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Voronovskaya estimates for convolution operators

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Abstract

We present a general method for establishing quantitative Voronovskaya-type estimates of convolution operators on homogeneous Banach spaces of periodic functions of one real variable or of functions on the real line. The method is based on properties of the Fourier transform of the kernel of the operator. We illustrate the elegance and the efficiency of this approach on two convolution operators—the Riesz typical means, and, in particular, the Fejér operator, and the generalized singular integral of Picard. A noteworthy feature of the former is the fact that, though the operator itself is saturated, the convergence in its Voronovskaya-type estimate can be of an arbitrary fast power-type provided that the function is smooth enough in a certain sense.

1 Introduction

Let X be a Banach space, $R \subseteq \mathbb{R}$ is not bounded above and $\mathcal{L}_\rho : X \to X$, $\rho \in R$, be a family of uniformly bounded linear operators such that

$$\operatorname{s-}\lim_{\rho\to\infty} \mathcal{L}_\rho f = f \quad \forall f \in X.$$

The limit above is taken in the norm of X, that is,

$$\lim_{\rho\to\infty} \|\mathcal{L}_\rho f - f\|_X = 0 \quad \forall f \in X.$$

Each such family $\{\mathcal{L}_\rho\}_{\rho\in R}$ and a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\lim_{\rho\to\infty} \varphi(\rho) = 0$ and $\varphi(\rho) \neq 0$ for every $\rho \in R$ define a differential-type linear operator D on some linear subspace of X by the relation

$$Df := \operatorname{s-}\lim_{\rho\to\infty} \frac{\mathcal{L}_\rho f - f}{\varphi(\rho)}.$$

We are interested how fast the convergence above is for f, which is smooth in a certain sense. That leads us to the problem of evaluating

$$\|\mathcal{L}_\rho f - f - \varphi(\rho) Df\|_X$$

for smooth f. It is especially important to consider this problem when φ is taken to be of the same magnitude as the optimal approximation order of $\mathcal{L}_\rho$. To recall, the optimal approximation order of an operator is the fastest rate it can approximate a function which it does not reproduce. With this choice of φ, D can be used to describe the saturation class of $\mathcal{L}_\rho$, that is, the set of functions which the operator approximates with the optimal approximation order (see e.g. [12, Definition 12.0.2]). We refer the reader for a short discussion on this general setting to [11, Sections 3.4 and 4]; detailed exposition can be found in [12, Chapters 12 and 13] and [13].

The most well-known result of this type is due to E. Voronovskaya and concerns the Bernstein operator [40]. To recall, the Bernstein operators or polynomials of degree $n \in \mathbb{N}_+$ of $f \in C[0,1]$ are defined by

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

Voronovskaya [40] (or see e.g. [15, p. 307]) showed that if $f''(x)$ exists at some point $x \in [0,1]$, then

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x); \tag{1.1}$$

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moreover, if $f \in C^2[0,1]$, then the convergence in (1.1) is uniform on $[0,1]$. The first result of this type for convolution operators seems to be established by Natanson [25] (or [26, Chapter X, § 3, Theorem 3], or [12, Section 2.5, Problem 7 and Proposition 12.2.9]) for the singular integral of the de La Vallée-Poussin (see [31, p. 355]).

Various estimates from above of the rate of the convergence in (1.1) have been established until now; see [22], [5, Section 4.3] and [21] for accounts of them. A characterization of this rate was established in [21]. Such results are usually referred to as Voronovskaya-type estimates or inequalities. General Voronovskaya-type estimates were established in [1, 16, 22, 29, 39]. A vast amount of Voronovskaya-type estimates for concrete operators has been published.

The purpose of this paper is to present a general approach to proving Voronovskaya-type estimates for convolution operators on homogeneous Banach spaces of 2π-periodic functions and of functions on $\mathbb{R}$. The proofs of the Voronovskaya-type inequalities, either general or for concrete operators, found in the literature, are most often essentially based on Taylor's formula or analogues thereof. The method we will apply uses the Fourier transform.

In the case of periodic functions it can be expressed by means of multipliers. Buchwalter [3, 4], but more fully Sunouchi [32, 33] (see [12, Section 12.6/Sec. 12.2]), used it to establish direct estimates for convolution operators of periodic functions. Trigub [35] (or [38, Sections 6.4.2 and 7.1.12]) developed independently a comparison principle for the error of multiplier operators based on the same basic approach. He and students of his used it to prove direct and strong converse inequalities for the rate of approximation of trigonometric operators. Recent instances of such results can be found in [36], the references cited there, and [24, 37]. An analogous comparison principle for convolution operators of functions on $\mathbb{R}^d$ was formulated by Shapiro [30, Section 9.4].

This approach was introduced to problems in approximation theory concerning convolution operators in $L_p(\mathbb{R})$, $1 \le p \le 2$, by Butzer [7, 8, 9, 10] (see [12, Section 12.6/Sec. 12.3]) and [12, Chapter 12]. Then the results were extended for $2 < p < \infty$ by duality [12, Chapter 13]. More background information on how these general ideas were applied can be found in [18, Section 3.3].

A very good feature of the Fourier transform method for treating approximation convolution operators is undoubtedly its elegance and efficiency. However, it can be used only to establish global estimates.

The contents of the paper are organized as follows. In Section 2 we will present the necessary preliminary notions and their properties. Section 3 contains the main results of the paper—general assertions yielding quantitative Voronovskaya-type estimates in the general class of the homogeneous Banach spaces of periodic functions and of functions on the real line. In the last section, we illustrate the general results on two convolution operators—the Riesz typical means and the generalized integral of Picard.

2 Convolution operators on homogeneous Banach spaces

We will consider a rather wide class of Banach spaces of real or complex-valued functions of one real variable. It includes the Lebesgue L_p-spaces, $1 \le p < \infty$ of 2π-periodic functions and the space of the continuous 2π-periodic functions, equipped with their standard norms, as well as their forms for functions defined on $\mathbb{R}$—$L_p(\mathbb{R})$, $1 \le p < \infty$, and the space of the uniformly continuous and bounded functions on $\mathbb{R}$, again with their standard norms:

$$\|f\|_{L_p(I)} := \left(\int_I |f(x)|^p \, dx \right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|f\|_{C(I)} := \sup_{x \in I} |f(x)|,$$

where $I = [-\pi, \pi]$ or $I = \mathbb{R}$. At some instances, for convenience, a normalizing factor is introduced to the integral.

The Sobolev spaces

$$W^r_{p,2\pi} := \{f \in AC^{r-1}_{loc}(\mathbb{R}),\ f^{(r)} \in L_p[-\pi,\pi],\ f \text{ is } 2\pi\text{-periodic}\}, \quad 1 \le p < \infty, \quad r \in \mathbb{N}_+,$$

where $AC^m_{loc}(\mathbb{R})$ denotes the space of the functions on $\mathbb{R}$, which are m times differentiable and $f, f', \dots, f^{(m)}$ are absolutely continuous on every interval $[a,b]$, with the norm

$$\|f\|_{W^r_{p,2\pi}} := \|f\|_{L_p[-\pi,\pi]} + \|f^{(r)}\|_{L_p[-\pi,\pi]},$$

and the space $C^r_{2\pi}$ of 2π-periodic functions, which are r times continuously differentiable on $\mathbb{R}$ with the norm

$$\|f\|_{C^r_{2\pi}} := \|f\|_{C[-\pi,\pi]} + \|f^{(r)}\|_{C[-\pi,\pi]}$$

as well as their analogues for functions on $\mathbb{R}$ are homogeneous Banach spaces, too. Thus the results below cover also the simultaneous approximation by convolution operators.

We will consider the homogeneous Banach spaces of periodic functions and of functions on $\mathbb{R}$ separately for simplicity. The cases of periodic functions and functions on $\mathbb{R}$ as well as their multivariate generalization can be treated together as it was done in [18].

We begin with the less abstract homogeneous Banach spaces of periodic functions.

