PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special Issue *Multivariate Approximation: Theory and Application, MATA2020*, Volume 14 · 2021 · Pages 18–25

A Reliable Algorithm for solution of Higher Dimensional Nonlinear $(1+1)$ and $(2+1)$ Dimensional Volterra-Fredholm Integral Equations

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Abstract

An approach to approximate solution of higher dimensional Volterra-Fredholm integral equations (VFIE) is presented in this paper. A well-established semi analytical method is extended to solution of VFIE for the first time, called Optimal Homotopy Asymptotic Method (OHAM). The efficiency and effectiveness of the proposed technique is tested upon $(1+1)$ and $(2+1)$ dimensional VFIE. Results obtained through OHAM are compared with multi quadric radial basis function method,radial basis function method, modified block-plus function method, Bernoulli collocation method, efficient pseudo spectral scheme, three dimensional block-plus function methods and $3D$ triangular function. The comparison clearly shows the effectiveness and reliability of the presented technique over these methods. Moreover, the use of OHAM is simple and straight forward.

Keywords: VF $(1+1)$ dimensional VFIE, $(2+1)$ dimensional IE, OHAM, Approximate solutions

1 Introduction

Differential and integral equations are observed in modeling various physical phenomena. The use of differential and integral equations in different fields of science and engineering attract the focus of the researchers to find its solutions, but exact solution of all problems is difficult to find, because of high nonlinearity. Therefore, researchers implement different numerical and approximate techniques for its approximate solutions. Almasieh. et al. applied multiquadric radial basis functions for solving 2 D VFIE [1] . Hafezet al. applied Bernoulli collocation method [2].Abdelkawy et al. obtained the approximate solution of 3D integral equations [3]. Mirzaee et al. introduced 3D triangular functions [4] andblock-pulse functions method for solution of 3D nonlinear mixed VFIE [5]. See further [6, 9]. Marinca et. al. [10, 13] introduced OHAM for the solution of differential equations. Different researchers successfully applied the proposed technique to different problems in science and engineering [14, 18]. In the present work, the proposed technique is extended to higher dimensional VFIE of the form:

$$u(z,t,i) = h(z,t,i) + \int_a^z \int_b^t \int_c^i k_1(z,t,i,r,s,v) M[u(r,s,v)] drdsdv + \int_m^e \int_n^w \int_o^\tau k_2(z,t,i,r,s,v) M[u(r,s,v)] drdsdv \quad (1)$$

$$u(z,t,i) = h(z,t,i) + \int_m^e \int_n^w \int_o^\tau (z,t,i,r,s,v) M[u(r,s,v)] drdsdv \quad (2)$$

Where h(z,t)and k(z,t,r,s) are the known analytical functions,$a, b, c, m, n, o \& w$ are constants and $z, t, i \& \tau$ are variables,M represent linear and nonlinear operator and u(z,t,i) is the solution to be determine. This paper is organized as follow. Section 1st is the introduction and some review of literature, basic idea of OHAM is in section 2. Section 3 represents some numerical examples. Section 4 is the results and discussions while section 5 is conclusions.

2 Basic Idea of OHAM

Taking a general integral equation of the form:

$$u(z,t,i) = h(z,t,i) + \int_a^z \int_b^t \int_c^i k(z,t,i,r,s,v) M[u(r,s,v)] drdsdv \quad (3)$$

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Construct an optimal homotopy $\upsilon(z,t;\rho):\xi\times[0,1]\longrightarrow R$ as

$$(1-\rho)\{L(u(z,t,i;\rho))-h(z,t,i)\}=\mathcal{H}(\rho)\{L(u(z,t,i;\rho))-h(z,t,i)-\delta(u)\}, \tag{4}$$

where L(u(z,t,i;ρ))denotes linear operator,δ(u) represent the integral operator and u(z,t,i)is the solution of the given equation to be determined.where $\rho\in[0,1],\mathcal{H}(\rho)=\sum_{m\geq1}c_m\rho^m$ for all $\rho\neq0$ is an auxiliary function, if $\rho=0$ then $\mathcal{H}(0)=0$ and c_m are the auxiliary constants.Expend u(z,t;ρ) about ρ by using Taylorâs series expansion, one can get:

$$\upsilon(z,t,i;\rho)=u_0(z,t,i)+\sum_{m\geq1}u_m(z,t,i)\rho^m \tag{5}$$

at ρ=1, the series in Equation (5) , becomes

$$\tilde{u}(z,t,i)=u_0(z,t,i)+\sum_{m\geq1}u_m(z,t,i) \tag{6}$$

Using Equation (5), and compare the coefficients of like power of ρ in Equation (4), one can get the following series of the problems:

$$\rho^0:u_0(z,t,i)-h(z,t,i)=0 \tag{7}$$

$$\rho^1:u_1(z,t,i)=c_1\delta(u_0)+(1+c_1)u_0(z,t,i)=0 \tag{8}$$

$$\rho^2:u_2(z,t,i)+c_1\delta(u_1)+c_2\delta(u_0)+c_2(h(z,t,i)-u_0(z,t,i))-(1+c_1)u_1(z,t,i)=0 \tag{9}$$

$$\rho^3:u_3(z,t,i)+c_3\delta(u_0)+c_2\delta(u_1)+c_1\delta(u_2)+c_3(h(z,t,i)-u_0(z,t,i))-(1+c_1)u_2(z,t,i)-c_1u_1(z,t,i)=0 \tag{10}$$

$$\rho^n:u_n(z,t,i)+\sum_{j=1}^{n-1}c_j\delta(u_{n-j})+c_n(h(z,t,i)-u_0(z,t,i))-(1+c_1)u_{n-1}(z,t,i)-\sum_{j=2}^{n-1}c_ju_{n-j}(z,t,i) \tag{11}$$

Using these solutions in Eq. (6) , one can get the approximate solution, containing auxiliary constants. One can obtain the values of these constants using collocation method and method of least square. We define the residual by putting the approximate solution Eq.(6) in Eq.(3) . To find values of the constants, c_m , $m=1,2,.......n$,we define

$$J(c_m)=\int_a^b\int_c^d\int_e^n R^2(z,t,i,c_m)dtdzdi \tag{12}$$

and then

$$\frac{\partial J_1}{\partial c_1}=0,\frac{\partial J_2}{\partial c_2}=0,\frac{\partial J_n}{\partial c_n}=0 \tag{13}$$

From system in Eq (13), one can easily obtain the values of constants. In collocation method,we take distinct points $\tau_i\in(a,b),i=1,2,....m$ in domain of the problem. The choice of selection of $\tau_i\in(a,b),i=1,2,....m$ is independent. Inserting τ_i into residual equation and put it equal to zero:

$$\Re(\tau_i,c_i)=0,i=1,2,3,....m \tag{14}$$

Solve system in Eq.(14) for constants

3 Illustrative Problems

In this section, some numerical problems are presented to show the efficiency and reliability of OHAM

Problem 3.1. consider 2D VIE [1]:

$$u(z,t)=e^tz^2-\frac{2z^3t^2}{3}+\int_0^z\int_{-1}^1 t^2e^{-r}\phi(r,s)drds,0\leq z\leq1, \tag{15}$$

with the exact solution $u(z,t)=e^tz^2$.

Using OHAM, discussed in pervious section. On can get different order problems and their solutions are given below:

Zero order problem and its solution is:

$$u_0(z,t)+\frac{-3e^tz^2+2z^3t^2}{3}=0 \tag{16}$$

$$u_0(z,t)+\frac{3e^t z^2-2z^3t^2}{3}=0 \tag{17}$$

1st order problem and its solution:

$$e^t z^2-\frac{2z^3t^2}{3}+e^t z^2c_1-\frac{2}{3}z^3t^2c_1+t^2\left(\int_0^z\int_{-1}^1 e^{-s}u_0(r,s)dsdr\right)c_1-u_0(z,t)-c_1u_o(z,t)+u_1(z,t)=0 \tag{18}$$

$$u_1(z,t)=\frac{z^3(-4e+(-5+e^2)z)t^2c_1}{6e} \tag{19}$$

By adding zero order and 1st order solution, we get 1st order OHAM solution

$$u(z,t)=\frac{e^t z^2-2z^3t^2}{3}+\frac{z^3(-4e+(-5+e^2)z)t^2c_1}{6e} \tag{20}$$

Using method of least square one can get the value of constant
$c_1=-1.248519006$.
With this constant the approximate solution Eq. (20), becomes

$$u(z,t)\approx\frac{e^t z^2-2z^3t^2}{3}-0.0765507z^3(-4e+(-5+e^2)z)t^2 \tag{21}$$

