variations of the original Shepard operator based on quadratic, cubic and linear polynomials, both in the univariate and bivariate framework. Finally, in Section 4 we discuss perspectives for future researches.

2 Shepard's method

Let $X = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$ be a set of n distinct points of a domain $\Omega \subset \mathbb{R}^2$ with associated function evaluations data $f_i = f(\boldsymbol{x}_i), i = 1, \dots, n$. The *classical Shepard operator* [50]

$$S_\mu[f](\boldsymbol{x}) = \sum_{i=1}^{n} A_{\mu,i}(\boldsymbol{x}) f_i, \quad \mu > 0,$$

is the linear combination of the functional values f_i with weight functions

$$A_{\mu,i}(\boldsymbol{x}) = \frac{|\boldsymbol{x} - \boldsymbol{x}_i|^{-\mu}}{\sum\limits_{k=1}^{n} |\boldsymbol{x} - \boldsymbol{x}_k|^{-\mu}}, \qquad i = 1, \dots, n$$

defined as the normalization of the inverse distance from the scattered points. $|\cdot|$ is the Euclidean norm. $\mu > 0$ is called the power parameter. Since the basis functions $A_{\mu,i}$ are cardinal, non-negative, and form a partition of unity, the interpolation operator S_μ is stable [38] in the sense that

$$\min_i f_i \le S_\mu[f](\boldsymbol{x}) \le \max_i f_i, \qquad \boldsymbol{x} = (x, y) \in \mathbb{R}^2,$$

but for $\mu > 1$ it has flat spots at all nodes, that is the gradient of the $S_\mu[f]$ is zero at every data point (for more details see [50, page 520]). Moreover, the algebraic degree of exactness (abbreviated by "dex" in the following) of the operator S_μ is 0, that is, it reproduces only constant polynomials, and its approximation order is at most $O(h)$, where h is the *mesh size* of the set of sample points [38]. The form of weight functions $A_{\mu,i}$ accords too much influence to data points that are far away from the point of

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approximation.To avoid this problem, Franke and Nielson [40] developed a modification of the original Shepard's method in order to make it more local, suggesting to use compact support basis functions

$$\widetilde{W}_{\mu,i}(\boldsymbol{x}) := \frac{W_{\mu,i}(\boldsymbol{x})}{\sum_{k=1}^{n} W_{\mu,k}(\boldsymbol{x})}$$

where

$$W_{\mu,i}(\boldsymbol{x}) := \left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_i|} - \frac{1}{R_{w_i}}\right)_+^{\mu}, \tag{1}$$

$(\cdot)_+$ is the positive part function, R_{w_i} is the radius of influence about node $\boldsymbol{x}_i$ chosen just large enough to include N_w nodes in the closed ball $B(\boldsymbol{x}, R_{w_i})$.

3 Combined Shepard operators

Several variations of the original Shepard method have been proposed, with the aim of solving the "flat-spot" problem. To correct this undesirable property, Shepard [50, page 520] suggested to add increments to the function values at nearby data points so that the interpolated surface would achieve desired partial derivatives at the data locations. As reported by [42], Shepard considered the special case $\mu = 2$, and proposed a technique for interpolating the given first partial derivatives utilizing a formula of the form

$$S_\mu[f](\boldsymbol{x}) = \sum_{i=1}^{n} A_{\mu,i}(\boldsymbol{x})\left(f_i + \frac{\partial f}{\partial x}(\boldsymbol{x}_i)(x-x_i) + \frac{\partial f}{\partial y}(\boldsymbol{x}_i)(y-y_i)\right).$$

Extensions to higher order derivative interpolation have been introduced by Farwig in 1986 [38]. Working in the s—dimensional space $\mathbb{R}^s$ (s is the dimension of the data locations), he considered the general case of the multivariate Taylor polynomial of any order

$$T_r[f, \boldsymbol{x}_i](\boldsymbol{x}) = \sum_{|\nu|=0}^{r} \frac{D^\nu f(\boldsymbol{x}_i)}{\nu!}(\boldsymbol{x}-\boldsymbol{x}_i)^\nu,$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_s)$ denotes a multi-index and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_s$, and defined the Shepard-Taylor operator

$$S_{T_r}[f](\boldsymbol{x}) = \sum_{i=1}^{n} A_{\mu,i}(\boldsymbol{x}) T_r[f, \boldsymbol{x}_i](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^s$$

which has algebraic degree of exactness r and interpolates on all data required for its definition, provided that $\mu \geq r+1$, for $\mu \in \mathbb{N}$, or $[\mu] \geq r$, otherwise. Under certain conditions of regularity of the domain Ω which contains the nodes and on the function f to be approximated, Farwig gave information on the rate of convergence of the Shepard-Taylor operator when the fill distance

$$h = \inf\left\{\rho > 0 : \text{ for every } \boldsymbol{x} \in D, B_\rho(\boldsymbol{x}) \text{ contains at least one element of } X\right\},$$

tends to 0. Note that the fill distance is the radius of the largest empty ball that can be placed among the data locations. Farwig showed that the rate of convergence of the Shepard-Taylor operator depends on the power parameter μ

$$\left\|S_{T_r}[f] - f\right\| = O(\varepsilon_\mu^r(h))$$

where

$$\varepsilon_\mu^r(h) = \begin{cases} |\log h|^{-1}, & \mu = s, \\ h^{\mu-s}, & \mu - s < r+1, \mu > s, \\ h^{\mu-s}|\log h|, & \mu - s = r+1, \\ h^{r+1}, & \mu - s > r+1. \end{cases}$$

3.1 Coman's approach for univariate instances

With the aim of increasing the approximation accuracy and to extend the interpolation property of the Shepard operator, the *Combined Shepard operators* [26] are defined by replacing each value f_i with the value of an interpolation operator at $\boldsymbol{x}_i$, $P[\cdot, \boldsymbol{x}_i](\boldsymbol{x})$, applied to f, with a certain degree of exactness greater than 0,

$$S_P[f](\boldsymbol{x}) = \sum_{i=1}^{n} A_{\mu,i}(\boldsymbol{x}) P[f, \boldsymbol{x}_i](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^s; \tag{2}$$

the combined operator $S_P[f]$ is no longer stable but $\text{dex}(S_P[f]) = \min_i \text{dex}(P[f, \boldsymbol{x}_i])$, i.e. it reproduces exactly polynomials of degree not greater than $\min_i \text{dex}(P[f, \boldsymbol{x}_i])$, and interpolates at $\boldsymbol{x}_i$ $P[f, \boldsymbol{x}_i]$ and all its successive derivatives of order not greater than $\mu - 1$, for $\mu \in \mathbb{N}$, or $[\mu]$, otherwise. Gheorge Coman and his collaborators introduced several combinations in univariate as well as in multivariate instances; among them, we mention:

- the Shepard-Lagrange interpolation operator [23];
- the Shepard-Hermite interpolation operator [24];

- the Shepard-Birkhoff interpolation operator [25, 27];
- the Shepard-Lidstone interpolation operator [17];
- the Shepard-Bernoulli interpolation operator [18], which uses the polynomial expansion introduced in [28].