2.1 Periodic functions

We denote the Banach space of all real or complex-valued 2π-periodic functions, which are Lebesgue integrable on $\mathbb{T} := [-\pi, \pi)$ by $L(\mathbb{T})$, the norm being given by

$$\|f\|_{L(T)} := \frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|\,dx.$$

Definition 2.1. (Katznelson [23, Definition I.2.10]) A *homogeneous Banach space* (abbreviated *HBS*) $B(\mathbb{T})$ on $\mathbb{T}$ (or of 2π-periodic functions) is a linear subspace of $L(\mathbb{T})$ having a norm $\|\circ\|_{B(\mathbb{T})}$ under which it is a Banach space such that:

(a) The translation is an isometry of $B(\mathbb{T})$ onto itself, i.e. for $f_t(x) := f(x-t)$, where $f \in B(\mathbb{T})$ and $t \in \mathbb{T}$, there hold $f_t \in B(\mathbb{T})$ and $\|f_t\|_{B(\mathbb{T})} = \|f\|_{B(\mathbb{T})}$;

(b) The translation is continuous on $B(\mathbb{T})$, i.e. for all $f \in B(\mathbb{T})$ and $t, t_0 \in \mathbb{T}$ there holds $\lim_{t\to t_0} \|f_t - f_{t_0}\|_{B(\mathbb{T})} = 0$;

(c) $B(\mathbb{T})$ is continuously embedded in $L(\mathbb{T})$, i.e. there exists a constant b such that for all $f \in B(\mathbb{T})$ there holds $\|f\|_{L(\mathbb{T})} \le b\|f\|_{B(\mathbb{T})}$.

The convolution of $f, g \in L(\mathbb{T})$ is defined by

$$f * g(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)\,dt, \quad x \in [-\pi, \pi).$$

Let $B(\mathbb{T})$ be a HBS of 2π periodic functions, $f \in L(\mathbb{T})$ and $g \in B(\mathbb{T})$. Then, as it was established in [30, Theorem 9.3.2.3] (or see [23, Chapter I, Section 2, Problem 13]), $f * g(x)$ is defined a.e. as the integral is the Lebesgue one, belongs to $B(\mathbb{T})$ and

$$\|f * g\|_{B(\mathbb{T})} \le \|f\|_{L(\mathbb{T})}\|g\|_{B(\mathbb{T})}. \tag{2.1}$$

We will consider approximation operators formed by means of the convolution with functions from a periodic approximate identity. We recall its definition.

Definition 2.2. (e.g. [12, Definitions 1.1.1 and 1.1.4] or [23, Definition I.2.2]) The family of functions $\{k_n(t)\}_{n\in\mathbb{N}_+}$ is called a *periodic approximate identity* if it satisfies the conditions:

(a) For all $n \in \mathbb{N}_+$ we have $k_n \in L(\mathbb{T})$ and

$$\frac{1}{2\pi}\int_{\mathbb{T}} k_n(t)\,dt = 1;$$

(b) There exists a constant M such that

$$\|k_n\|_{L(\mathbb{T})} \le M \quad \forall n \in \mathbb{N}_+;$$

(c) For each $0 < \delta < \pi$, there holds

$$\lim_{n\to\infty}\int_{\{t:\delta\le|t|\le\pi\}} |k_n(t)|\,dt = 0.$$

The function $k_n(t)$ is called a *kernel*.

Let $L(\mathbb{R})$ denote the Banach space of the Lebesgue integrable functions on $\mathbb{R}$, equipped with the norm

$$\|f\|_{L(\mathbb{R})} := \int_{\mathbb{R}} |f(x)|\,dx.$$

Each function $k \in L(\mathbb{R})$ such that

$$\int_{\mathbb{R}} k(t)\,dt = 1$$

generates a kernel k_n, $n \in \mathbb{N}_+$, on $\mathbb{T}$, which satisfies Definition 2.2, by (see [12, Proposition 3.1.12] or [23, VI.1.15])

$$k_n(t) := 2\pi\sum_{j=-\infty}^{\infty} nk(n(t+2j\pi)). \tag{2.2}$$

Given a periodic approximate identity $\{k_n(t)\}_{n\in\mathbb{N}_+}$, we consider the bounded linear operators of $J_n : B(\mathbb{T}) \to B(\mathbb{T})$, $n \in \mathbb{N}_+$, defined by

$$J_n f(x) := k_n * f(x), \quad x \in [-\pi, \pi).$$

By virtue of Young's inequality (2.1) and Definition 2.2(b), we have

$$\|k_n * f\|_{B(\mathbb{T})} \le \|k_n\|_{L(\mathbb{T})}\|f\|_{B(\mathbb{T})} \le M\|f\|_{B(\mathbb{T})} \quad \forall n;$$

thus $\{J_n\}_{n\in\mathbb{N}_+}$ is a family of uniformly bounded linear operators mapping $B(\mathbb{T})$ into $B(\mathbb{T})$. In addition, we have (see [23, Theorem I.2.11], [12, Theorems 1.1.5] and [38, Section 1.3]; cf. [27, Theorem 5] and [28, p. 67])

$$\text{s-}\lim_{n\to\infty} J_n f = f \quad \forall f \in B(\mathbb{T}).$$

2.2 Functions defined on the real line

Similarly to the periodic case, a class of Banach spaces which naturally generalizes the classical $L_p(\mathbb{R})$, $1 \le p < \infty$, and the space of uniformly continuous and bounded functions on $\mathbb{R}$ can be introduced.

Definition 2.3. (Shapiro [30, Definition 9.3.1.1]) A *homogeneous Banach space* (abbreviated *HBS*) $B(\mathbb{R})$ *on* $\mathbb{R}$ is a Banach space of Lebesgue measurable real or complex-valued functions on $\mathbb{R}$ with norm $\|\circ\|_{B(\mathbb{R})}$, satisfying the conditions:

(a) The translation is an isometry of $B(\mathbb{R})$ onto itself, i.e. for $f_t(x) := f(x-t)$, where $f \in B(\mathbb{R})$ and $t \in \mathbb{R}$, there hold $f_t \in B(\mathbb{R})$ and $\|f_t\|_{B(\mathbb{R})} = \|f\|_{B(\mathbb{R})}$;

(b) The translation is continuous on $B(\mathbb{R})$, i.e. for all $f \in B(\mathbb{R})$ and $t, t_0 \in \mathbb{R}$ there holds $\lim_{t\to t_0} \|f_t - f_{t_0}\|_{B(\mathbb{R})} = 0$;

(c) The functions of $B(\mathbb{R})$ are locally Lebesgue integrable, as there exists a constant b such that for all $f \in B(\mathbb{R})$ there holds

$$\int_0^1 |f(x)|\,dx \le b\,\|f\|_{B(\mathbb{R})}.$$

Let $B(\mathbb{R})$ be a HBS on $\mathbb{R}$. The convolution of $f \in L(\mathbb{R})$ and $g \in B(\mathbb{R})$ is defined by

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy,$$

as the integral is the Lebesgue one. As it was established in [30, Theorem 9.3.2.3], $f * g(x)$ is defined a.e., belongs to $B(\mathbb{R})$ and

$$\|f * g\|_{B(\mathbb{R})} \le \|f\|_{L(\mathbb{R})}\,\|g\|_{B(\mathbb{R})}. \tag{2.3}$$

As in the periodic case, we will consider approximation operators formed by means of the convolution with functions from an approximate identity. Its definition is similar. Let $\mathbb{R}_+ := (0, \infty)$.

Definition 2.4. (e.g. [12, Definitions 3.1.1 and 3.1.4]) The family $\{k_\rho(t)\}_{\rho\in\mathbb{R}_+}$ is called an *approximate identity on* $\mathbb{R}$ if it satisfies the conditions:

(a) For all $\rho > 0$ we have $k_\rho \in L(\mathbb{R})$ and

$$\int_{\mathbb{R}} k_\rho(t)\,dt = 1;$$

(b) There exists a constant M such that

$$\|k_\rho\|_{L(\mathbb{R})} \le M \quad \forall \rho > 0;$$

(c) For each $\delta > 0$, there holds

$$\lim_{\rho\to\infty} \int_{\{t:|t|\ge\delta\}} |k_\rho(t)|\,dt = 0.$$

The function $k_\rho(t)$ is called a *kernel*. More generally, the parameter ρ can vary in some set $R \subseteq \mathbb{R}$, which is not bounded above.