Table 1: comparison of absolute errors (AE) of OHAM and RBFâS Method [1]

(z,t)	OHAM	EXACT	AE in [1]	$(N=5)$	AE of $1^{st} order OHAM$
(0,0)	0	0	1.6379×10^{-6}	2.4657×10^{-6}	0
(0.1,0.1)	0.0111	0.0111	7.6810×10^{-5}	1.4696×10^{-5}	1.4739×10^{-6}
(0.2,0.2)	0.0489	0.0489	4.1947×10^{-4}	3.3702×10^{-4}	4.1313×10^{-5}
(0.3,0.3)	0.1215	0.1215	2.7146×10^{-3}	2.4510×10^{-3}	2.6928×10^{-4}
0.4,0.4)	0.2387	0.2387	9.9376×10^{-3}	1.0059×10^{-2}	9.4746×10^{-4}
(0.5,0.5)	0.4122	0.4122	2.9926×10^{-2}	3.0543×10^{-2}	2.3199×10^{-3}
(0.6,0.6)	0.6560	0.6560	7.5715×10^{-2}	7.5897×10^{-2}	8.5612×10^{-3}
(0.7,0.7)	0.9867	0.9867	1.6434×10^{-1}	1.6356×10^{-1}	4.3506×10^{-3}
(0.8,0.8)	1.4244	1.4244	3.1791×10^{-1}	3.1748×10^{-1}	6.3296×10^{-3}
(0.9,0.9)	1.9922	1.9922	5.6869×10^{-1}	5.6961×10^{-1}	6.3479×10^{-3}

Problem 3.2. Taking $(1+1)$-dimensional nonlinear VFIE [1, 2]:

$$u(t,z)=h(z,t)-\int_0^z\int_0^1 t^2e^{-4s}(u(r,s))^2drds0\leq z,t\leq 1 \tag{22}$$

where $h(z,t)=\left(e^{2t}z^2-\frac{1}{512}e^{-4z}(-1+e^4)\left(-3+3e^{4z}-4z\left(3+2z+(3+4z(1+z))\right)\right)t^2\right)$, and the exact solution $u(z,t)=z^2e^{2t}$ apply the proposed method, one can obtain the following solution and auxiliary constant.
Auxiliary constant
$c_1=-1.0002912$.
and approximate solution

$u(z,t)\approx e^{2t}z^2-\frac{1}{512}e^{-4z}(-1+e^4)\left(-3+3e^{4z}-4z\left(3+2z+(3+4z(1+z))\right)\right)t^2+2.922386\times10^{-10}e^{-12z}$ $(e^{12z}(1078756769-3805380e^2+2851e^4)-34992e^{8z}(31191+9e^4-480e^2(1+4z+8z^2)+640z(195+2z(195+256z(1+z))))+2187(-1+e)e^{4z}(1+e)(-5661+3e^2(93+8z(45+4z(21+8z(3+2z)))))-8z(5805+4z(5877+8z(1651+2z(1177+160z(7+4z))))))-16(-1+e^4)(18631+24z(6035+481+(481+z(1195+35(709+12z(77+z(73+6z(8+3z)))))))))t^2$

Problem 3.3. Consider$(2+1)$-dimensional linear VFIE [3]:

$$u(z,t,i)=h(z,t,i)+\frac{1}{2}\int_0^z\int_0^1\int_0^1(ti+rv)u(r,s,v)dvdsdr, 0\leq z,t,i\leq 1 \tag{23}$$

where $h(z,t,i)=\frac{t^4}{72}(4i(-1+cos(1))-3(sin(1)-cos(1)))+z^2t^2sin(i)$ with close form solution $u(z,t,i)=z^2t^2sin(i)$ By apply