All above operators need the declaration of one or more fictive nodes in order to realize the combination in an appropriate way. In the univariate case, for example, the Shepard-Lagrange operator [23] of degree of exactness $m \geq 1$

$$S_{L_m}[f](x) = \sum_{i=1}^{n} A_{\mu,i}(x) L[f, x_i, \ldots, x_{i+m}](x),$$

where $L[f, x_i, \ldots, x_{i+m}](x)$ is the Lagrange interpolation operator on $x_i, \ldots, x_{i+m}$, requires to set $x_i := x_{n-i}$ for each $i = n+1, \ldots, n+m$. We will discuss about the bivariate case in Section 3.4.1 where we consider the more general problem of Hermite-Birkhoff interpolation.

Shepard-Bernoulli operators, introduced in [15],

$$S_{B_m}[f](x) = \sum_{i=1}^{n} A_{\mu,i}(x) P_m[f, x_i, x_{i+1}](x) \tag{3}$$

represent further combinations where $P_m[f, x_i, x_{i+1}](x)$ is the generalized Taylor polynomial of f in $[x_i, x_{i+1}]$,

$$P_m[f, x_i, x_{i+1}](x) = f(x_i) + \sum_{k=1}^{m} \frac{S_k\left(\dfrac{x - x_i}{h_i}\right)}{k!} h_i^{k-1} \left(f^{(k-1)}(x_{i+1}) - f^{(k-1)}(x_i)\right), \tag{4}$$

which uses differences of derivatives at the end points of the interval and $S_k(x) = B_k(x) - B_k$ being $B_k(x)$ the k-th Bernoulli polynomial and B_k the k-th Bernoulli number and $h_i = x_{i+1} - x_i$. As for the previously listed univariate combined Shepard operators, the rate of convergence of the Shepard-Bernoulli operators is equal to the rate of convergence of the Shepard-Taylor operators. Numerical examples, in [15], demonstrate the accuracy of these combinations in special situations, in particular, when they are applied to the problem of interpolating the discrete solutions of initial value problems for ordinary differential equations. The problem of generalizing such kind of univariate operators to the multivariate case leads to new approaches to Hermite-Birkhoff interpolation of scattered data (See Section 3.4.1).

3.2 QSHEP2D, CSHEP2D and other operators

In 1988 Renka [47] considered the case in which the polynomial $P[f, x_i](x)$ is a quadratic polynomial written as a Taylor series about the point x_i

$$Q_i[f](\boldsymbol{x}) = a_{i1}(x - x_i)^2 + a_{i2}(x - x_i)(y - y_i) + a_{i3}(y - y_i)^2 + a_{i4}(x - x_i) + a_{i5}(y - y_i) + f_i$$

with constant term f_i and coefficients $a_{ij}, j = 1, \ldots, 5$ which minimize the weighted sum of squares error

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} W_{k,i} \left[Q_i[f](\boldsymbol{x}_k) - f_k\right]^2$$

with weights $W_{k,i} = \left(\frac{1}{|\boldsymbol{x}_k - \boldsymbol{x}_i|} - \frac{1}{R_{q_i}}\right)_+^2$, R_{q_i} is the radius of influence about node $\boldsymbol{x}_i$ chosen just large enough to include N_q nodes in the closed ball $B(\boldsymbol{x}_i, R_{q_i})$. The polynomial $P[f, \boldsymbol{x}_i](x)$ is now the quadratic polynomial $Q_i[f](\boldsymbol{x})$ which interpolates f_i and fits the data values on a set of nearby nodes in a weighted least-square sense and the QSHEP2D operator [48] is defined by

$$S_Q[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x}) Q_i[f](\boldsymbol{x}). \tag{5}$$

In order to increase the precision of the QSHEP2D operator, in 1999 Renka [49] introduced the CSHEP2D operator

$$S_C[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x}) C_i[f](\boldsymbol{x}). \tag{6}$$

This operator has cubic precision and is realized similarly to $S_Q[f]$. In this case the polynomial $C_i[f](\boldsymbol{x})$ is the cubic polynomial which interpolates f_i and fits the data values on a set of nearby nodes in a weighted least-square sense. The coefficients of $C_i[f](\boldsymbol{x})$ are chosen to minimize the weighted sum of squares error

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} W_{k,i} \left[C_i[f](\boldsymbol{x}_k) - f_k\right]^2$$

with weights $W_{k,i} = \left(\frac{1}{|\boldsymbol{x}_k - \boldsymbol{x}_i|} - \frac{1}{R_{q_i}}\right)_+^3$.

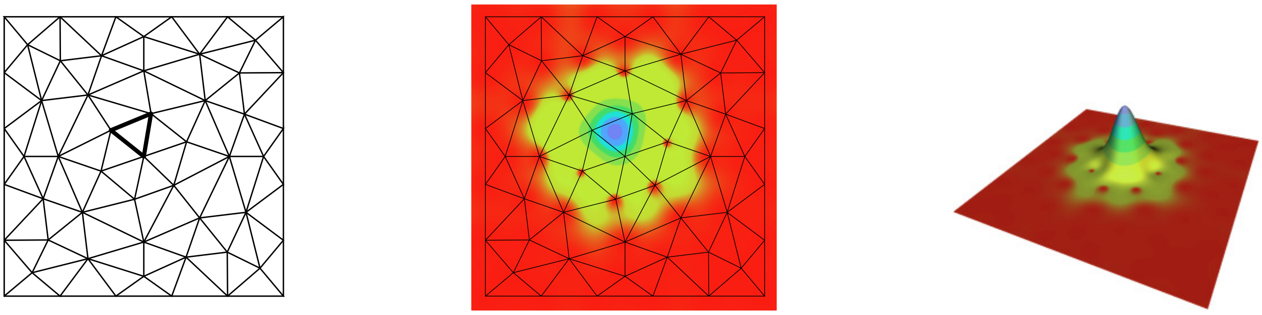

Figure 1: Basis function $B_{2,j}(x)$ for the indicated triangle t_j with respect to a Delaunay triangulation T.

In 2010 Thacker et al. [52] pointed out that the primary disadvantage of the quadratic and cubic variant of the Shepard method is that, for large data sets, a considerable amount of preprocessing is needed to determine the closest points and calculate the local approximation. This consideration motivates the choice of a linear polynomial which requires a smaller numbers of coefficients to be computed to construct the local least square fit. For higher dimensions they therefore propose the use of the LSHEP2D operator

$$S_L[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W_{\mu,i}}(\boldsymbol{x}) L_i[f](\boldsymbol{x})$$

where $L_i[f](\boldsymbol{x})$ is the linear polynomial which interpolates f_i and fits the data values on a set of nearby nodes in a weighted least-square sense.

3.3 Little's approach

The use of local polynomial interpolants based on the vertices of triangles is not new in the literature. In the framework of scattered data interpolation, a method by Little [44], called triangular Shepard method, has been introduced in 1982.