An easy and important for the application way to generate an approximate identity on $\mathbb{R}$ is the following. Let $k \in L(\mathbb{R})$ be such that

$$\int_{\mathbb{R}} k(t)\,dt = 1.$$

Then, as it can be straightforwardly seen, $k_\rho(t) := \rho k(\rho t)$, $\rho > 0$, defines an approximate identity on $\mathbb{R}$.

Just as in the periodic case, given an approximate identity $\{k_\rho(t)\}_{\rho\in\mathbb{R}_+}$ on $\mathbb{R}$, we can define the bounded linear operators $J_\rho : B(\mathbb{R}) \to B(\mathbb{R})$, $\rho > 0$, defined by

$$J_\rho f(x) := k_\rho * f(x), \quad x \in \mathbb{R}.$$

By virtue of Young's inequality (2.3) and Definition 2.4(b), we have

$$\|k_\rho * f\|_{B(\mathbb{R})} \le \|k_\rho\|_{L(\mathbb{R})}\,\|f\|_{B(\mathbb{R})} \le M\|f\|_{B(\mathbb{R})} \quad \forall \rho > 0;$$

thus $\{J_\rho\}_{\rho\in\mathbb{R}_+}$ is a family of uniformly bounded linear operators mapping $B(\mathbb{R})$ into $B(\mathbb{R})$. In addition, we have (see [12, Theorems 1.1.5 and 3.1.6], [23, Lemma I.2.2 and Theorem I.2.11], [30, Corollary 9.2.4.1 and Section 9.3.3] and [38, Section 1.3])

$$\text{s-}\lim_{\rho\to\infty} J_\rho f = f \quad \forall f \in B(\mathbb{R}). \tag{2.4}$$

3 Voronovskaya-type estimates

We will demonstrate that Voronovskaya-type estimates for convolution operators in homogeneous Banach spaces follow easily form properties of the Fourier transform of its kernel.

For $f \in L(\mathbb{T})$ its Fourier coefficients are given by

$$\hat{f}(m) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx, \quad m \in \mathbb{Z},$$

and for $f \in L(\mathbb{R})$ its Fourier transform by

$$\hat{f}(u) := \int_{\mathbb{R}} f(x)e^{-iux}dx, \quad u \in \mathbb{R}.$$

The theorems below are application of the simple idea on which [18, Theorems 1.1 and 1.2] are based. This approach was used in [18] to derive sufficient conditions that imply direct and one-term strong converse inequalities by means of K-functionals for convolution operators in homogeneous Banach spaces. These conditions concern the Fourier transform of the kernel of the convolution operator. They were further considered in [19, 20].

3.1 Periodic functions

Theorem 3.1. *Let $B(\mathbb{T})$ be a HBS of periodic functions, $\{k_n\}_{n\in\mathbb{N}_+}$ satisfy Definition 2.2 and*

$$J_n f := k_n * f, \quad f \in B(\mathbb{T}), \quad n \in \mathbb{N}_+.$$

Let $D, G : Y \to B(\mathbb{T})$, where $Y \subseteq B(\mathbb{T})$, be such that for every $f \in Y$ there hold:

$$\widehat{Df}(m) = \psi(m)\hat{f}(m), \quad m \in \mathbb{Z}, \tag{3.1}$$

and

$$\widehat{Gf}(m) = \theta(m)\hat{f}(m), \quad m \in \mathbb{Z}, \tag{3.2}$$

with some $\psi, \theta : \mathbb{Z} \to \mathbb{C}$.

Finally, let there exist $\varphi, \tau : \mathbb{N}_+ \to \mathbb{R}_+$ and $\ell_n \in L(\mathbb{T})$, $n \in \mathbb{N}_+$, such that

$$\widehat{k_n}(m) - 1 - \varphi(n)\psi(m) = \tau(n)\theta(m)\widehat{\ell_n}(m), \quad m \in \mathbb{Z}, \tag{3.3}$$

and

$$\|\ell_n\|_{L(\mathbb{T})} \le L, \quad n \in \mathbb{N}_+, \tag{3.4}$$

with some absolute constant $L > 0$.

Then for all $f \in Y$ and $n \in \mathbb{N}_+$ there holds

$$\|J_n f - f - \varphi(n)Df\|_{B(\mathbb{T})} \le L\,|\tau(n)|\,\|Gf\|_{B(\mathbb{T})}.$$

Proof. Equation (3.3) imlpies for any $f \in Y$

$$\widehat{k_n}(m)\hat{f}(m) - \hat{f}(m) - \varphi(n)\psi(m)\hat{f}(m) = \tau(n)\widehat{\ell_n}(m)\theta(m)\hat{f}(m), \quad m \in \mathbb{Z};$$

hence by virtue of the formula $\widehat{g * h} = \hat{g}\hat{h}$, $g, h \in L(\mathbb{T})$, and (3.1)-(3.2), we arrive at

$$(k_n * f - f - \varphi(n)Df)\widehat{\ }(m) = \tau(n)(\ell_n * Gf)\widehat{\ }(m), \quad m \in \mathbb{Z}.$$

In view of the uniqueness of the Fourier coefficients, the last relation yields

$$k_n * f(x) - f(x) - \varphi(n)Df(x) = \tau(n)\ell_n * Gf(x), \quad x \in \mathbb{T}.$$

(Certainly, the above equality generally holds a.e.)

Consequently, taking into account Young's inequality (2.1) and (3.4), we get the assertion of the theorem. □

In the case of approximate identity with a kernel of the form (2.2), the statement of the theorem is more concise.

Corollary 3.2. *Let $B(\mathbb{T})$ be a HBS of periodic functions, $k \in L(\mathbb{R})$ with*

$$\int_{\mathbb{R}} k(t)\,dt = 1,$$

k_n be defined by (2.2), and

$$J_n f := k_n * f, \quad f \in B(\mathbb{T}), \quad n \in \mathbb{N}_+.$$

Let $D, G : Y \to B(\mathbb{T})$, where $Y \subseteq B(\mathbb{T})$, be such that for every $f \in Y$ there hold:

$$\widehat{Df}(m) = \psi(m)\hat{f}(m), \quad m \in \mathbb{Z},$$

and

$$\widehat{Gf}(m) = \theta(m)\hat{f}(m), \quad m \in \mathbb{Z},$$

where $\psi, \theta : \mathbb{R} \to \mathbb{C}$ satisfy the relations

$$\psi(nu) = n^{\alpha}\psi(u) \quad \text{and} \quad \theta(nu) = n^{\beta}\theta(u), \quad n \in \mathbb{N}_+, \quad u \in \mathbb{R}, \tag{3.5}$$

with some $\alpha, \beta \in \mathbb{R}$.

Finally, let there exist $\ell \in L(\mathbb{R})$ such that

$$\hat{k}(u) - 1 - \psi(u) = \theta(u)\hat{\ell}(u), \quad u \in \mathbb{R}. \tag{3.6}$$

Then for all $f \in Y$ and $n \in \mathbb{N}_+$ there holds

$$\left\| J_n f - f - \frac{1}{n^{\alpha}} Df \right\|_{B(\mathbb{T})} \le \frac{\|\ell\|_{L(\mathbb{R})}}{n^{\beta}} \|Gf\|_{B(\mathbb{T})}.$$

Proof. We set

$$\ell_n(t) := 2\pi \sum_{j=-\infty}^{\infty} n\ell(n(t + 2j\pi)).$$

By [12, Proposition 5.1.28], we have $\widehat{k_n}(m) = \hat{k}(m/n)$ and $\widehat{\ell_n}(m) = \hat{\ell}(m/n)$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}_+$.

