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Asymptotics of integrals of Vandermonde determinants

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Abstract

Let ν be a finite measure on a non-polar compact $K \subset \mathbb{C}$. Let D be the Vandermonde determinant in $z_1, ...z_n$. and Z_n the integral of $|D|^2$ over K^n with respect to the product measure $d\,\nu(z_1)...d\,\nu(z_n)$. Then the leading term asymptotics of Z_n are the same if and only if the measure is regular (in the sense of Stahl-Totik).

To my friend and colleague, Norm Levenberg on his 60^{th} birthday

1 Introduction

Let K be a compact non-polar set in the complex plane. Let ν be a finite measure on K. We consider the integrals

$$Z_n = \int_{K^n} |D(z_1, z_2, .., z_n)|^2 d\,\nu(z_1)...d\,\nu(z_n).$$

Here $D(z_1, .., z_n) = \prod_{1\le i<j\le n}(z_j - z_i)$ denotes the Vandermonde determinant.

These integrals arise, for example, in the study of normal matrix models where the eigenvalues given by $z_1, ..., z_n$ are confined to the compact set K. In this context the constants Z_n are referred to as normalizing constants as the expression

$$\frac{1}{Z_n}|D(z_1, z_2, .., z_n)|^2 d\,\nu(z_1)...d\,\nu(z_n)$$

is a probability distribution on K^n - the joint distribution of the eigenvalues.

They also arise in the study of orthogonal polynomials as they are related to the study of Bergman function asymptotics or, in the real case, asyptotics for the Christoffel-Darboux kernel.

Asymptotic expansions for the Z_n are of considerable interest. In this paper we will deal only with leading term asymptotics for Z_n. The formula (1) below is well-known in many cases. In this paper we will establish Theorem 1.1 which gives necessary and sufficient conditions on the measure ν so that the asymptotics for Z_n

$$\lim_n \frac{1}{n^2}\log Z_n = \log(\mathrm{cap}(K)) \tag{1}$$

hold. Here cap denotes logarithmic capacity and $\mathrm{cap}(K) > 0$ since K is non-polar.

We let $q_n(z)$ denote the monic polynomial of degree n and of minimal $L^2(\nu)$ norm. That is

$$\int_K |q_n(z)|^2 d\,\nu(z) = \inf_{p\in\mathcal{N}_n}\{\int_K |p(z)|^2 d\,\nu(z)\}.$$

where $\mathcal{N}_n$ denotes the space of monic polynomials of degree n.

We let

$$e_n =: \Big(\int_K |q_n(z)|^2 d\,\nu(z)\Big)^{1/2} = ||q_n||_{L^2(\nu)},$$

and we let $r_n(z)$ be a monic polynomial of degree n of minimal sup norm on K. That is

$$||r_n||_K = \inf_{p\in\mathcal{N}_n}\Big(||p||_K\Big).$$

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The polynomial r_n is, in general, not unique.

However it is known (see [4]) that

$$\lim_n ||r_n||_K^{1/n} = \text{cap}(K). \tag{2}$$

Now,

$$e_n = ||q_n||_{L^2(\nu)} \leq ||r_n||_{L^2(\nu)}$$

so that

$$\limsup_n \frac{1}{n} \log e_n \leq \log \text{cap}(K) \tag{3}$$

and, in particular the sequence $\{\frac{1}{n} \log e_n\}$ is bounded above.

Measures for which

$$\lim_n \frac{1}{n} \log e_n = \log \text{cap}(K) \tag{4}$$

are important.

Definition 1.1. (see [5]) The measure ν is *regular* if

$$\lim_n \frac{1}{n} \log ||q_n||_{L^2(\nu)} = \log(\text{cap}(K)). \tag{5}$$

We will write $\nu \in \text{REG(K)}$.

Theorem 1.1.

$$\lim_n \frac{1}{n^2} \log Z_n = \log(cap(K))$$

if and only if $\nu \in REG(K)$.

To establish the theorem, we must show that (1) holds if and only if (5) holds.