Let $T = \{t_1, t_2, \ldots, t_m\}$ be a triangulation of X, where $t_j = [\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3}]$, $\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \boldsymbol{x}_{j_3} \in X$. The *triangular Shepard basis functions* with respect to the triangulation T are defined as the normalization of the product of inverse distances from the vertices of the triangles

$$B_{\mu,j}(\boldsymbol{x}) = \frac{\prod_{k=1}^{3} \left|\boldsymbol{x} - \boldsymbol{x}_{j_k}\right|^{-\mu}}{\sum_{k=1}^{m} \prod_{l=1}^{3} \left|\boldsymbol{x} - \boldsymbol{x}_{k_l}\right|^{-\mu}}, \qquad j = 1, \ldots, m, \quad \mu > 0. \tag{7}$$

Definition 3.1. For each $\mu > 0$ the *triangular Shepard operator* is defined by

$$K_\mu[f](\boldsymbol{x}) = \sum_{j=1}^{m} B_{\mu,j}(\boldsymbol{x}) L_j(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \tag{8}$$

where $L_j(\boldsymbol{x}) = f_{j_1}\lambda_{j_1}(\boldsymbol{x}) + f_{j_2}\lambda_{j_2}(\boldsymbol{x}) + f_{j_3}\lambda_{j_3}(\boldsymbol{x})$ is the linear interpolation polynomial on the triangle t_j, $j = 1, \ldots, m$.

The definition of the triangular Shepard operator is then based on a triangulation of the nodes and an extension of Shepard's point-based basis functions to triangle-based basis functions. The latter are then used in combination with linear polynomials that locally interpolate the given data at the vertices of each triangle. The method reproduces linear polynomials without using any derivative data and Little noticed that it surpasses Shepard's method greatly in aesthetic behavior. However, Little did not give indications on the choice of the triangulation and on the approximation order of the triangular Shepard method.

For a Delaunay triangulation T (See Figure 1) the triangular Shepard basis functions with power parameter μ look like the classical Shepard basis functions with power parameter 3μ and hence are very similar to the local Shepard basis functions when $\boldsymbol{x}$ is far away from the vertices of the triangle. Nevertheless, we can consider a triangulation T with overlapping or disjoint triangles (See Figure 2). For such triangulations the triangular Shepard basis functions have a different behaviour, especially near the vertices of the triangles. The triangle-based basis functions (7) satisfy the following properties:

- like Shepard's basis functions, they are positive and form a partition of unity

$$B_{\mu,j}(\boldsymbol{x}) \geq 0, \qquad \sum_{j=1}^{m} B_{\mu,j}(\boldsymbol{x}) = 1;$$

- the cardinality property is now related to triangle instead of point

$$B_{\mu,j}(\boldsymbol{x}_i) = 0, \text{ for each } \boldsymbol{x}_i \text{ which is not a vertex of } t_j;$$

- the gradient of $B_{\mu,j}$ vanishes at each node $\boldsymbol{x}_i$ which is not a vertex of t_j

$$\nabla B_{\mu,j}(\boldsymbol{x}_i) = 0, \text{ for each } \boldsymbol{x}_i \text{ which is not a vertex of } t_j;$$

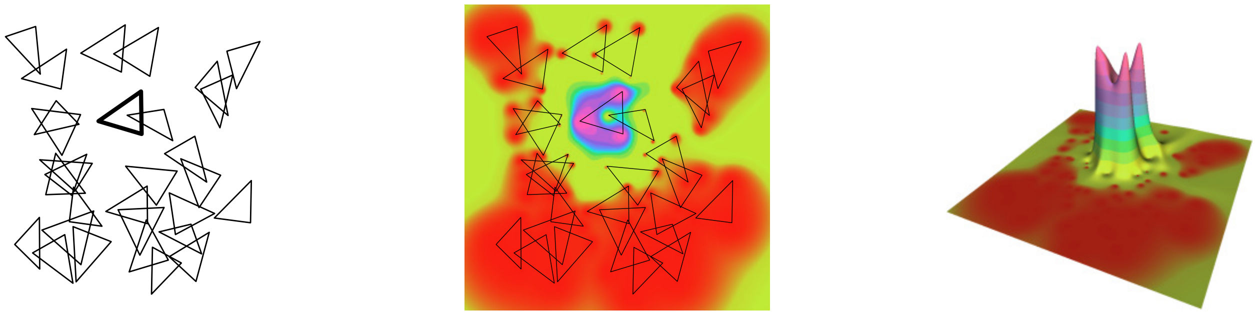

Figure 2: Basis function $B_{2,j}(x)$ for the indicated triangle t_j with respect to a general triangulation T with overlapping triangles.

- the sum of the gradient of $B_{\mu,j}$ over $\boldsymbol{x}_i$ which is not a vertex of t_j vanishes

$$\sum_{\boldsymbol{x}_i \text{ not a vertex of } t_j} \nabla B_{\mu,j}(\boldsymbol{x}_i) = 0.$$

Recently, in [37] , we have deeply studied the approximation order of the triangular Shepard method theoretically and numerically and we have given a procedure for the choice of the triangulation. We have obtained the following theoretical result on the approximation order of the triangular Shepard method

Theorem 3.1. *Let Ω be a compact convex domain which contains X. Let $C^{1,1}(\Omega)$ be the class of differentiable functions $f : \Omega \to \mathbb{R}$ whose partial derivatives are Lipschitz-continuous of order* 1*, equipped with the seminorm*

$$\|f\|_{1,1} = \sup\left\{ \frac{\left| \frac{\partial f}{\partial x^{1-\alpha}\partial y^{\alpha}}(\boldsymbol{u}) - \frac{\partial f}{\partial x^{1-\alpha}\partial y^{\alpha}}(\boldsymbol{v})\right|}{|\boldsymbol{u}-\boldsymbol{v}|} : \boldsymbol{u},\boldsymbol{v} \in \Omega, \boldsymbol{u} \neq \boldsymbol{v}, \alpha \in \{0,1\} \right\}.$$

If $f \in C^{1,1}(\Omega)$ then for each $\mu > 4/3$ we have

$$\|f - K_\mu[f]\| \le CM\|f\|_{1,1}h^2,$$

where C is a positive constant which depends only on T and M is a positive constant which depends on the distribution of points and triangles.

3.4 Complete Hermite-Birkhoff interpolation on scattered data

The Hermite interpolation problem on scattered data consists in determining a continuous function such that its values and derivatives up to the order r_i at each interpolation node $\boldsymbol{x}_i$ (the order may depend on the node) match the values assumed at that node by an unknown continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and its derivatives, respectively. The Birkhoff interpolation problem on scattered data is a generalization of Hermite one, where not all functionals or derivative values to be interpolated are supplied: for this reason this kind of interpolation is also called lacunary interpolation. Let us assume that at each sample point $\boldsymbol{x}_k, k = 1,\dots,n$ the general set

$$\mathcal{I}_k(f) = \left\{ f^{(p,q)}(\boldsymbol{x}_k), (p,q) \in I_k \right\},\ I_k \subset \mathbb{N}^2 \text{ finite set} \tag{9}$$

of Birkhoff data about the function f is given.

The problem of Hermite-Birkhoff interpolation on scattered data has already been considered by Wu [56], Sun [51], Narcowich and Ward [46] and others in the framework of radial basis functions (see [39, Ch. 36] or [53, Ch. 16.2] and the references therein). In particular, in [53, Ch. 16.2] it is shown that it is always possible to reconstruct a function from Hermite-Birkhoff data by an interpolant which uses certain C^{2k} radial basis functions in order to interpolate C^k data. The extra radial basis functions regularity is the price one needs to pay to ensure invertibility of the interpolation matrix [39, Ch. 36, p. 334], which can be also very large in the applications. In any case, whenever large data sets are considered, above methods can be combined with fast evaluation methods like, for instance, partition of unity methods [19, 20, 21].