Therefore, (3.5) and (3.6) with $u = m/n$ yield (3.3) with $\varphi(n) := n^{-\alpha}$ and $\tau(n) := n^{-\beta}$.

By [12, (3.1.24), (i)], we have $\|\ell_n\|_{L(\mathbb{T})} \le \|\ell\|_{L(\mathbb{R})}$ for all $n \in \mathbb{N}_+$; thus (3.4) holds with $L = \|\ell\|_{L(\mathbb{R})}$

Now, the assertion of the corollary follows from Theorem 3.1. □

3.2 Functions defined on the real line

Let B be a linear space. For an operator D defined on a subset of B with values in B, we set $D^{-1}(B) := \{f \in B : Df \in B\}$.

Let $B(\mathbb{R})$ be a HBS on $\mathbb{R}$ and D be an operator, which is defined on subsets of $B(\mathbb{R})$ and $L(\mathbb{R})$. If $B(\mathbb{R}) \neq L(\mathbb{R})$, we still denote the action of the operator D on elements of $B(\mathbb{R})$ and $L(\mathbb{R})$ by the same letter. We can do so because usually in the applications D is defined on every HBS on $\mathbb{R}$ by the same operation. We say that *D commutes with the convolution* if

$$Df * g = f * Dg \quad \forall f \in D^{-1}(B(\mathbb{R})) \quad \forall g \in D^{-1}(L(\mathbb{R})). \tag{3.7}$$

Theorem 3.3. *Let $B(\mathbb{R})$ be a HBS on $\mathbb{R}$, $\{k_\rho\}_{\rho \in \mathbb{R}_+}$ satisfy Definition 2.4 and*

$$J_\rho f := k_\rho * f, \quad f \in B(\mathbb{R}), \quad \rho > 0.$$

Let $D, G : Y \to B(\mathbb{R})$, where $Y \subseteq B(\mathbb{R})$, and $D, G : Y_L \to L(\mathbb{R})$, where $Y_L \subseteq L(\mathbb{R})$ is dense in $L(\mathbb{R})$. In addition, let each of the operators D and G commutes with the convolution in the sense of (3.7). Further, let for every $\eta \in Y_L$ there hold:

$$\widehat{D\eta}(u) = \psi(u)\hat{\eta}(u), \quad u \in \mathbb{R}, \tag{3.8}$$

and

$$\widehat{G\eta}(u) = \theta(u)\hat{\eta}(u), \quad u \in \mathbb{R}, \tag{3.9}$$

with some $\psi, \theta : \mathbb{R} \to \mathbb{C}$.

Finally, let there exist $\varphi, \tau : \mathbb{R}_+ \to \mathbb{R}_+$ and $\ell_\rho \in L(\mathbb{R})$, $\rho > 0$, such that

$$\widehat{k_\rho}(u) - 1 - \varphi(\rho)\psi(u) = \tau(\rho)\theta(u)\widehat{\ell_\rho}(u), \quad u \in \mathbb{R}, \tag{3.10}$$

and

$$\|\ell_\rho\|_{L(\mathbb{R})} \le L, \quad \rho > 0,$$

with some absolute constant $L > 0$.

Then for all $f \in Y$ and $\rho > 0$ there holds

$$\|J_\rho f - f - \varphi(\rho)Df\|_{B(\mathbb{R})} \le L\,|\tau(\rho)|\,\|Gf\|_{B(\mathbb{R})}.$$

Proof. Just as in the proof of Theorem 3.1 we derive from (3.8)-(3.10) the relation

$$k_\rho * \eta(x) - \eta(x) - \varphi(\rho) D\eta(x) = \tau(\rho)\ell_\rho * G\eta(x) \quad \text{a.e. in } \mathbb{R}$$

for every $\eta \in Y_L$.

Consequently, for all $\eta \in Y_L$ and $f \in Y$ there holds

$$(k_\rho * \eta) * f(x) - \eta * f(x) - \varphi(\rho)(D\eta) * f(x) = \tau(\rho)(\ell_\rho * G\eta) * f(x) \quad \text{a.e. in } \mathbb{R}.$$

Taking into account that D and G commute with the convolution, we arrive at

$$\eta * \big(k_\rho * f - f - \varphi(\rho) Df\big)(x) = \eta * (\tau(\rho)\ell_\rho * Gf)(x) \quad \text{a.e. in } \mathbb{R} \tag{3.11}$$

for all $\eta \in Y_L$ and $f \in Y$.

Since Y_L is dense in $L(\mathbb{R})$, (3.11) and Young's inequality (2.3) imply

$$\eta * \big(k_\rho * f - f - \varphi(\rho) Df\big)(x) = \eta * (\tau(\rho)\ell_\rho * Gf)(x) \quad \text{a.e. in } \mathbb{R}$$

for all $\eta \in L(\mathbb{R})$ and $f \in Y$. In particular, we have

$$k_\sigma * \big(k_\rho * f - f - \varphi(\rho) Df - \tau(\rho)\ell_\rho * Gf\big)(x) = 0 \quad \text{a.e. in } \mathbb{R}$$

for any $\sigma > 0$, that is,

$$J_\sigma\big(k_\rho * f - f - \varphi(\rho) Df - \tau(\rho)\ell_\rho * Gf\big) = 0 \quad \forall \sigma > 0.$$

Now, (2.4) with σ in place of ρ yields

$$k_\rho * f - f - \varphi(\rho) Df = \tau(\rho)\ell_\rho * Gf; \tag{3.12}$$

hence the assertion of the theorem follows by virtue of (2.3). □

Remark 1. The elements of any HBS on $\mathbb{R}$ are tempered distributions (see [30, pp. 207–208] or [20, pp. 12–13]). Therefore, if ψ is infinitely many times continuously differentiable on $\mathbb{R}$ and of most polynomial growth at infinity, the last theorem can be stated and verified quite similarly to Theorem 3.1 using the basics of the Fourier analysis for tempered distributions.

Similarly, to the periodic case we get the following form of the assertion of Theorem 3.3.

Corollary 3.4. *Let $B(\mathbb{R})$ be a HBS on $\mathbb{R}$, $k \in L(\mathbb{R})$ with*

$$\int_{\mathbb{R}} k(t)\,dt = 1$$

$k_\rho(t) := \rho k(\rho t)$, $t \in \mathbb{R}$, and

$$J_\rho f := k_\rho * f, \quad f \in B(\mathbb{R}), \quad \rho > 0.$$

Let $D, G : Y \to B(\mathbb{R})$, where $Y \subseteq B(\mathbb{R})$, and $D, G : Y_L \to L(\mathbb{R})$, where $Y_L \subseteq L(\mathbb{R})$ is dense in $L(\mathbb{R})$. In addition, let each of the operators D and G commutes with the convolution in the sense of (3.7). Further, let for every $\eta \in Y_L$ there hold:

$$\widehat{D\eta}(u) = \psi(u)\hat{\eta}(u), \quad u \in \mathbb{R}, \tag{3.13}$$

and

$$\widehat{G\eta}(u) = \theta(u)\hat{\eta}(u), \quad u \in \mathbb{R}, \tag{3.14}$$

where $\psi, \theta : \mathbb{R} \to \mathbb{C}$ satisfy the relations

$$\psi(\rho u) = \rho^\alpha \psi(u) \quad \text{and} \quad \theta(\rho u) = \rho^\beta \theta(u), \quad \rho > 0, \quad u \in \mathbb{R}, \tag{3.15}$$

with some $\alpha, \beta \in \mathbb{R}$.