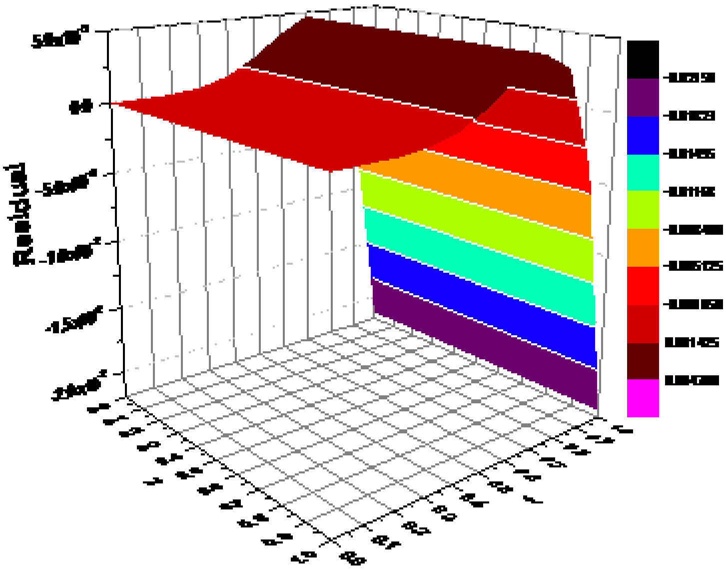

Figure 1: Residual Plot of problem 1.

Table 2: Absolute errors (AE) of OHAM, MQâs [1] and BC Method[2]

(z,t)	OHAM	EXACT	AE [1] ($c = 0.6$)	AE [2]]	OHAM(AE)
(0,0)	0	0	6.4387×10^{-3}	0	0
(0.1,0.1)	0.0121	0.0121	6.9441×10^{-3}	1.0555×10^{-4}	4.3680×10^{-15}
(0.2,0.2)	0.0597	0.0597	4.3133×10^{-2}	6.7607×10^{-4}	2.5081×10^{-8}
(0.3,0.3)	0.1640	0.1640	9.8257×10^{-2}	1.5111×10^{-3}	8.6414×10^{-7}
0.4,0.4)	0.3561	0.3561	1.6816×10^{-1}	1.6686×10^{-3}	8.5050×10^{-6}
(0.5,0.5)	0.6795	0.6796	2.6325×10^{-1}	1.3417×10^{-5}	4.3547×10^{-5}
(0.6,0.6)	1.1951	1.1952	3.8837×10^{-1}	4.0520×10^{-3}	3.8811×10^{-4}
(0.7,0.7)	1.9867	1.9871	5.3340×10^{-1}	9.4556×10^{-3}	3.8811×10^{-4}
(0.8,0.8)	3.1691	3.1699	6.9169×10^{-1}	1.3304×10^{-2}	2.8416×10^{-4}
(0.9,0.9)	4.8987	4.09002	8.8598×10^{-1}	1.1352×10^{-2}	1.5441×10^{-3}

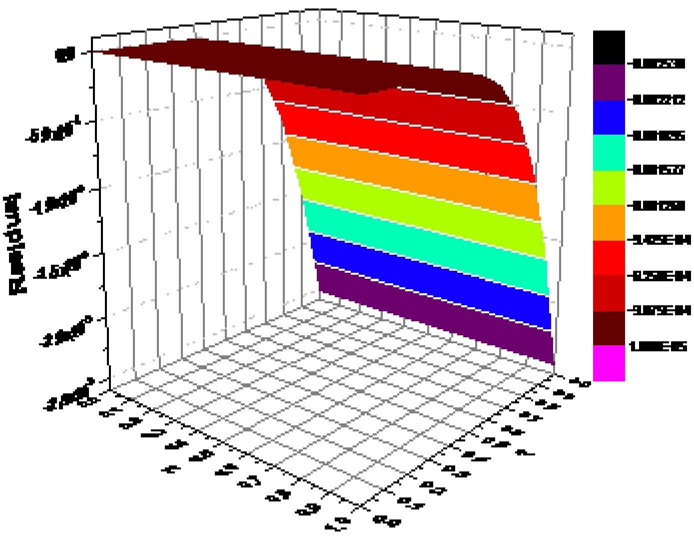

Figure 2: Residual Plot for problem 2.

OHAM one can obtain the following approximate solutions. Values of auxiliary constants is obtained by using collocation method: $c_1 = -1.0459773, c_2 = -0.0037576, c_3 = -0.0002483$.
With these constants the 3rd order OHAM solution