In section 2 we show that this question reduces to the following: given that certain means of the sequence (7) converge then the sequence converges to the same limit. Criteria for this to happen are developed in section 3 and we show that the sequence $\{\frac{1}{n} \log ||q_n||_{L^2(\nu)}\}$ satisfies those criteria.

In section 4 we discuss regular measures and a closely related notion of Bernstein-Markov measures.

2 Convergence of means

We now express the constants Z_n in terms of the

$$e_n =: \Big(\int_K |q_n(z)|^2 d\,\nu(z) \Big)^{1/2} = ||q_n||_{L^2(\nu)}.$$

Doing row operations on the Vandermonde determinants we obtain

$$Z_n = \int_{K^n} \det(q_i(z_j))\overline{\det(q_i(z_j))} d\,\nu(z_1) d\,\nu(z_2)...d\,\nu(z_n)$$

where the indices in the determinant are $i = 0, ..., n-1$ and $j = 1, ..., n$.

Expanding the determinants and using the fact that the q_j's are mutually orthogonal we have

$$Z_n = n! ||q_0||^2_{L^2(\nu)} .... ||q_{n-1}||^2_{L^2(\nu)}.$$

Hence

$$\frac{1}{n^2} \log Z_n = \frac{\log n!}{n^2} + \frac{2}{n^2} \sum_{i=0}^{n-1} \log ||q_i||_{L^2(\nu)}.$$

So (1) is equivalent to

$$\lim_n \Big(\frac{2}{n^2} \sum_{i=0}^{n-1} \log ||q_i||_{L^2(\nu)} \Big) = \log \text{cap}(K). \tag{6}$$

Thus we must show that (5) and (6) are equivalent.

Consider the sequence

$$\log e_1, \overbrace{\frac{1}{2} \log e_2, \frac{1}{2} \log e_2}^{two \quad terms}, ... \overbrace{\frac{1}{n} \log e_n, ... \frac{1}{n} \log e_n}^{n \quad terms}. \tag{7}$$

The mean of the above terms is

$$\frac{2}{n(n+1)} \Big(\log e_1 + \log e_2 + ... + \log e_n \Big).$$

and (6) is equivalent to

$$\lim_n \frac{2}{n(n+1)} \Big(\log e_1 + \log e_2 + ... + \log e_n \Big) = \log \text{cap}(K). \tag{8}$$

It is a standard fact that if a sequence converges then this implies that the arithmetic means converge to the same limit. Thus (5) implies (6). We must show that the converse holds. Of course, it is not true, in general, that if certain arithmetic means of a sequence converge then the sequence converges. The fact that this holds in the case of theorem 1.1 will be due to special properties of the sequence $\{e_n\}$.

3 Properties of the sequence $\{e_n\}$

In theorem 3.1 we will establish a necessary condition for a mean of a sequence to converge while the sequence itself diverges. In theorem 3.2 we will show that the sequence $\{\frac{1}{n}\log e_n\}$ does not satisfy that condition, so we may conclude that the sequence converges.

We will use the concept of a sequence of zero density defined as follows:

Definition 3.1. (see [7]) Let $J \subset \mathbb{N}$ be a subsequence. Then J is of *zero density* if

$$\lim_n \Big(\frac{\operatorname{card}(J \cap \{1,2,\dots,n\}}{n}\Big) = 0.$$

Theorem 3.1. *Let $\{a_n\}$ be a sequence of real numbers, satisfying:*

$$\limsup_n a_n \le s$$

and

$$\lim_n \frac{2}{n(n+1)}\Big(\sum_{j=1}^{n} j a_j\Big) = s.$$

Let J be a subsequence of $\mathbb{N}$ such that for some $\eta > 0$

$$\limsup_{n\in J} a_n \le s - \eta.$$

Then J is of zero density.

Proof. The conclusion is not affected if finitely many terms are changed in J so we may assume that $a_n \le s - \eta$ for $n \in J$.

The hypothesis imply that the sequence is bounded above and we assume that $a_n \le L$ for all n and some constant L. Furthermore the existence of the limsup of the sequence implies that given any $\epsilon > 0$ there exists a $B(\epsilon)$ such that for $n \ge B(\epsilon)$ we have $a_n \le s + \epsilon$.