Finally, let there exist $\ell \in L(\mathbb{R})$, $\rho > 0$, such that

$$\hat{k}(u) - 1 - \psi(u) = \theta(u)\hat{\ell}(u), \quad u \in \mathbb{R}. \tag{3.16}$$

Then for all $f \in Y$ and $\rho > 0$ there holds

$$\left\| J_\rho f - f - \frac{1}{\rho^\alpha} Df \right\|_{B(\mathbb{R})} \le \frac{\|\ell\|_{L(\mathbb{R})}}{\rho^\beta} \|Gf\|_{B(\mathbb{R})}.$$

Proof. The assertion of the corollary is derived from Theorem 3.3 just as Corollary 3.2 from Theorem 3.1, as now we set $\ell_\rho(t) := \rho\ell(\rho t)$ and use that $\widehat{k_\rho}(u) = \hat{k}(u/\rho)$ and $\widehat{\ell_\rho}(u) = \hat{\ell}(u/\rho)$. □

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4 Examples

4.1 The Riesz means

As the first example let us consider the Riesz typical means of the Fourier series of $f \in B(\mathbb{T})$. They are defined for $f \in B(\mathbb{T})$ and $n \in \mathbb{N}_0$ by

$$R_{\alpha,n}f(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} r_{\alpha,n}(t)f(x-t)\,dt, \quad x \in [-\pi,\pi),$$

where the kernel $r_{\alpha,n}$, $\alpha > 0$, is given by

$$r_{\alpha,n}(t) := \sum_{k=-n}^{n}\left(1-\left|\frac{k}{n+1}\right|^{\alpha}\right)e^{ikt}.$$

For $\alpha = 1$ we have the Fejér means.

In case when f is real-valued, we usually represent $R_{\alpha,n}$ in the form

$$R_{\alpha,n}f(x) = \sum_{k=0}^{n}\left(1-\left(\frac{k}{n+1}\right)^{\alpha}\right)A_k(f,x),$$

where

$$A_0(f,x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt,$$
$$A_k(f,x) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt\cos kx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt\sin kx, \quad k > 0.$$

The differential operator that is naturally associated with the Riesz typical means is the Riesz derivative.

Following its definition in $L_p(\mathbb{T})$, $1 \le p < \infty$, and $C(\mathbb{T})$ (see e.g. [12, Definition 11.5.10]), we can introduce it in any HBS of periodic functions in the following way.

Definition 4.1. Let $B(\mathbb{T})$ be a HBS of 2π-periodic functions, $f \in B(\mathbb{T})$ and $\alpha > 0$. If there exists $g \in B(\mathbb{T})$ such that $|m|^{\alpha}\hat{f}(m) = \hat{g}(m)$, $m \in \mathbb{Z}$, then g is called the Riesz derivative of f of order α and is denoted by $D^{\{\alpha\}}f$.

We set

$$W^{\{\alpha\}}(B(\mathbb{T})) := \{f \in B(\mathbb{T}) : \exists D^{\{\beta\}}f \in B(\mathbb{T}),\ 0 < \beta \le \alpha\}.$$

The optimal approximation order of $R_{\alpha,n}$ in $L_p(\mathbb{T})$, $1 \le p < \infty$, and $C(\mathbb{T})$ is $n^{-\alpha}$ (see [12, Proposition 12.2.7]). Further, the limit

$$\text{s-}\lim_{n\to\infty} n^{\alpha}(R_{\alpha,n}f - f)$$

exists in the norm of $L_p(\mathbb{T})$, $1 \le p < \infty$, or $C(\mathbb{T})$ iff $D^{\{\alpha\}}f \in L_p(\mathbb{T})$, or $D^{\{\alpha\}}f \in C(\mathbb{T})$, respectively, as in this case the limit is $-D^{\{\alpha\}}f$ (see [12, Theorem 12.2.6 and pp. 445–446]).

General assertions that describe the operator defined by

$$\text{s-}\lim_{n\to\infty}\frac{J_n f - f}{\varphi(n)},$$

where J_n is a convolution operator on a HBS on $\mathbb{T}$ and $\lim_{n\to\infty}\varphi(n) = 0$, were stated in [19, Theorem 4.1 and Propositions 6.2 and 6.4] by means of the Fourier transform.

A characterization of the rate of convergence of $R_{\alpha,n}$ by a K-functional defined by a slight modification of $D^{\{\alpha\}}$ in any Banach space of periodic functions, satisfying Definition 2.1, (a) and (c), was established in [17, Theorem 2.2] (see also [18, Theorem 5.2]).

The following Voronvskaya-type inequality holds true.

Theorem 4.1. *Let $B(\mathbb{T})$ be a HBS of 2π-periodic functions, and $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha < \beta$. Then for all $f \in W^{\{\beta\}}(B(\mathbb{T}))$ and all $n \in \mathbb{N}_0$ there holds*

$$\left\| R_{\alpha,n}f - f + \frac{1}{(n+1)^{\alpha}}D^{\{\alpha\}}f \right\|_{B(\mathbb{T})} \le \frac{2+8\beta-4\alpha}{(n+1)^{\beta}}\,\|D^{\{\beta\}}f\|_{B(\mathbb{T})}. \tag{4.1}$$

Remark 2. Let us note that, by virtue of [18, Theorem 5.2], the operator $R_{\alpha,n}$ is saturated with saturation rate of $n^{-\alpha}$ in any HBS of periodic functions, that is, $\|R_{\alpha,n}f - f\|_{B(\mathbb{T})}$ cannot tend to 0 faster than $n^{-\alpha}$ as $n \to \infty$ unless we have the trivial situation $R_{\alpha,n}f = f$ for all n, that is, $f = \text{const}$.

On the other hand, inequality (4.1) can be written in the form

$$\|(n+1)^{\alpha}(R_{\alpha,n}f - f) + D^{\{\alpha\}}f\|_{B(\mathbb{T})} \le \frac{2+8\beta-4\alpha}{(n+1)^{\beta-\alpha}}\,\|D^{\{\beta\}}f\|_{B(\mathbb{T})},$$

which shows that, if we set $V_{\alpha,n}f := (n+1)^{\alpha}(R_{\alpha,n}f - f)$ for $f \in W^{\{\alpha\}}(\mathbb{T})$, then the error in norm $\|V_{\alpha,n}f + D^{\{\alpha\}}f\|_{B(\mathbb{T})}$ can tend to 0 as fast as any negative power of n as $n \to \infty$, provided that f possesses a Riesz derivative of order, which is high enough.

Bustamante and Flores-de-Jesús [6, Theorem 4.5 (see also Corollary and 4.1Theorem 4.6)] established a Voronovskaya-type estimate for the Fejér means $R_{1,n}$ in $L_p(\mathbb{T})$, $1 \le p < \infty$, and $C(\mathbb{T})$ with explicit constants in terms of the best approximation of f and its conjugate. This result implies, among others, the property we described above in the case $\alpha = 1$.

In the proof of Theorem 4.1, we will make use of the criterion stated below (see e.g. [12, Corollary 6.3.9 and (6.3.9)], or [23, Theorem I.4.1 and its proof]). For a function v on $\mathbb{Z}$, we set $\Delta \mathrm{v}(m) := \mathrm{v}(m+1) - \mathrm{v}(m)$, $m \in \mathbb{Z}$, and

$$\Delta^2 \mathrm{v}(m) := \Delta(\Delta \mathrm{v})(m) = \mathrm{v}(m+2) - 2\mathrm{v}(m+1) + \mathrm{v}(m), \quad m \in \mathbb{Z}.$$

Theorem A. *If v_n, $n \in \mathbb{N}_0$, are even functions on $\mathbb{Z}$ such that $\lim_{m\to\infty} \mathrm{v}_n(m) = 0$ and there exists $M > 0$ such that*

$$\sum_{m=0}^{\infty}(m+1)|\Delta^2 \mathrm{v}_n(m)| \le M \quad \forall\, n \in \mathbb{N}_0,$$

then there exist (even) functions $v_n \in L(\mathbb{T})$ such that $\hat{v}_n(m) = \mathrm{v}_n(m)$, $m \in \mathbb{Z}$, and $\|v_n\|_{L(\mathbb{T})} \le M$ for all $n \in \mathbb{N}_0$.

Proof of Theorem 4.1. We apply Theorem 3.1 with $k_n(t) := r_{\alpha,n}(t)$, $Y := W^{\{\beta\}}(\mathbb{T})$, $D := -D^{\{\alpha\}}$, $G := D^{\{\beta\}}$, $\varphi(n) := (n+1)^{-\alpha}$ and $\tau(n) := (n+1)^{-\beta}$.