$u(z,t,i) \approx \frac{1}{14929920} t^2(-13.41521(-17cos(1)360t^2(cos(1)-sin(1))+9sin(1)+12z(2+40t^2(-1+cos(1)-5cos(1)+3sin(1)))+ 236.31880(360-759cos(1)+8640t^2(cos(1)-sin(1))+399sin(1)+4i(268+2880t^2(-1+cos(1))-673cos(1)+405sin(1)))- 1.14437(36520-76363cos(1)+622080t^2(cos(1)-sin(1))+39843sin(1)+12i(8992+69120t^2(-1=cos(1))-22687cos(1)+ 13695sin(1)))-150.62073(36(8-17cos(1)+360t^2(cos(1)-sin(1))+9sin(1)+12i(2+40t^2(-1+cos(1))-5cos(1)+3cos(1)+ 3sin(1)))-0.00376(360-759cos(1)+8640t^2(cos(1)-sin(1))=399sin(1)+4i(268+2880t^2(-1+cos(1)-673cccccos(1)= 405sin(1))))+13824(-0.00776+15(t^2(-4i+(3+4i)cos(1)-3sin(1)+72z^2sin(1))))$

Table 3: absolute errors (AE) of OHAM w.r.t different orders

(z,t,i)	OHAM	EXACT	$1^stOrderAE$	$2^{nd}OrderAE$	$3^{rd}OrderAE$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2.9964×10^{-2}	2.9964×10^{-2}	38403×10^{-4}	2.8749×10^{-7}	3.4695×10^{-18}
$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	96642×10^{-4}	96642×10^{-4}	87051×10^{-5}	1.4024×10^{-7}	1.0842×10^{-19}
$(\frac{1}{16},\frac{1}{16},\frac{1}{16})$	9.5305×10^{-7}	9.5305×10^{-7}	3.8632×10^{-6}	1.1395×10^{-8}	3.2822×10^{-21}
$(\frac{1}{32},\frac{1}{32},\frac{1}{32})$	2.9798×10^{-8}	2.9798×10^{-8}	8.8463×10^{-7}	2.9507×10^{-9}	6.8160×10^{-22}
$(\frac{1}{64},\frac{1}{64},\frac{1}{64})$	9.3129×10^{-10}	9.3129×10^{-10}	2.1071×10^{-7}	7.5027×10^{-10}	3.5424×10^{-18}

Table 4: comparison of OHAM with EPS Method [3]

(z,t,i)	OHAM	EXACT	EPS method [3]	$3^{rd}OrderOHAM$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2.9964×10^{-2}	2.9964×10^{-2}	1.2×10^{-11}	3.4695×10^{-18}
$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	96642×10^{-4}	96642×10^{-4}	6.1×10^{-8}	1.0842×10^{-19}
$(\frac{1}{16},\frac{1}{16},\frac{1}{16})$	9.5305×10^{-7}	9.5305×10^{-7}	2.5×10^{-10}	3.2822×10^{-21}
$(\frac{1}{32},\frac{1}{32},\frac{1}{32})$	2.9798×10^{-8}	2.9798×10^{-8}	1.1×10^{-11}	6.8160×10^{-22}
$(\frac{1}{64},\frac{1}{64},\frac{1}{64})$	9.3129×10^{-10}	9.3129×10^{-10}	4.0×10^{-13}	3.5424×10^{-18}

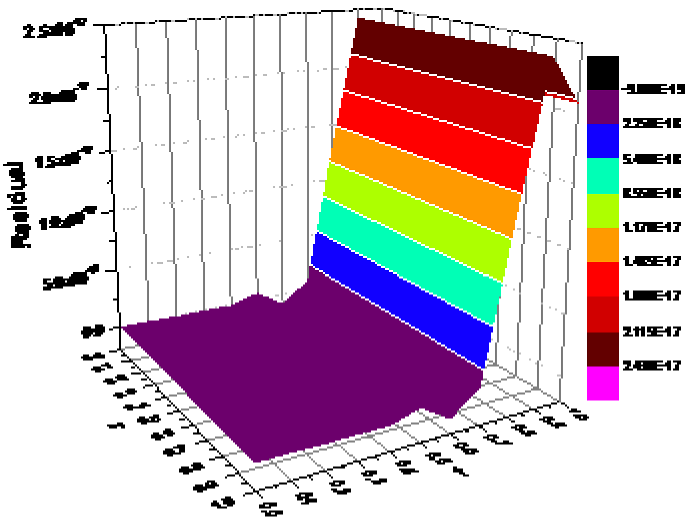

Figure 3: Residual Plot taking $i = 1/16$ of problem 3.