The proof will proceed by contradiction. Suppose that J is not of zero density. Then there is a subsequence $\{n_k\} \subset J$ such that

$$\lim_k \Big(\frac{\operatorname{card}(J \cap \{1,2,\dots,n_k\}}{n_k}\Big) \ge \alpha > 0$$

where $\alpha \le 1$.

We let

$$J_k = J \cap \{1,2,\dots,n_k\}$$

and

$$J_k^c = \{1,2,\dots,n_k\} \setminus J$$

We may assume, again possibly changing finitely many terms from J, given $\gamma > 0$ that

$$\alpha + \gamma \ge \frac{1}{n_k}\operatorname{card}(J_k) \ge \alpha - \gamma \tag{9}$$

for all k.

Consider the sum $\sum_{j=1}^{n_k} j a_j$. We will separately estimate the sum of those terms with $j \in J_k$ and $j \in J_k^c$.

$$\sum_{j\in J_k} j a_j \le \Big(\sum_{j\in J_k} j\Big)(s-\eta) = \Big(\sum_{j\in J_k} j\Big)s - \Big(\sum_{j\in J_k} j\Big)\eta. \tag{10}$$

And, since $\operatorname{card}(J_k) \ge (\alpha-\gamma)n_k$, we have (where [x] denotes the greatest integer $\le x$)

$$\sum_{j\in J_k} j \ge \sum_{j=1}^{[(\alpha-\gamma)n_k]} j = \frac{([(\alpha-\gamma)n_k])([(\alpha-\gamma)n_k]+1)}{2}. \tag{11}$$

Also,

$$\sum_{j\in J_k^c} j a_j \le \Big(\sum_{1}^{B} j a_j\Big) + \sum_{j>B, j\in J_k^c} j a_j \le \frac{B(B+1)L}{2} + \sum_{j\in J_k^c} j(s+\epsilon). \tag{12}$$

And since $\operatorname{card}(J_k^c) \le n_k(1-(\alpha-\gamma))$

$$\sum_{j\in J_k^c} j \le \sum_{[n_k(\alpha-\gamma)]}^{n_k} j = \frac{n_k(n_k-1)}{2} - \frac{[n_k(\alpha-\gamma)]([n_k(\alpha-\gamma)]-1)}{2}. \tag{13}$$

Combining 10 and 12 we have

$$\sum_{j=1}^{n_k} j a_j \leq \Big(\sum_{j=1}^{n_k} j\Big)s - \eta\Big(\sum_{j\in J_k} j\Big) + \epsilon \sum_{j\in J_k^c} j + \frac{B(B+1)L}{2}. \tag{14}$$

Thus

$$\begin{aligned}\limsup_k \frac{2}{n_k^2}\sum_{j=1}^{n_k} j a_j &\leq s - \eta\Big(\sum_{j\in J_k} j\Big) + \epsilon \sum_{j\in J_k^c} j \\ &\leq s - \frac{(\alpha-\gamma)^2\eta + \epsilon\big((1-(\alpha-\gamma)^2}{2}\big).\end{aligned} \tag{15}$$

Given s, α, η one may choose γ, ϵ so that the right side if the inequality is $< s$. This contradiction establishes the result. □

Theorem 3.2. *Suppose that for some subsequence $J \subset \mathbb{N}$ that*

$$\sup_{n\in J} \frac{1}{n} \log e_n \leq \log(cap(K)) - \eta \tag{16}$$

for some $\eta > 0$. Then there is a subsequence J_1 of $\mathbb{N}$, not of zero density, such that (16) holds (possibly with a different η).