The definition of the Riesz derivative (Definition 4.1) shows that (3.1) and (3.2) are satisfied with $\psi(m) := -|m|^{\alpha}$ and $\theta(m) := |m|^{\beta}$.

Since for $m \in \mathbb{Z}$

$$\widehat{r_{\alpha,n}}(m) - 1 + \frac{|m|^{\alpha}}{(n+1)^{\alpha}} = \frac{|m|^{\beta}}{(n+1)^{\beta}} \begin{cases} 0, & |m| \le n, \\ \left|\dfrac{n+1}{m}\right|^{\beta-\alpha} - \left|\dfrac{n+1}{m}\right|^{\beta}, & |m| > n, \end{cases}$$

in order to verify (3.3) and (3.4), we need to show that there exist functions $\ell_n \in L(\mathbb{T})$, $n \in \mathbb{N}_0$, such that

$$\widehat{\ell_n}(m) = \begin{cases} 0, & |m| \le n, \\ \left|\dfrac{n+1}{m}\right|^{\beta-\alpha} - \left|\dfrac{n+1}{m}\right|^{\beta}, & |m| > n, \end{cases} \tag{4.2}$$

and $\|\ell_n\|_{L(\mathbb{T})}$ is bounded on n.

DeVore [14, p. 67–68] showed that for any $\alpha > 0$ the sequence

$$y_n(m) := \begin{cases} 1, & |m| \le n, \\ \left|\dfrac{n+1}{m}\right|^{\alpha}, & |m| > n, \end{cases}$$

satisfies the assumptions of Theorem A; hence there exist functions $v_n \in L(\mathbb{T})$, $n \in \mathbb{N}_0$, such that $\widehat{v_n}(m) = y_n(m)$, $m \in \mathbb{Z}$, and $\|v_n\|_{L(\mathbb{T})} \le M$ for all n with some positive constant M. Using this result, respectively, with β and $\beta - \alpha$ in place of α, we deduce the existence of functions $a_n, b_n \in L(\mathbb{T})$, $n \in \mathbb{N}_0$, such that

$$\widehat{a_n}(m) = \begin{cases} 1, & |m| \le n, \\ \left|\dfrac{n+1}{m}\right|^{\beta}, & |m| > n, \end{cases} \quad \text{and} \quad \widehat{b_n}(m) = \begin{cases} 1, & |m| \le n, \\ \left|\dfrac{n+1}{m}\right|^{\beta-\alpha}, & |m| > n, \end{cases} \tag{4.3}$$

as, moreover, $\|a_n\|_{L(\mathbb{T})} \le M_1$ and $\|b_n\|_{L(\mathbb{T})} \le M_2$ for all n with some positive constants M_1 and M_2.

Setting $\ell_n := b_n - a_n$, we get functions in $L(\mathbb{T})$ with uniformly bounded $L(\mathbb{T})$-norms, whose Fourier coefficients satisfy (4.2).

However, since we are interested in getting a concrete and preferably small upper bound of $\|\ell_n\|_{L(\mathbb{T})}$, we will carry out the calculations again with some alternations.

Let $\gamma > 0$. We set for $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$

$$\mathrm{v}_n(m) := \begin{cases} 1, & |m| \le n, \\ \left|\dfrac{n+1}{m}\right|^{\gamma}, & |m| > n. \end{cases}$$

Since the function $x^{-\gamma}$ is convex on $[1, \infty)$, then $\Delta^2 \mathrm{v}_n(m) \ge 0$ for all $m > n$; hence for any $N \in \mathbb{N}_+$ with $N \ge n+1$ there holds

$$\begin{aligned} \sum_{m=n+1}^{N} (m+1)|\Delta^2 \mathrm{v}_n(m)| &= \sum_{m=n+1}^{N} (m+1)\big(\mathrm{v}_n(m+2) - 2\mathrm{v}_n(m+1) + \mathrm{v}_n(m)\big) \\ &= (N+1)\mathrm{v}_n(N+2) - (N+2)\mathrm{v}_n(N+1) - (n+1)\mathrm{v}_n(n+2) + (n+2)\mathrm{v}_n(n+1) \\ &\le 1 + (n+1)\left(1 - \left(\frac{n+1}{n+2}\right)^{\gamma}\right). \end{aligned}$$

Thus, we have

$$\sum_{m=n+1}^{\infty}(m+1)|\Delta^2 \mathrm{v}_n(m)| \le 1+(n+1)\left(1-\left(\frac{n+1}{n+2}\right)^{\gamma}\right), \quad n\in\mathbb{N}_0. \tag{4.4}$$

Next, we calculate

$$\begin{aligned} &\Delta^2 \mathrm{v}_n(m)=0, \quad 0\le m\le n-1, \quad n\ge 1,\\ &|\Delta^2 \mathrm{v}_n(n)| = 1-\left(\frac{n+1}{n+2}\right)^{\gamma}. \end{aligned} \tag{4.5}$$

Relations (4.4)-(4.5) yield

$$\sum_{m=0}^{\infty}(m+1)|\Delta^2 \mathrm{v}_n(m)| \le 1+2(n+1)\left(1-\left(\frac{n+1}{n+2}\right)^{\gamma}\right). \tag{4.6}$$

By elementary calculus we derive the estimate

$$1-x^{\gamma}\le c_{\gamma}(1-x), \quad x\in\left[\frac{1}{2},1\right],$$

where

$$c_{\gamma} := \begin{cases} 2\left(1-2^{-\gamma}\right)\le 2\gamma, & 0<\gamma<1,\\ \gamma, & \gamma\ge 1. \end{cases} \tag{4.7}$$

We apply it with $x=(n+1)/(n+2)$ in (4.6) to arrive at

$$\sum_{m=0}^{\infty}(m+1)|\Delta^2 \mathrm{v}_n(m)| \le 1+4\gamma, \quad n\in\mathbb{N}_0. \tag{4.8}$$

Now, by virtue of Theorem A and (4.8) with $\gamma=\beta$ and $\gamma=\beta-\alpha$, we get

$$\|a_n\|_{L(\mathbb{T})}\le 1+4\beta \quad \text{and} \quad \|b_n\|_{L(\mathbb{T})}\le 1+4(\beta-\alpha), \quad n\in\mathbb{N}_0,$$

where $a_n, b_n\in L(\mathbb{T})$, $n\in\mathbb{N}_0$, satisfy (4.3).

Thus, as we observed earlier, the Fourier coefficients of the functions $\ell_n := b_n - a_n \in L(\mathbb{T})$, $n\in\mathbb{N}_0$, are given by (4.2) and

$$\|\ell_n\|_{L(\mathbb{T})}\le \|a_n\|_{L(\mathbb{T})}+\|b_n\|_{L(\mathbb{T})}\le 2+8\beta-4\alpha, \quad n\in\mathbb{N}_0.$$

The assertion of the theorem follows from Theorem 3.1. □

Remark 3. In the proof of Theorem 4.1 we actually established that

$$\left\|R_{\alpha,n}f - f + \frac{1}{(n+1)^{\alpha}}D^{\{\alpha\}}f\right\|_{B(\mathbb{T})} \le \frac{2(1+c_{\beta}+c_{\beta-\alpha})}{(n+1)^{\beta}}\,\|D^{\{\beta\}}f\|_{B(\mathbb{T})},$$

for all $f\in W^{\beta}(B(\mathbb{T}))$ and $n\in\mathbb{N}_0$, where $0<\alpha<\beta$ and c_{γ} is defined in (4.7).

Let us explicitly state the Voronovskaya-type estimate for the the Fejér means $\sigma_n := R_{1,n}$.