Problem 3.4. consider $(2+1)$-dimensional linear mixed VFIE[4]:

$$u(z,t,i) = h(z,t,i) + \frac{1}{20}\int_0^z\int_0^t\int_0^i iu(r,s,v)dvdsdr + \frac{1}{10}\int_0^1\int_0^1\int_0^1 (z+r)u(r,s,v)dvdsdr, 0 \le z,t,i \le 1 \quad (24)$$

where h(z,t,i) is selected in such a way that solution becomes $h(z,t,i) = sin(t+i)$, apply the proposed method we get the following solution:

Optimum value of auxiliary constant determined by using method of least square:
$c_1 = -1.1157439$.
with this constant the solution becomes:

$u(z,t,i) \approx -\frac{1}{5}(1+2z)sin(\frac{1}{2})^2 sin(1)+\frac{1}{20}zi(sin(t)+sin(i)-sin(t+i))+sin(t+i)+0.0002324(3z(-6+zi(2+i^2)+14cos(1)-4cos(2))$-$3z^2 i(2+i^2)cos(t)-4(3-7cos(1)+2cos(2)+106sin(1)-51sin(2)))(+6z(-144sin(1)+70sin(2)+i(-z(1+ti)cos(i)+zcos(t+i)-8(1+z)tzsin(\frac{1}{2})^2 sin(1)+40sin(t)+40sin(i)+z(t-i)+(-40+ti)sin(t+i))))$

Table 5: Absolute errors (AE) of 1^{st} order OHAM with MBPFs Method [4]

(z,t,i)	OHAM	EXACT	AE in [4] $(m=8)$ $(k=2)$	AE of OHAM
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	0.8414	0.8415	2.9263×10^{-2}	3.2343×10^{-5}
$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	0.4792	0.4794	5.2143×10^{-2}	2.1138×10^{-4}
$(\frac{1}{8},\frac{1}{8},\frac{1}{8})$	0.3660	0.3663	5.9035×10^{-2}	2.9095×10^{-4}
$(\frac{1}{16},\frac{1}{16},\frac{1}{16})$	0.1243	0.1247	6.1123 $(\times10^{-2}$	3.3020×10^{-4}
$(\frac{1}{32},\frac{1}{32},\frac{1}{32})$	0.0621	0.0625	3.1017 $(\times10^{-2}$	3.4921×10^{-4}
$(\frac{1}{64},\frac{1}{64},\frac{1}{64})$	0.0308	0.0312	6.2231×10^{-2}	3.5870×10^{-4}

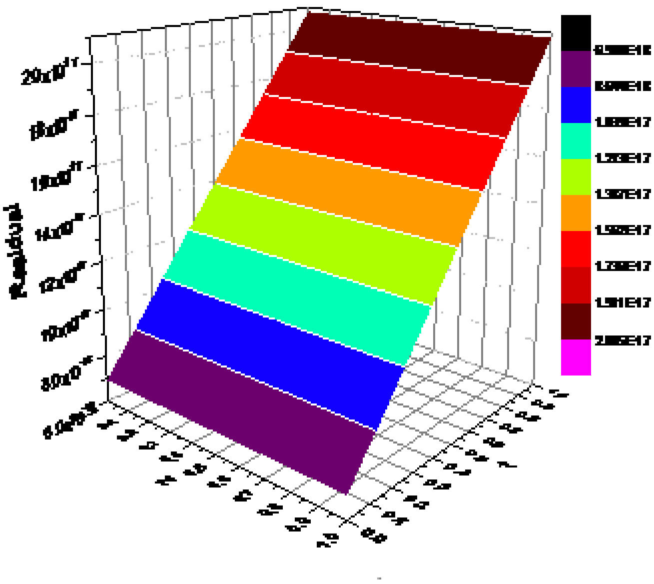

Figure 4: Residual Plot taking $i = 1/8$ of problem 4.

Problem 3.5. Consider$(2+1)$-dimensional linear mixed VFIE[4, 5]:

$$u(z,t,i) = \left(z^2ti + \frac{11z^6(3+4t^2)i}{5760}\right) + \frac{1}{4}\int_0^t\int_0^1\int_0^1 (z+v)(t^2+r)is(u(r,s,v))^2 drdsdv, 0 \le z,t,i \le 1, \tag{25}$$

with the exact solution $u(z,t,i) = z^2ti$,
Apply the proposed methodone can get the following solution:
Optimum value of auxiliary constant determined by using method of least square:
$c_1 = -1.0122278$.
Approximate solution

$u(z,t,i) \approx z^2ti - \frac{11z^6(3+4t^2)i}{5760} + 3.4574495\times10^{-11}z^6(55910400 - 885248z^4 + 4345z^8)(3+4t^2)i$.