3.4.1 Hermite-Birkhoff interpolation of scattered data: Coman's approach

In [25], G. Coman suggests to use the data in (9) to determine a global interpolant with a higher algebraic degree of exactness. Coman's approach requires to fix an appropriate order of the sample points $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$ and, starting from each $\boldsymbol{x}_k$, to specify an appropriate subsequence of consecutive points

$$\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \dots, \boldsymbol{x}_{k+\nu_k-1}, \quad \nu_k \in \mathbb{N}, \tag{10}$$

that guarantees the existence and uniqueness of local polynomial interpolants $B_k^{d_k}[f](x,y)$ on these points. Before introducing Theorem 3.2, we assume that, for each $k = 1,\dots,n$,

1. the set
$$\mathcal{X}_{k,\nu_k} = \left\{ (x_{k+j}, y_{k+j}) : j = 0,1,\dots,\nu_k - 1 \right\}, \quad \nu_k \in \mathbb{N}, \quad 0 < \nu_k < n-1,$$
of ν_k consecutive sampled points of the sequence (10), is fixed, with the agreement that $\boldsymbol{x}_{N+i} = \boldsymbol{x}_i$, $i \in \mathbb{N}$;

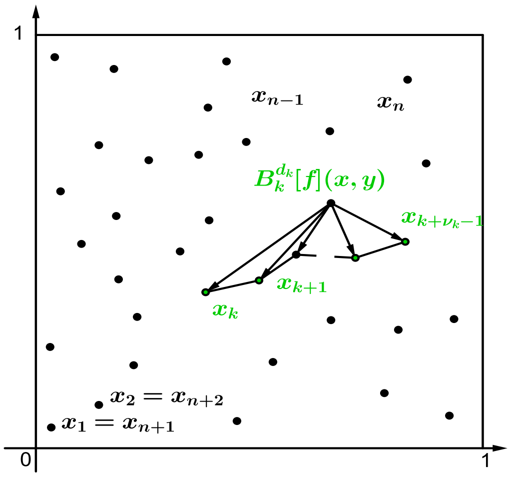

Figure 3: The Shepard operator of Birkhoff type $S_B f$ exists, interpolates on all data and has algebraic degree of exactness $d = \min_{k=1,\ldots,n} d_k$ if, and only if, for each $i = 1, \ldots, n$ the local interpolation problem with data sites $\mathcal{X}_{k,\nu_k}$ and data values $\mathcal{I}_{k,\nu_k}(f)$ has the unique solution $B_k^{d_k}[f](x,y)$ in the polynomial space $\mathcal{P}_{x,y}^{d_k}$

2. the set
$$\mathcal{I}_{k,\nu_k}(f) = \cup_{j=0}^{\nu_k-1} \mathcal{I}_{k+j}(f)$$
is the union set of the Birkhoff data about f at the nodes of $\mathcal{X}_{k,\nu_k}$;
3. $B_k^{d_k}[f](x,y)$ is the bivariate polynomial of total degree d_k that *interpolates all data* from $\mathcal{I}_{k,\nu_k}(f)$, i.e.
$$\left.\frac{\partial^{p+q} B_k^{d_k}[f](x,y)}{\partial x^p \partial y^q}\right|_{(x_j,y_j)} = f^{(p,q)}(x_j,y_j),\ (x_j,y_j) \in \mathcal{X}_{k,\nu_k} \text{ and } (p,q) \in \mathcal{I}_j(f);$$
4. $m_k = \dim\left(\mathcal{P}_{x,y}^{d_k}\right) = \frac{(d_k+1)(d_k+2)}{2}$.

Under these assumptions, we can state the following

Theorem 3.2 ([25]). *If for each $k = 1, \ldots, n$,*

1. *(a) $\sharp\left(\mathcal{I}_{k,\nu_k}(f)\right) = m_k$,*
   (b) the polynomial $B_k^{d_k}[f](x,y)$ exists,

then the Shepard operator of Birkhoff type
$$S_B[f](x,y) = \sum_{k=1}^{n} A_{\mu,k}(x,y) B_k^{d_k}[f](x,y) \tag{11}$$
interpolates on all given data provided that $\mu > \max_{k=1}^{n} \max_{(p,q)\in I_k}\{p+q\}$ and has algebraic degree of exactness $\min_{k=1}^{n} d_k$.

In [25] the author is aware of the difficulties of the general problem he considers, which mainly consists in finding an appropriate order of the sampled points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ and $\nu_k \in \mathbb{N}, k = 1, \ldots, n$ such that conditions *1a* and *1b* hold. Nevertheless, he provided two interesting special cases in which he weakens the conditions on the set $\mathcal{I}_{k,\nu_k}(f)$ by allowing it to be a proper subset of $\cup_{j=0}^{\nu_k-1}\mathcal{I}_{k+j}(f)$ and in this way he solves two well known Hermite-Birkhoff interpolation problems on scattered data. In both examples he assumes that $\sharp\left(\mathcal{I}_{k,\nu_k}(f)\right) = m = \frac{(r+1)(r+2)}{2}$ for each $k = 1, \ldots, n$, i.e. that all Birkhoff polynomials have the same degree r.

Example 3.1. As first example, set $m_k = 6$ (and therefore $r_k = 2$), $\nu_k = 3$,
$$I_k = \{(0,0),(2,0),(1,1),(0,2)\}$$
for each $k = 1, \ldots, n$, and consider the bivariate Lidstone type data [16],
$$\mathcal{I}_{k,3}(f) = \left\{f(x_k,y_k), f^{(2,0)}(x_k,y_k), f^{(1,1)}(x_k,y_k), f^{(0,2)}(x_k,y_k), f(x_{k+1},y_{k+1}), f(x_{k+2},y_{k+2})\right\}.$$
Through the analysis of Vandermonde determinant G. Coman showed that if the points of $\mathcal{X}_{k,3} = \{\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \boldsymbol{x}_{k+2}\}$ do not lie on a line l_k, for all $k = 1, \ldots, n$, then the Birkhoff polynomial $B_k^2[f](x,y)$ exists for each $k = 1, \ldots, n$ and can be recovered from the data of $\mathcal{I}_{k,3}(f)$ by the method of unknown coefficients.

Example 3.2. As second example, set $m_k = 6$ (and therefore $r_k = 2$), $\nu_k = 4$,

$$I_k = \{(0,0),(1,0),(0,1)\}$$

for each $k = 1,\dots,n$, and consider bivariate Hermite type data [33],

$$\mathcal{I}_{k,4}(f) = \left\{f(x_k,y_k), f^{(1,0)}(x_k,y_k), f^{(0,1)}(x_k,y_k), f(x_{k+1},y_{k+1}), f(x_{k+2},y_{k+2}), f(x_{k+3},y_{k+3})\right\}.$$

In this case, it is possible to state that if the points of $\mathcal{X}_{k,4} = \{\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \boldsymbol{x}_{k+2}, \boldsymbol{x}_{k+3}\}$ are such that the line l_{k+i} determined by $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+i}$ is different from the line l_{k+j} determined by $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+j}$ for each $i,j = 1,2,3$, $i \neq j$ for all $k = 1,\dots,n$, then the Hermite polynomial $H_k^2[f](x,y)$ exists and can be recovered from the data of $\mathcal{I}_{k,3}(f)$ by the method of unknown coefficients.