Corollary 4.2. *Let $B(\mathbb{T})$ be a HBS of 2π-periodic functions, and $\beta>1$. Then for all $f\in W^{\{\beta\}}(B(\mathbb{T}))$ and all $n\in\mathbb{N}_0$ there holds*

$$\left\|\sigma_n f - f + \frac{1}{n+1}D^{\{1\}}f\right\|_{B(\mathbb{T})} \le \frac{8\beta-2}{(n+1)^{\beta}}\,\|D^{\{\beta\}}f\|_{B(\mathbb{T})}.$$

We recall that, if $B(\mathbb{T})=L_p(\mathbb{T})$, $1\le p<\infty$, or $B(\mathbb{T})=C(\mathbb{T})$, then $f\in W^{\{1\}}(B(\mathbb{T})))$ iff $\tilde{f}\in AC(\mathbb{T})$ and $\left(\tilde{f}\right)'\in L_p(\mathbb{T})$, $1\le p<\infty$, or $\left(\tilde{f}\right)'\in C(\mathbb{T})$, respectively, as $D^{\{1\}}f=\left(\tilde{f}\right)'$ (see e.g. [12, Section 9.3, Problem 3]). Here $AC(\mathbb{T})$ denotes the space of the absolutely continuous 2π-periodic functions, and $\tilde{f}$ is the conjugate of f.

Let us mention that Corollary 3.2 can also be used to prove the Voronovskaya-type inequality for $R_{\alpha,n}$ stated in Theorem 4.1.

Besides the results mentioned at the end of Remark 2, a point-wise Voronovskaya-type estimate for $R_{\alpha,n}$ with $\alpha\in\mathbb{N}_+$ was established in [34, § 3]. There a direct inequality and a matching one-term strong converse inequality in the uniform norm by means of the classical periodic modulus of smoothness were established for even α (a characterization for odd α was given, too). A global Voronovskaya-type estimate for the Fejér means in Banach space of periodic functions, satisfying Definition 2.1, (a) and (c), was proved in [17, (11.6)].

4.2 The generalized singular integral of Picard

The generalized singular integral of Picard of the function $f \in B(\mathbb{R})$ is defined by

$$C_{\alpha,\rho} f(x) := \rho \int_{\mathbb{R}} c_\alpha(\rho t) f(x-t)\, dt, \quad x \in \mathbb{R},$$

where the kernel c_α, $\alpha > 0$, is given by its Fourier transform

$$\widehat{c_\alpha}(u) = (1 + |u|^\alpha)^{-1}.$$

For $\alpha = 2$ we get the classical singular integral of Picard. In this case we have $c_2(t) = (1/2)\exp(-|t|)$.

The optimal approximation order of $C_{\alpha,\rho}$ in $L_p(\mathbb{R})$, $1 \le p < \infty$, and in $UCB(\mathbb{R})$—the space of the uniformly continuous and bounded functions on $\mathbb{R}$, is $\varphi(\rho) := \rho^{-\alpha}$ (see [12, Propositions 12.4.2 and 13.2.1, and Section 13.2, Problems 1 and 2]).

The differential operator that is naturally associated with the singular integral of Picard is the Riesz derivative. We will formulate its definition in the strong sense of convergence in the norm of HBSs on $\mathbb{R}$.

For $0 < \alpha < 1$ and $h \in \mathbb{R}$ we set

$$n_{h,\alpha}(x) := \frac{1}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \left(\frac{\operatorname{sgn}(x+h)}{|x+h|^{1-\alpha}} - \frac{\operatorname{sgn} x}{|x|^{1-\alpha}} \right), \quad x \in \mathbb{R},$$

where $\Gamma(\alpha)$ stands for the Gamma function. Then $n_{h,\alpha} \in L(\mathbb{R})$.

Also, for $h \in \mathbb{R}$, $h \ne 0$, we set

$$n_h(x) := \begin{cases} \dfrac{1}{\pi x}, & |x| \ge |h|, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$n_{h,0}(x) := n_h(x+h) - n_h(x), \quad x \in \mathbb{R}.$$

We have $n_{h,0} \in L(\mathbb{R})$.

Let, as usual, $[\alpha]$ denote the largest integer not greater than $\alpha \in \mathbb{R}_+$.

Definition 4.2. Let $B(\mathbb{R})$ be a HBS on $\mathbb{R}$, $f \in B(\mathbb{R})$ and $0 < \alpha \le 1$. If the limit

$$\text{s-}\lim_{h\to 0} \frac{f * n_{h,1-\alpha}}{h}$$

exists in the norm of $B(\mathbb{R})$, then we call it the strong Riesz derivative of order α of f and denote it by $D_s^{\{\alpha\}} f$.

For $\alpha > 1$ the strong Riesz derivative of order α of $f \in B(\mathbb{R})$ is defined inductively by

$$D_s^{\{\alpha\}} f := \begin{cases} D_s^{\{1\}}\left(D_s^{\{\alpha-1\}} f\right), & \alpha \in \mathbb{N}_+, \\ D_s^{\{\alpha-[\alpha]\}}\left(D_s^{\{[\alpha]\}} f\right), & \alpha \notin \mathbb{N}_+. \end{cases}$$

We set

$$W^{\{\alpha\}}(B(\mathbb{R})) := \{ f \in B(\mathbb{R}) : \exists\, D_s^{\{\beta\}} f \in B(\mathbb{R}),\ 0 < \beta \le \alpha \}.$$

As is known (see [12, (10.5.8) and Theorems 11.2.6 and 11.2.9] and [18, Proposition 4.2]), we have that $D_s^{\{\alpha\}} f \in L(\mathbb{R})$ if and only if there exists $g \in L(\mathbb{R})$ such that $|u|^\alpha \hat{f}(u) = \hat{g}(u)$, $u \in \mathbb{R}$. Therefore, the Schwartz space of rapidly decreasing functions on $\mathbb{R}$ is contained in $W^{\{\alpha\}}(L(\mathbb{R}))$ for every $\alpha > 0$; hence $W^{\{\alpha\}}(L(\mathbb{R}))$ is dense in $L(\mathbb{R})$. We also have

$$(D_s^{\{\alpha\}} f)\widehat{\ }(u) = |u|^\alpha \hat{f}(u), \quad f \in W^{\{\alpha\}}(L(\mathbb{R})). \tag{4.9}$$

The operator $D_s^{\{\alpha\}}$ commutes with the convolution. This is so because $D_s^{\{\alpha\}}$ is defined by means of strong convergence and, as Young's inequality (2.3) implies, the convolution is a bounded linear operator on each of its arguments, the other being fixed.

As it follows from [12, Theorem 12.3.11], the limit

$$\text{s-}\lim_{\rho\to\infty} \rho^\alpha (C_{\alpha,\rho} f - f)$$

exists in the norm of $L_p(\mathbb{R})$, $1 < p \le 2$, iff $D_s^{\{\alpha\}} f \in L_p(\mathbb{R})$; in this case the limit is equal to $-D_s^{\{\alpha\}} f$. The spaces $L_p(\mathbb{R})$, $p = 1$ or $2 < p < \infty$, and $UCB(\mathbb{R})$ can be treated by [12, Theorems 12.3.1, Propositions 13.2.1 and 13.2.2, and Section 13.2, Problems 1 and 2]; in particular, the operators $C_{\alpha,\rho}$ were considered in [12, Section 12.3, Problem 6, and pp. 503–504].

A general sufficient condition, which implies

$$\text{s-}\lim_{\rho\to\infty} \frac{J_\rho f - f}{\varphi(\rho)},$$

where J_ρ is a convolution operator in a HBS on $\mathbb{R}$ and $\lim_{\rho\to\infty}\varphi(\rho)=0$, was given in [20, Theorem 4.6] (see also [20, Theorems 4.8 and 4.9] for its necessity).

A characterization of the rate of convergence of $C_{\alpha,\rho}$ in terms of a K-functional defined by means of $D^{\{\alpha\}}$ was established [18, Theorem 4.4] in an arbitrary HBS on $\mathbb{R}$.

We will prove the following Voronovskaya-type inequality for the Picard operator in any HBS on $\mathbb{R}$.