Lemma 3.3. *Given $\epsilon > 0$ there is an integer $t_0(\epsilon)$ such that for all n and all $t \geq t_0$ we have*

$$e_{n+t} \leq (cap(K)+\epsilon)^t e_n.$$

Proof. Choose t_0 so that $||r_t||_K \leq (\text{cap}(K)+\epsilon)^t$ for $t \geq t_0$. Then,

$$\begin{aligned} e_{n+t} = \Big(\int_K |q_{n+t}(z)|^2 d\nu\Big)^{1/2} &\leq \Big(\int_K |q_n(z) r_t(z)|^2 d\nu\Big)^{1/2} \\ &\leq ||r_t||_K e_n \end{aligned}$$

and the lemma follows. □

Proof. (of theorem 3.3)

Now choose $\epsilon > 0$ so that for some $\eta_1 > 0$, and all ρ such that $1/2 \leq \rho \leq 2/3$ we have

$$\rho[\log(\text{cap}(K)+\epsilon)] + (1-\rho)[\log(\text{cap}(K)) - \eta] \leq \log(\text{cap}(K)) - \eta_1.$$

For $n \in \mathbb{N}$ and $n \geq t_0$, $n \leq t \leq 2n$ we have from lemma 3.3

$$\frac{1}{n+t}\log e_{n+t} \leq \frac{t}{n+t}\log(\text{cap}(K)+\epsilon) + \frac{n}{n+t}\log(\text{cap}(K)) - \eta.$$

So

$$\frac{1}{n+t}\log e_{n+t} \leq \log(\text{cap}(K)) - \eta_1.$$

So if J_1 includes all such indices $(n+t)$ as above, then J_1 is not of zero density but

$$\sup_{n\in J_1} \frac{1}{n}\log e_n \leq \log(\text{cap}(K)) - \eta_1.$$

□

Proof. (of theorem 1.1)

The proof will be by contradiction.

We assume (8) holds. If (5) does not hold there must be a subsequence $J \subset \mathbb{N}$ such that for some $\eta > 0$ we have $\sup_{n\in J} \log \frac{1}{n} \log e_n \leq \log cap(K) - \eta$. By theorem 3.1, J must be of zero density but by theorem 3.2 we may choose such a J not of zero density.

□

4 Regular measures

In this section we will deal with properties of regular measures. These measures are extensively studied in [5].

Definition 4.1. Let μ be a finite measure on a non-polar compact set $K \subset \mathbb{C}$. We say that μ is a Bernstein-Markov measure if given $\epsilon > 0$ there is a constant $C > 0$ such that

$$||p||_K \leq C(1+\epsilon)^n ||p||_{L^2(\mu)} \tag{17}$$

for all holomorphic polynomials p of degree $\leq n$ and all n.

We will write $\mu \in$ BM(K).

Lemma 4.1.

$$BM(K) \subset REG(K).$$

Proof. We must show that all Bernstein-Markov measures are regular. Let μ be Bernstein-Markov on K. Given $\epsilon > 0$ there is a constant C such that

$$||r_n||_K \leq ||q_n||_K \leq C(1+\epsilon)^n ||q_n||_{L^2(\mu)}.$$

It follows that

$$\liminf_n ||q_n||_{L^2(\mu)}^{1/n} \geq \lim_n ||r_n||_K^{1/n} = \text{cap}(K).$$

Combining this with (3) we obtain (1). □

Asymptotics for the Bergman function are valid for regular measures. Let μ be a finite measure on a non-polar compact set $K \subset \mathbb{C}$. We let $d\mu_{eq}(K)$ denote the equilibrium measure of K (see [4]).

The Bergman function is defined by

$$S_n(z) := \sum_{j=0}^{n} |p_j(z)|^2 \tag{18}$$

where the $p_j(z)$ are orthonormal polynomials in $L^2(\mu)$ and deg $p_j = j$.

For $K \subset \mathbb{R}, S_n$ as a function of the real variable, is known as the Christoffel-Darboux function.

Theorem 4.2. *Let $\mu \in REG(K)$. Then*

$$\lim_n \frac{1}{n} S_n(z) d\mu = d\mu_{eq}(K) \quad weak^*.$$

Proof. Theorem 2.2 in [3] gives the above result when $\mu \in BM(K)$. However that proof only uses the Bernstein-Markov property of μ to establish the asymptotics of Z_n as given in theorem 1. Those asymptotics are valid when $\mu \in REG(K)$ by the "easy" part of theorem 1.1 i.e. (5) implies (1).