3.4.2 Hermite-Birkhoff interpolation of scattered data: Allasia and Bracco approach

If the data in $\mathcal{I}_k(f)$ are of Hermite-Birkhoff type, then the Taylor-Birkhoff polynomial [10]

$$TB_{r_k}[f,\boldsymbol{x}_k](\boldsymbol{x}) = \sum_{j\in I_k} \frac{f^{(j)}(\boldsymbol{x}_k)}{j!}(\boldsymbol{x}-\boldsymbol{x}_k)^j$$

interpolates all data in $\mathcal{I}_k(f)$ but reproduces polynomials of total degree $p \le r_k$ if and only if, $\mathcal{I}_k(f)$ contains all partial derivative values of order less than or equal p. Consequently, the Shepard-Taylor-Birkhoff operators

$$S_{TB}[f](\boldsymbol{x}) = \sum_{i=1}^{r_i} \widetilde{W}_{\mu,i}(\boldsymbol{x}) TB_{r_i}[f,\boldsymbol{x}_i](\boldsymbol{x}) \tag{12}$$

interpolate all data but may have a lower algebraic degree of precision and, in extreme cases, may not even have dex $= 0$, if at some nodes $\boldsymbol{x}_k$ the functional value $f(\boldsymbol{x}_k)$ do not belong to $\mathcal{I}_k(f)$. This fact badly affects the approximation performances of the operator.

3.4.3 Hermite-Birkhoff interpolation of scattered data: our approach

Under certain conditions of completeness of the Hermite-Birkhoff data, we have introduced [36] a new interpolation scheme which removes the weaknesses and holds the strengths of the aforesaid general methods. The proposed method:

- reproduces polynomials just as well as the Coman method;
- is as simple to implement as the Allasia and Bracco method.

We assume that for each $k = 1,2,\dots,n$ the set

$$\mathcal{I}_k(f) = \left\{f^{(p,q)}(\boldsymbol{x}_k), (p,q) \in I_k\right\}, \ I_k \subset \mathbb{N}^2 \text{ finite set}$$

contains partial derivative values at $\boldsymbol{x}_k$ of total order up to r_k, $0 \le r_k \le r$ of a differentiable function $f \in C^r(\Omega)$, $r \ge 0$ and satisfies the following condition

(C) If some partial derivative value of order s belongs to $\mathcal{I}_k(f)$, then all partial derivative values of order s belong to $\mathcal{I}_k(f)$.

Definition 3.2. Under the above assumptions, the dataset $\mathcal{I}(f) = \cup_{k=1}^n \mathcal{I}_k(f)$ is called a set of *complete Hermite-Birkhoff data*.

Our approach to complete Hermite-Birkhoff interpolation is based on the following three steps:

1. we associate the sample point $\boldsymbol{x}_i \in X$ with a triangle $\Delta(i)$ with a vertex in $\boldsymbol{x}_i$ and other two vertices in certain interpolation nodes $\boldsymbol{x}_j, \boldsymbol{x}_k$ in the closed ball $B(\boldsymbol{x}_i, R_{\Delta(i)}) \subset \Omega$;
2. we identify a polynomial space $\mathcal{P}_{\boldsymbol{x}}^{d_i}$, $d_i \in \mathbb{N}$, and a polynomial $P^{\Delta(i)}[f](\boldsymbol{x}) \in \mathcal{P}_{\boldsymbol{x}}^{d_i}$, based on the vertices of the triangle $\Delta(i)$, which is the unique solution of a Hermite-Birkhoff interpolation problem with interpolation dataset obtainable from $\mathcal{I}_i(f) \cup \mathcal{I}_j(f) \cup \mathcal{I}_k(f)$;
3. we choose the pair $\left(\Delta(i), P^{\Delta(i)}[f](\boldsymbol{x})\right)$ such that the error of approximation of $P^{\Delta(i)}[f](\boldsymbol{x})$ is the smallest in $B(\boldsymbol{x}_i, R_{\Delta(i)})$. (See Figure 4)

We call $\boldsymbol{x}_i$ the *referring vertex* of $\Delta(i)$ and we denote it also by $\boldsymbol{x}_0$; starting from $\boldsymbol{x}_0$ and moving counterclockwise we denote the vertices $\boldsymbol{x}_j, \boldsymbol{x}_k$ also by $\boldsymbol{x}_1, \boldsymbol{x}_2$; without loss of generality we assume $\boldsymbol{x}_1 = \boldsymbol{x}_j$. Under the assumption (C) we can rephrase the interpolation data in $\mathcal{I}_i(f) \cup \mathcal{I}_j(f) \cup \mathcal{I}_k(f)$ in terms of derivatives along the directed sides of the triangle $\Delta(i)$. A convenient representation of $P^{\Delta(i)}[f]$ uses such kind of derivatives, as coefficients, and a polynomial basis given in barycentric coordinates relative to the triangle $\Delta(i)$, that is the polynomials in $\mathcal{P}_{\boldsymbol{x}}^{d_i}$ are expressed as polynomials in $\mathcal{P}_{\boldsymbol{\lambda}}^{d_i}$, the space of homogeneous polynomials of degree d_i in the variables $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2)$:

$$P^{\Delta(i)}[f] = \sum_\rho Q_{0,\rho} D_0^{\boldsymbol{\alpha}_\rho} f(\boldsymbol{x}_0) + \sum_\sigma Q_{1,\sigma} D_1^{\boldsymbol{\beta}_\sigma} f(\boldsymbol{x}_1) + \sum_\tau Q_{2,\tau} D_2^{\boldsymbol{\gamma}_\tau} f(\boldsymbol{x}_2), \tag{13}$$

where $Q_{0,\rho} = Q_{0,\rho}(\lambda_0,\lambda_1,\lambda_2)$, $Q_{1,\sigma} = Q_{1,\sigma}(\lambda_0,\lambda_1,\lambda_2)$, $Q_{2,\tau} = Q_{2,\tau}(\lambda_0,\lambda_1,\lambda_2)$ are homogeneous polynomials in barycentric coordinates of degree d_i and

$$\left\{\boldsymbol{\alpha}_\rho\right\}_\rho = I_i, \quad \left\{\boldsymbol{\beta}_\sigma\right\}_\sigma \subset I_j, \quad \left\{\boldsymbol{\gamma}_\tau\right\}_\tau \subset I_k, \tag{14}$$

$$\sharp(\{\rho\}) + \sharp(\{\sigma\}) + \sharp(\{\tau\}) = \binom{d_i+2}{d_i} = \dim\left(\mathcal{P}_{\boldsymbol{x}}^{d_i}\right). \tag{15}$$

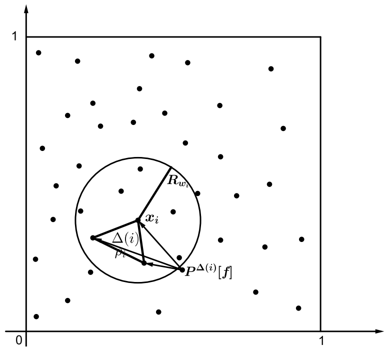

Figure 4: The point-triangle association is realized in order to reduce the error of approximation of a suitable polynomial $P^{\Delta(i)}[f](\boldsymbol{x})$, which is based on the vertices of the triangle $\Delta(i)$ and is *the unique solution* in a polynomial space $\mathcal{P}_{\boldsymbol{x}}^{d_i}$ of a three point Hermite-Birkhoff interpolation problem with data obtainable from $\mathcal{I}_i(f) \cup \mathcal{I}_j(f) \cup \mathcal{I}_k(f)$.