Theorem 4.3. *Let $B(\mathbb{R})$ be a HBS on $\mathbb{R}$ and $\alpha>0$. Then for all $f\in W^{2\alpha}(B(\mathbb{R}))$ and all $\rho>0$ there holds*

$$\left\| C_{\alpha,\rho}f - f + \frac{1}{\rho^\alpha} D_s^{\{\alpha\}} f \right\|_{B(\mathbb{R})} \le \frac{\|c_\alpha\|_{L(\mathbb{R})}}{\rho^{2\alpha}} \|D_s^{\{2\alpha\}} f\|_{B(\mathbb{R})}. \tag{4.10}$$

We have

$$\|c_\alpha\|_{L(\mathbb{R})}=1, \quad 0<\alpha\le 2,$$

and

$$\|c_\alpha\|_{L(\mathbb{R})} \le \frac{2(2\alpha-1)}{\alpha}\csc\frac{\pi}{\alpha} \le 2\alpha-1, \quad \alpha>2. \tag{4.11}$$

Proof. We apply Corollary 3.4 with $k:=c_\alpha$, $Y:=W^{\{2\alpha\}}(B(\mathbb{R}))$, $Y_L:=W^{\{2\alpha\}}(L(\mathbb{R}))$, $D:=-D_s^{\{\alpha\}}$ and $G:=D_s^{\{2\alpha\}}$.

The operators D and G satisfy (3.7).

Further, by virtue of (4.9), we have (3.13) and (3.14) with $\psi(u):=-|u|^\alpha$ and $\theta(u):=|u|^{2\alpha}$. Clearly, they satisfy (3.15) with $\beta:=2\alpha$.

Finally, we have

$$\widehat{c_\alpha}(u)-1+|u|^\alpha = \frac{|u|^{2\alpha}}{1+|u|^\alpha} = |u|^{2\alpha}\,\widehat{c_\alpha}(u), \quad u\in\mathbb{R}, \tag{4.12}$$

which shows that (3.16) holds with $\ell=c_\alpha$.

Now, Corollary 3.4 implies (4.10).

It remains to estimate $\|c_\alpha\|_{L(\mathbb{R})}$. As is known (see e.g. [12, Problem 6.4.5, (ii)]), $c_\alpha(x)\ge 0$ on $\mathbb{R}$ for $\alpha\in(0,2]$. Therefore,

$$\|c_\alpha\|_{L(\mathbb{R})}=\widehat{c_\alpha}(0)=1, \quad 0<\alpha\le 2.$$

For $\alpha>2$, we use the Fourier transform inversion formula (e.g. [12, Proposition 5.1.10, or Problem 6.4.5, (i)]) to get

$$c_\alpha(x)=\frac{1}{\pi}\int_0^{+\infty}\frac{\cos xu}{1+u^\alpha}\,du, \quad x\in\mathbb{R}. \tag{4.13}$$

Therefore,

$$|c_\alpha(x)|\le\frac{1}{\pi}\int_0^{+\infty}\frac{du}{1+u^\alpha}, \quad x\in\mathbb{R}. \tag{4.14}$$

We make the change of the variable given by $v=(1+u^\alpha)^{-1}$ in the integral on the right hand-side to arrive at

$$\begin{aligned}\int_0^{+\infty}\frac{du}{1+u^\alpha} &= \frac{1}{\alpha}\int_0^1 v^{-1/\alpha}(1-v)^{1/\alpha-1}dv\\ &= \frac{1}{\alpha}\mathrm{B}\left(1-\frac{1}{\alpha},\frac{1}{\alpha}\right)=\frac{1}{\alpha}\frac{\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma(1)}\\ &= \frac{1}{\alpha}\frac{\pi}{\sin\frac{\pi}{\alpha}}=\frac{\pi}{\alpha}\csc\frac{\pi}{\alpha}.\end{aligned} \tag{4.15}$$

Above $\mathrm{B}(z_1,z_2)$ denotes the beta function and we used the identities

$$\mathrm{B}(z_1,z_2)=\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$$

and

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}, \quad z\notin\mathbb{Z}. \tag{4.16}$$

We deduce from (4.14)-(4.15) the estimate

$$|c_\alpha(x)|\le\frac{1}{\alpha}\csc\frac{\pi}{\alpha}, \quad x\in\mathbb{R}. \tag{4.17}$$

Let $x\ge 1$. Integrating by parts twice the integral on the right hand-side of (4.13), we arrive at the formula

$$c_\alpha(x)=\frac{\alpha(\alpha-1)}{\pi x^2}\int_0^{+\infty}\frac{u^{\alpha-2}}{(1+u^\alpha)^2}\cos xu\,du-\frac{2\alpha^2}{\pi x^2}\int_0^{+\infty}\frac{u^{2\alpha-2}}{(1+u^\alpha)^3}\cos xu\,du,$$

where we have taken into account that

$$\left(\frac{1}{1+u^\alpha}\right)' = -\alpha\,\frac{u^{\alpha-1}}{(1+u^\alpha)^2}$$

and

$$\left(\frac{1}{1+u^\alpha}\right)'' = -\alpha(\alpha-1)\,\frac{u^{\alpha-2}}{(1+u^\alpha)^2} + 2\alpha^2\,\frac{u^{2\alpha-2}}{(1+u^\alpha)^3}.$$

Consequently,

$$|c_\alpha(x)| \le \frac{\alpha(\alpha-1)}{\pi x^2}\int_0^{+\infty}\frac{u^{\alpha-2}}{(1+u^\alpha)^2}du + \frac{2\alpha^2}{\pi x^2}\int_0^{+\infty}\frac{u^{2\alpha-2}}{(1+u^\alpha)^3}du. \tag{4.18}$$

Just as above, making the change of the variable $v = (1+u^\alpha)^{-1}$ in each of the integrals above, we get

$$\int_0^{+\infty}\frac{u^{\alpha-2}}{(1+u^\alpha)^2}du = \frac{1}{\alpha}\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right) \tag{4.19}$$

and

$$\int_0^{+\infty}\frac{u^{2\alpha-2}}{(1+u^\alpha)^3}du = \frac{1}{2\alpha}\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(2-\frac{1}{\alpha}\right). \tag{4.20}$$

Next, by means of the identities $\Gamma(z+1) = z\Gamma(z)$ and (4.16), we evaluate the products of the Γ-function on the right hand-side:

$$\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right) = \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right) = \frac{\pi}{\alpha}\csc\frac{\pi}{\alpha}$$

and

$$\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(2-\frac{1}{\alpha}\right) = \left(1-\frac{1}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right) = \frac{(\alpha-1)\pi}{\alpha^2}\csc\frac{\pi}{\alpha}.$$

We substitute these expressions in (4.19)-(4.20) and derive from (4.18) the estimate

$$|c_\alpha(x)| \le \frac{2(\alpha-1)}{\alpha x^2}\csc\frac{\pi}{\alpha}, \quad x \ge 1. \tag{4.21}$$

Using that c_α is an even function (see (4.13)), and estimating $|c_\alpha(x)|$ by (4.17) for $x \in [0,1]$, and by (4.21) for $x \ge 1$, we arrive at

$$\|c_\alpha\|_{L(\mathbb{R})} = 2\int_0^{+\infty}|c_\alpha(x)|\,dx \le \frac{2(2\alpha-1)}{\alpha}\csc\frac{\pi}{\alpha}.$$

Finally, since $\sin\gamma \ge 2\gamma/\pi$ for $\gamma \in [0,\pi/2]$, we get the second estimate in (4.11). □

Remark 4. Let us note that, in view of (3.12), (4.12) yields the functional equation

$$C_{\alpha,\rho}f - f + \frac{1}{\rho^\alpha}D_s^{\{\alpha\}}f = \frac{1}{\rho^{2\alpha}}C_{\alpha,\rho}D_s^{\{2\alpha\}}f, \quad f \in W^{2\alpha}(B(\mathbb{R})), \quad \rho > 0.$$

Vorononoskaya-type estimates using the ordinary derivative and the Caputo fractional derivative were given in [2, Chapters 10-13] for the classical singular integral of Picard. These estimates are point-wise or in the L_p-norm.

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