□

Related results may be found in ([1], corollary 4.4) and ([6], corollary 1).

We record a number facts about regular and Bernstein-Markov measures:

- If the set K is a regular set in the sense of potential theory, (i.e. it is regular for the exterior Dirichlet problem (see [4]) then every Bernstein-Markov measure is regular, i.e. BM(K) = REG(K). (see [5], theorem 3.2.3 (v))
- A specific example of a measure that is regular but not Bernstein-Markov (so the set K is necessarily not regular) is given in ([5], example 3.5.3).
- There exists a Bernstein-Markov measure on any compact set (see [3], corollary 3.5).
- A convenient sufficient condition that a measure be regular on a regular compact set K is the following "mass-density " condition: μ is regular if there are constants $T > 0, R_0 > 0$ such that $\mu(\Delta(z,R)) \geq R^T$ for all $R \leq R_0$ and all $z \in K$ where $\Delta(z,R)$ denotes the disc center z and radius R. In particular, one dimensional Lebesgue measure on a compact interval is Bernstein-Markov, as is planar Lebesgue measure on a compact disc.

5 $\beta > 0$

We consider the constants

$$Z_{n,\beta} = \int_{K^n} |D(z_1,...z_n)|^\beta d\,\nu(z_1)...d\,\nu(z_n) \tag{19}$$

where $\beta > 0$ is a real parameter.

We will prove the following result:

Theorem 5.1. *Let $K \subset \mathbb{C}$ be compact and $\nu \in REG(K)$. Then*

$$\lim_n \frac{1}{n^2} \log Z_{n,\beta} = \frac{\beta}{2} \log cap(K).$$

Proof.

$$\int_{K^n} |D(z_1,...,z_n)|^\beta d\,\nu(z_1)...d\,\nu(z_n) \leq sup_{K^n}|D(z_1,...,z_n)|^\beta\, \nu(K)^n$$

so it follows from the asymptotics for $sup_{K^n}|D(z_1,...,z_n)|$ given by the Fekete-Szego theorem (see [4], theorem 5.5.2) that

$$\limsup_n \frac{1}{n^2} \log Z_{n,\beta} \leq \frac{\beta}{2} \log \mathrm{cap}(K). \tag{20}$$

Now, by corollary 3.4.2 of [5] since $\nu \in$ REG(K) we have

$$\lim_n (||q_{n,\beta}||_{L^\beta(\nu)})^{\frac{1}{n}} = \mathrm{cap}(K)$$

where $q_{n,\beta}$ is a monic polynomial of degree n of minimal $L^\beta(\nu)$ norm. It follows that given $\epsilon > 0$ there exists n_0 such that

$$\Big(\int_K |q(z)|^\beta d\,\nu(z)\Big)^{\frac{1}{n\beta}} \geq \mathrm{cap}(K) - \epsilon \tag{21}$$

for all monic polynomials q of degree $n \geq n_0$.

Then, doing the integral in z_n in the integral below and using (21) we have, by Fubini's theorem:

$$Z_{n,\beta} = \int_K |(z_n - z_1)...(z_n - z_{n-1})|^\beta d\,\nu(z_n) \int_{K^{n-1}} |D(z_1,...z_{n-1}|^\beta d\,\nu(z_1)...d\,\nu(z_{n-1})$$
$$\geq (\mathrm{cap}(K) - \epsilon)^{(n-1)\beta} Z_{n-1,\beta}.$$

Repeating this procedure, we get

$$Z_{n,\beta} \geq (\mathrm{cap}(K) - \epsilon)^{\Big((n-1)+...+(n-n_0)\Big)\beta} Z_{n_0-1,\beta}.$$

Since $\epsilon > 0$ is arbitrary we have

$$\liminf_n \frac{1}{n^2} \log Z_{n,\beta} \geq \frac{\beta}{2} \log \mathrm{cap}(K). \tag{22}$$

Combining (20) and (22) completes the proof.

□

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