The use of derivatives along the directed sides of the triangle $\Delta(i)$ and of barycentric coordinates, in the case of polynomial Hermite-Birkhoff interpolation on triangles, is suggested by Lorentz [45] as well as by Chui and Lai [22], since both of them are affine invariants. As a consequence, the non vanishing of the Vandermonde determinant of each particular interpolation problem does not depend on the position of the vertices of the triangle and can be checked once and for all. The unique solution of the related problem, if it exists, can be easily computed in barycentric coordinates by symbolic computation software like Mathematica and stored for later use.

If the polynomial solution $P^{\Delta(i)}[f]$ of a Hermite-Birkhoff interpolation problem on the triangle $\Delta(i)$ has degree p and f belongs to the class $C^{p,1}(\Omega)$ of functions p-times continuously differentiable in Ω with all partial derivatives of order p Lipschitz-continuous in Ω, then, by a repeated application of the truncated Taylor expansion with integral remainder to each term $D_j^{\beta} f,\ j = 1, 2, |\beta| \leq p$ in the expression of $P^{\Delta(i)}[f]$ we can prove that

$$P^{\Delta(i)}[f](\boldsymbol{x}) = T_p[f, \boldsymbol{x}_i](\boldsymbol{x}) + \delta^{\Delta(i)}[f](\boldsymbol{x}),$$

where $\delta^{\Delta(i)}[f]$ is a polynomial expressed by difference quotients of derivatives of order p. We get the following estimate

$$\left|\delta^{\Delta(i)}(\boldsymbol{x})\right| \leq \sum_{k=0}^{p} c_k \rho^{p+1-k} \left(\rho^2 S\right)^k \|\boldsymbol{x} - \boldsymbol{x}_0\|_2^k \, |f|_{p,1}.$$

where

$$|f|_{p,1} = \sup_{i=0,\dots,p} \left\{ \frac{\left| \frac{\partial^p f}{\partial x^{p-i} \partial y^i}(\boldsymbol{x}_1) - \frac{\partial^p f}{\partial x^{p-i} \partial y^i}(\boldsymbol{x}_2) \right|}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}, \boldsymbol{x}_1 \neq \boldsymbol{x}_2 \text{ in } \Omega \right\},$$

$$\rho = max\left\{\|\boldsymbol{x}_0 - \boldsymbol{x}_1\|_2, \|\boldsymbol{x}_0 - \boldsymbol{x}_2\|_2, \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2\right\}$$

and

$$S^{-1} = 2 \times Area\,(\Delta(i)).$$

Remark 1. The term $\rho^2 S$ depends only on the shape of the triangle and from the above inequality we get

$$\lim_{\substack{\rho \to 0 \\ \rho^2 S = const}} \|P^{\Delta(i)}[f] - T_p[f, \boldsymbol{x}_i]\|_{C(\Omega)} = 0.$$

Finally, we define the Shepard Hermite-Birkhoff operator

$$S_{HB}[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x}) P^{\Delta(i)}[f](\boldsymbol{x}). \tag{16}$$

In conclusion, the main idea is to use the local interpolants on suitable triangles and use barycentric coordinates instead of cartesian coordinates. This allows us to determine the regularity of a Hermite-Birkhoff interpolation problem by mapping it to one of a finite number of reference triangles with associated interpolation conditions for which the question of regularity has already been settled. This approach has the following advantages

- it only requires the solution of a relatively small number of small linear systems to solve local Hermite-Birkhoff interpolation problems in polynomial spaces;
- the resulting Hermite-Birkhoff interpolant of f is *local*, that is its value at a point $\boldsymbol{x} \in \Omega$ depends only on the values of f at a small number of neighboring nodes;
- it allows to reconstruct a continuous function from Hermite-Birkhoff data by an interpolant which uses C^{r+1} functions in order to interpolate C^r data.

3.4.4 Special cases: Hermite type data

Let us assume that all functional evaluations and supplementary derivative data up to a fixed order r are given at each node. In this case, it is possible to enhance the degree of exactness of the Shepard-Taylor operator to $p=r+q, q>0$, maintaining the interpolation properties of the Shepard-Taylor operator S_{T_r} and reaching the accuracy of approximation of S_{T_p}. In fact, Chui and Lai [22] formulated certain Hermite-type interpolation conditions on the vertices of the triangle $\Delta(i)$ which ensure uniqueness of interpolation in $\mathcal{P}^p_{\boldsymbol{x}}$ and gave an explicit expression for the polynomial solution $P^{\Delta(i)}[f]:=H^{\Delta(i)}_{p,r}$, which requires some additional notations in order to be specified.

Let us denote by $\mathbb{Z}^2_+$ the set of all pairs with non-negative integer components in the euclidean space $\mathbb{R}^2$.

Definition 3.3. A subset M^2 of $\mathbb{Z}^2_+$ is called a lower set if for each $\beta,\gamma\in\mathbb{Z}^2_+$, $\beta\in M^2$ and $0\le\gamma\le\beta$ it results $\gamma\in M^2$.

Let $\Gamma^2_r:=\left\{\beta\in\mathbb{Z}^2_+:|\beta|\le r\right\}$, $\Lambda^3_r:=\left\{\alpha\in\mathbb{Z}^3_+:|\alpha|=r\right\}$ and A^r_i the raising map from Γ^2_r to Λ^3_r defined by

$$A^r_0\beta=(r-|\beta|,\beta_1,\beta_2),\quad A^r_1\beta=(\beta_1,r-|\beta|,\beta_2),\quad A^r_2\beta=(\beta_1,\beta_2,r-|\beta|),\quad \beta\in\mathbb{Z}^2_+.$$

Definition 3.4. A collection of subsets M^2_0, M^2_1, M^2_2 of Γ^2_r is said to form a partition of Λ^3_r if

1. $A^r_jM^2_j\cap A^r_kM^2_k=\emptyset$ for $j\neq k$, and
2. $\displaystyle\bigcup_{j=0}^{2}A^r_jM^2_j=\Lambda^3_r$.

The following Theorem was proven in [22, Theorem 3.1.4] in the general case of a simplex in $\mathbb{R}^s$.