4 Result and Discussion:

In this work, reliable algorithm of OHAM is successfully applied to VFIE of higher dimensional. Table 3 contain results of different order solution of OHAM for Problem 3.Tables $1,2,4,5 and 6$ show comparison of OHAM solution with other method which

Table 6: Absolute errors (AE) of 1^{st} order OHAM with other method [4, 5]

(z,t,i)	OHAM	EXACT	AE in [4]($k=2$) ($m=3$)	AE in [5] ($m=3$)	AE in OHAM
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	0.0625	0.0625	7.78×10^{-2}	2.288×10^{-3}	6.6998×10^{-7}
$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	3.9063×10^{-3}	3.9063×10^{-3}	2.12×10^{-2}	2.877×10^{-3}	4.6084×10^{-9}
$(\frac{1}{8},\frac{1}{8},\frac{1}{8})$	2.4414×10^{-4}	2.4414×10^{-4}	5.12×10^{-2}	7.848×10^{-4}	3.4090×10^{-11}
$(\frac{1}{16},\frac{1}{16},\frac{1}{16})$	1.5259×10^{-5}	1.5259×10^{-5}	7.98×10^{-2}	1.014×10^{-3}	2.6233×10^{-13}
$(\frac{1}{32},\frac{1}{32},\frac{1}{32})$	9.5367×10^{-7}	9.5367×10^{-7}	1.11×10^{-2}	1.028×10^{-3}	2.0415×10^{-15}
$(\frac{1}{64},\frac{1}{64},\frac{1}{64})$	5.9605×10^{-8}	5.9605×10^{-8}	1.50×10^{-2}	1.029×10^{-3}	1.5934×10^{-17}

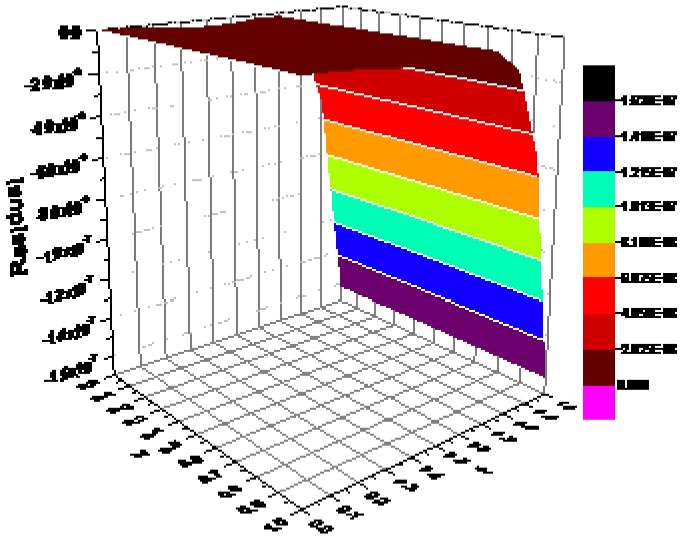

Figure 5: Residual Plot for 1st order OHAM solution taking $i=1/8$ for problem 5.

clearly show the reliability of OHAM over these methods.Plots of residual errors for problems $(1-5)$ are presented in Figures $(1-5)$, respectively. it is clear from table 3 that approximate solution gets closure and closure to exact solution when order of approximation increases.

5 Conclusion

In this work, it is proved that OHAM is a consistent and efficient tool for strongly nonlinear problem and higher dimensional VFIE. The proposed method is tested with solution of$(1+1)$ and $(2+1)$ dimensional VFIE. Results reveal that the presented technique is an accurate and reliable one. The fast convergence and accuracy of the proposed technique is a valid reason for researcher to use OHAM for different nonlinear problem arising in different field of science and technology. In the next work, we will extent the proposed technique for strongly nonlinear two-dimensional problem of fractional order.

6 Acknowledgment

Praveen Agarwal was paying thanks to the SERB (project TAR/2018/000001), DST(project DST/INT/DAAD/P-21/2019, and INT/RUS/RFBR/308) and NBHM (DAE)(project 02011/12/2020 NBHM(R.P)/RD II/7867).

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