Theorem 3.3. *Let $M^2_0=\left\{\beta\in\mathbb{Z}^2_+:|\beta|\le r\right\}$ and M^2_1, M^2_2 lower sets forming a partition of Λ^3_r. Then for any given set of data $\left\{f_{j,\beta}\in\mathbb{R}:\beta\in M^2_j, j=0,1,2\right\}$ there exists a unique polynomial $H^{\Delta(i)}_{p,r}$ of total degree $p=r+q, q>0$ satisfying*

$$D^\beta_j H^{\Delta(i)}_{p,r}(\boldsymbol{x}_j)=f_{j,\beta},\quad \beta\in M^2_j, j=0,1,2.$$

Moreover, $H^{\Delta(i)}_{p,r}(\boldsymbol{x})$ may be formulated in Bézier representation of total degree p with respect to the simplex $\Delta(i)$ as follows

$$H^{\Delta(i)}_{p,r}(\boldsymbol{x})=\sum_{j=0}^{2}\sum_{\beta\in M^2_j}\left\{\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\frac{(p-|\gamma|)!}{p!}f_{j,\gamma}\right\}\phi^p_{A^p_j\beta}(\lambda_0(\boldsymbol{x}),\lambda_1(\boldsymbol{x}),\lambda_2(\boldsymbol{x}))\tag{17}$$

where

$$\phi^p_{A^p_j\beta}(\lambda_0(\boldsymbol{x}),\lambda_1(\boldsymbol{x}),\lambda_2(\boldsymbol{x}))=\frac{p!}{\left(A^p_j\beta\right)!}\lambda_0(\boldsymbol{x})^{\left(A^p_j\beta\right)_0}\lambda_1(\boldsymbol{x})^{\left(A^p_j\beta\right)_1}\lambda_2(\boldsymbol{x})^{\left(A^p_j\beta\right)_2}.$$

For each fixed $\mu>0$ and $p=1,2,\dots$ the bivariate Shepard-Hermite [33] operator is defined by

$$S_{H_{p,r}}[f](\boldsymbol{x})=\sum_{i=1}^{n}\widetilde{W}_{\mu,i}(\boldsymbol{x})H^{\Delta(i)}_{p,r}[f](\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega$$

where $H^{\Delta(i)}_{p,r}[f](\boldsymbol{x})$ is the Hermite interpolating polynomial (17) on the triangle $\Delta(i)$, $i=1,\dots,n$.

3.4.5 Special cases: Lidstone data

We assume that, together with function evaluations, all even order partial derivatives up to a fixed order $2p-2, p\in\mathbb{N}$, are given at each sample point. We call such kind of data *Lidstone type data*, in honor of G. J. Lidstone, who, in 1929 [43], provided an explicit expression of a polynomial which approximates a given function in the neighborhood of two points instead of one (say them a and b), generalizing in such a way the Taylor polynomial. This polynomial, known as Lidstone interpolating polynomial [5], uses function evaluations and all even order derivatives up to the order $2p-2$ at a and b and is expressed in terms of Lidstone polynomials

$$\begin{cases}\Lambda_0(x)=x,\\ \Lambda''_k(x)=\Lambda_{k-1}(x), & k\ge1,\\ \Lambda_k(0)=\Lambda_k(1)=0, & k\ge1.\end{cases}$$

The interest for this kind of expansion lies in the fact that it finds application to several problems of numerical analysis such as approximation of solutions of some boundary value problems, polynomial approximation, construction of splines with application to finite elements, etc. [1, 2, 3, 4, 5]. In a remark made in [5, p. 37] reference was made to the lack of literature on the extension of some results on the approximation of univariate functions by means of Lidstone polynomials to functions of two independent variables over non-rectangular domains. Costabile and Dell'Accio [30] answered this question by providing a new polynomial approximation formula, which uses function evaluations and even order derivatives at the vertices of the simplex and is the univariate Lidstone expansion when restricted to each side. The combination of the Lidstone approximation formula on the triangle

$$\begin{aligned}L^{\Delta(i)}_p[f](\boldsymbol{x})=\sum_{k=0}^{p-1}\Bigg(\sum_{j=0}^{p-1-k}&\left(D^{(2j,2k)}_2f(\boldsymbol{x}_0)\Lambda_j(1-\lambda_1-\lambda_2)+D^{(2j,2k)}_2f(\boldsymbol{x}_2)\Lambda_j(\lambda_1+\lambda_2)\Lambda_k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)\right)+\\ +\sum_{j=0}^{p-1-k}&\left(D^{(2j,2k)}_1f(\boldsymbol{x}_0)\Lambda_j(1-\lambda_1-\lambda_2)+D^{(2j,2k)}_1f(\boldsymbol{x}_1)\Lambda_j(\lambda_1+\lambda_2)\Lambda_k\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\right)\Bigg)(\lambda_1+\lambda_2)^{2k}\end{aligned}\tag{18}$$

	$f(\boldsymbol{x}_i)$	$\frac{\partial f}{\partial x}(\boldsymbol{x}_i)$	$\frac{\partial f}{\partial y}(\boldsymbol{x}_i)$	$\frac{\partial^2 f}{\partial x^2}(\boldsymbol{x}_i)$	$\frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{x}_i)$	$\frac{\partial^2 f}{\partial y^2}(\boldsymbol{x}_i)$
$\triangleright$	•					
$\Diamond$	•	•	•			
$\circ$	•			•	•	•
$\times$		•	•	•	•	•
$\square$		•	•			
$\triangle$				•	•	•
$*$	•	•	•	•	•	•

Table 1: The association symbol-data: the dot "•" indicates the presence of the data.

Interpolation data	$\triangleright$	$\Diamond$	$\circ$	$\times$	$\square$	$\triangle$	$*$
Number of nodes	151	152	154	154	167	154	157

Table 2: Variation, in number, of randomly chosen complete data for the 1089 nodes.

with the local Shepard operator provides an interpolation operator, namely the Shepard-Lidstone operator [16],

$$S_{L_p}[f](\boldsymbol{x}) = \sum_{i=1}^{n} \widetilde{W}_{\mu,i}(\boldsymbol{x}) L_p^{\Delta(i)}[f](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$

which satisfies bivariate Lidstone interpolation conditions and reproduces polynomials up to the degree $2p-1$.

3.4.6 Special cases: Complementary Lidstone data

Despite classical Lidstone interpolation has a long history [11, 12, 13, 41, 43, 54, 55] Complementary Lidstone Interpolation has been only recently introduced by Costabile, Dell'Accio and Luceri in [35, 31] and drawn on by Agarwal, Pinelas and Wong in two successive papers [6, 7]. Like Lidstone expansion, Complementary Lidstone expansion finds application to approximation of solutions of some boundary value problems [8, 9]. Complementary Lidstone Interpolation naturally complements Lidstone Interpolation: both interpolation polynomials are based on two points (say them a and b) and interpolate all data required for their definition, but while the Lidstone Interpolation polynomial requires the use of odd order Bernoulli polynomials [54], function evaluations and even order derivative data in both points, the Complementary Lidstone Interpolation polynomial requires even order Bernoulli polynomials [29], function evaluation at a (or b) and odd order derivative data in both boundary points. To generalize this kind of interpolation in the context of bivariate scattered data, we introduced in [34] three point interpolation polynomials $CL_{1,p}^{\Delta(i)}$, $CL_{2,p}^{\Delta(i)}$, $CL_{3,p}^{\Delta(i)}$ on the triangle by opportunely modifying that one proposed in [32]. These polynomials interpolate all odd order derivatives, up to order $2p-1$, at the referring vertex of the triangle, some odd order derivatives, up to order $2p-1$, at the remaining vertices and function evaluation only in a vertex. Similarly to the Lisdtone polynomials, the combination of these polynomials with the local Shepard operator provides an interpolation operator which satisfies bivariate Complementary-Lidstone interpolation conditions and reproduces polynomials of degree less or equal $2p$ (for more details see [34]).

3.5 Numerical results

To test the accuracy of approximation of the Shepard Hermite-Birkhoff operators (16) in the bivariate interpolation of large sets of scattered data, we carried out a series of experiments by setting $N_w = 13$ nodes in the ball $B(\boldsymbol{x}_i, R_{w_i})$, in order to define the basis functions $\widetilde{W}_{\mu,i}(\boldsymbol{x})$, and $N_t = N_w$ nodes in $B(\boldsymbol{x}_i, R_{t_i})$, in order to associate to each node the triangle $\Delta(i)$. As for the numerical experiments we consider the set of Renka's test functions (see [49]) generally used in the bivariate interpolation of large sets of scattered data. The numerical results are obtained by using a set of 1089 regularly distributed interpolation nodes in the unit square $R = [0,1] \times [0,1]$. The interpolation conditions, displayed in Figure 5, are randomly chosen and the association symbol-data is as reported in Table 1, in particular, the interpolation data vary in number, as reported in Table 2. In Table 3 we report the maximum error $e_{\max}$, the average error e_{mean}, and the mean square error e_{MS} for the Shepard Hermite-Birkhoff operator (16) and for the Shepard-Taylor-Birkhoff (12). The pointwise errors e_i were determined in absolute value at the $n_e = 100 \times 100$ points of a regular grid of R.

4 Future challenges

The obtained numerical results encourages us to develop and analyze, in future work, improvements of the triangular Shepard operator K_2 (8), in order to increase the accuracy of approximation of the original method or to improve its efficiency. As for the Shepard method, the approximation order of the operator K_2 can be improved by combining the triangular Shepard basis functions (7) with interpolation polynomials on the triangle of degree at least 2. It would be desirable that these interpolation polynomials would have the characteristic of depending symmetrically from the three vertices of each triangle, likewise the linear case. If we assume that at each sample point, in addition to the functional evaluations, the values of the first order derivatives are given, then the quadratic Bernoulli polynomial on the triangle

$$\begin{aligned} P_j(x) = {} & f_{j_1}\lambda_{j,j_1}(x) + f_{j_2}\lambda_{j,j_2}(x) + f_{j_3}\lambda_{j,j_3}(x) + \tfrac{1}{2}\lambda_{j,j_1}\lambda_{j,j_2}\left(D_2^{(1,0)} f_{j_2} - D_2^{(1,0)} f_{j_1}\right) \\ & + \tfrac{1}{2}\lambda_{j,j_1}\lambda_{j,j_3}\left(D_1^{(0,1)} f_{j_1} - D_1^{(0,1)} f_{j_3}\right) + \tfrac{1}{2}\lambda_{j,j_2}\lambda_{j,j_3}\left(D_3^{(0,1)} f_{j_3} - D_3^{(0,1)} f_{j_2}\right) \end{aligned}$$

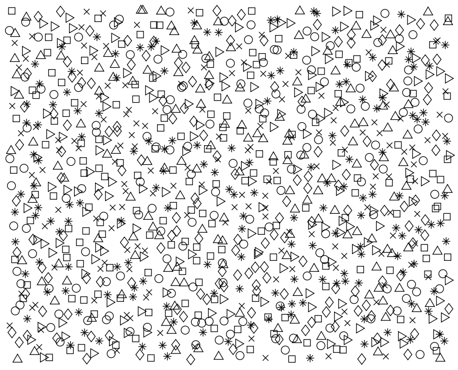

Figure 5: The set of 1089 regularly distributed interpolation nodes in the unit square $[0,1]\times[0,1]$ with specified interpolation data. The association symbol-data is as reported in Table 1.

	$e_{\max}$	e_{mean}	e_{MS}
$S_{TB}[f_1]$	1.03	1.09e-1	2.02e-1
$S_{HB}[f_1]$	2.39e-2	1.37e-4	6.29e-4
$S_{TB}[f_2]$	2.22e-1	5.17e-2	8.71e-2
$S_{HB}[f_2]$	4.25e-3	7.70e-5	2.65e-4
$S_{TB}[f_3]$	3.63e-1	4.92e-2	7.91e-2
$S_{HB}[f_3]$	8.57e-4	7.73e-6	2.56e-5
$S_{TB}[f_4]$	3.29e-1	7.55e-2	1.08e-1
$S_{HB}[f_4]$	9.82e-4	4.30e-6	2.18e-5
$S_{TB}[f_5]$	3.21e-1	2.59e-2	5.82e-2
$S_{HB}[f_5]$	1.61e-2	3.97e-5	3.43e-4
$S_{TB}[f_6]$	3.86e-1	1.29e-1	1.71e-1
$S_{HB}[f_6]$	1.75e-4	1.76e-6	6.84e-6
$S_{TB}[f_7]$	2.81	3.87e-1	6.04e-1
$S_{HB}[f_7]$	2.52e-1	1.89e-3	6.93e-3
$S_{TB}[f_8]$	2.38	2.32e-1	4.30e-1
$S_{HB}[f_8]$	4.59e-1	9.41e-4	9.69e-3
$S_{TB}[f_9]$	1.76e+2	1.42e+1	3.00e+1
$S_{HB}[f_9]$	2.02e+1	5.07e-2	4.33e-1
$S_{TB}[f_{10}]$	8.38e-1	8.70e-2	1.36e-1
$S_{HB}[f_{10}]$	4.46e-1	8.96e-4	9.38e-3

Table 3: Comparison of the two interpolation operators, S_{HB} and S_{TB}, by using the 1089 nodes with complete interpolation data as in Figure 5.

is suitable for this purpose. In fact, it is a *quadratic* polynomial, with a *symmetric* expansion, which interpolates the functional evaluations at each vertex of t_j and the differences of the derivatives along the directed side of the triangle. If we assume that at each sample point only the functional evaluations are given, then the unknown coefficients (in red) of the quadratic Bernoulli polynomial on the triangle

$$\begin{aligned} P_j(x) = f_{j_1}\lambda_{j,j_1}(x) + f_{j_2}\lambda_{j,j_2}(x) + f_{j_3}\lambda_{j,j_3}(x) + \tfrac{1}{2}\lambda_{j,j_1}\lambda_{j,j_2}\left(D_2^{(1,0)}f_{j_2} - D_2^{(1,0)}f_{j_1}\right) \\ + \tfrac{1}{2}\lambda_{j,j_1}\lambda_{j,j_3}\left(D_1^{(0,1)}f_{j_1} - D_1^{(0,1)}f_{j_3}\right) + \tfrac{1}{2}\lambda_{j,j_2}\lambda_{j,j_3}\left(D_3^{(0,1)}f_{j_3} - D_3^{(0,1)}f_{j_2}\right) \end{aligned}$$

can be computed in a least square sense, in analogy with the QSHEP2D operator. In our case, however, the number of coefficients to be determined is 3 instead of 5.

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