Equivariant Weierstrass Preparation and Values of L-functions at Negative Integers

Dedicated to Professor Kazuya Kato

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Abstract. We apply an equivariant version of the p-adic Weierstrass Preparation Theorem in the context of possible non-commutative generalizations of the power series of Deligne and Ribet. We then consider CM abelian extensions of totally real fields and by combining our earlier considerations with the known validity of the Main Conjecture of Iwasawa theory we prove, modulo the conjectural vanishing of certain μ-invariants, a (corrected version of a) conjecture of Snaith and the 'rank zero component' of Kato's Generalized Iwasawa Main Conjecture for Tate motives of strictly positive weight. We next use the validity of this case of Kato's conjecture to prove a conjecture of Chinburg, Kolster, Pappas and Snaith and also to compute explicitly the Fitting ideals of certain natural étale cohomology groups in terms of the values of Dirichlet L-functions at negative integers. This computation improves upon results of Cornacchia and Østvær, of Kurihara and of Snaith, and, modulo the validity of a certain aspect of the Quillen-Lichtenbaum Conjecture, also verifies a finer and more general version of a well known conjecture of Coates and Sinnott.

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1. Introduction

In a beautiful series of papers in 1993 Kato formulated and studied a 'Generalized Iwasawa Main Conjecture' for motives over number fields with respect to certain commutative coefficient rings [24, 25, 26]. This conjecture refined the 'Tamagawa number conjectures' previously formulated by Bloch and Kato in [3] and by Fontaine and Perrin-Riou in [20], and led naturally to the subsequent formulation by Flach and the first named author in [9] of a Tamagawa number conjecture for motives over number fields with respect to more general coefficient rings which, in particular, need not be commutative.

The above approach has already led to some remarkable new insights and results in a number of rather different contexts. We recall, for example, that it has led to a universal approach to and refinement of the 'refined Birch and Swinnerton-Dyer Conjectures' for abelian varieties with complex multiplication which were formulated by Gross (cf. [9, Rem. 10 and §3.3, Examples c), d) e)]) and of all of the 'refined abelian Stark conjectures' which were formulated by Gross, by Tate, by Rubin and by Darmon (cf. [6, 7]). At the same time, the approach has led to a natural re-interpretation and refinement of all of the central conjectures of classical Galois module theory (cf. [8, 5, 2]) and has also more recently become the focus of attempts to formulate a natural 'Main Conjecture of non-abelian Iwasawa theory' (see, for example, the forthcoming articles of Huber and Kings and of Weiss and the first named author in this regard). It is certainly a great pleasure, on the occasion of Kato's fiftieth birthday, to offer in this manuscript some additional explicit evidence in support of his Generalized Iwasawa Main Conjecture and, more generally, to demonstrate yet further the enormous depth and significance of the approach that he introduced in [24, 25, 26].

To describe the main results of the present manuscript in some detail we now fix a totally real number field k and a finite Galois extension K of k which is either totally real or a CM field, and we set $G := \mathrm{Gal}(K/k)$. We also fix a rational prime number p and an algebraic closure $\mathbb{Q}_p^c$ of the field of p-adic rationals $\mathbb{Q}_p$.

We recall that if G is abelian, then a key ingredient of Wiles' proof [39] of the Main Conjecture of Iwasawa theory is the construction by Deligne and Ribet of an element of the power series ring $\mathbb{Z}_p[G][[T]]$ which is uniquely characterized by its relation to the p-adic L-series associated to the extension K/k.

In this manuscript we first relax the restriction that G is abelian and discuss the possible existence of elements of the (non-commutative) power series ring $\mathbb{Z}_p[G][[T]]$ which are related in a precise manner to the p-adic Artin L-functions associated to irreducible $\mathbb{Q}_p^c$-valued characters of G. In particular, under a certain natural hypothesis on G (which does not require G to be abelian), and assuming the vanishing of certain μ-invariants, we apply an appropriate version of the Weierstrass Preparation Theorem for the ring $\mathbb{Z}_p[G][[T]]$ to derive relations between two hypothetical generalizations of the power series of Deligne and Ribet.

In the remainder of the manuscript our main aim is to show that, if G is abelian, then the above considerations can be combined with the constructions of Deligne and Ribet and the theorem of Wiles to shed light on a number of interesting questions. To describe these applications we assume for the rest of this introduction that G is abelian.
Our first application is to the 'Wiles Unit Conjecture' which is formulated by Snaith in [34, Conj. 6.3.4]. Indeed, by using the above approach we are able to show that the validity of a slightly amended version of Snaith's conjecture follows directly from the (in general conjectural) vanishing of certain natural μ-invariants, and also to show that the original version of Snaith's conjecture does not hold in general (cf. Remark 5).
To describe the next application we fix an integer r with $r > 1$. Under the aforementioned hypothesis concerning μ-invariants we shall prove the Generalized Iwasawa Main Conjecture of [25, Conj. 3.2.2] for the pair $(h^0(\mathrm{Spec}\,(K))(1-r), \mathfrak{A}_r)$, where $\mathfrak{A}_r$ is a natural ring which annihilates the space $\mathbb{Q} \otimes_{\mathbb{Z}} K_{2r-1}(K)$. If $k = \mathbb{Q}$, then (by a result of Ferrero and Washington in [18]) the appropriate μ-invariants are known to vanish and hence we obtain in this way a much more direct proof of the relevant parts of the main result (Cor. 8.1) of our earlier paper [11]. We remark however that the proofs of all of our results in this area involve a systematic use of the equivariant Iwasawa theory of complexes which was initiated by Kato in [25] and subsequently extended by Nekovář in [30].
As a further application, we combine our result on the Generalized Iwasawa Main Conjecture with certain explicit cohomological computations of Flach and the first named author in [8] to prove (modulo the aforementioned hypothesis on μ-invariants) that the element $\Omega_{r-1}(K/k)$ of $\mathrm{Pic}(\mathbb{Z}[G])$ which is defined by Chinburg, Kolster, Pappas and Snaith in [13] belongs to the kernel of the natural scalar extension morphism $\mathrm{Pic}(\mathbb{Z}[G]) \to \mathrm{Pic}(\mathfrak{A}_r)$.
As a final application we then combine our approach with a development of a purely algebraic observation of Cornacchia and the second named author in [15] to compute explicitly certain Fitting ideals which are of arithmetical interest. To be more precise in this regard we assume that p is odd, we fix a finite set of places S of K which contains all archimedean places and all places which either ramify in K/k or are of residue characteristic p and we write $\mathcal{O}_{K,S}$ for the ring of S-integers of K. Writing τ for the complex conjugation in G we let e_r denote the idempotent $\frac{1}{2}(1 + (-1)^r\tau)$ of $\mathbb{Z}_p[G]$. Then, under the aforementioned hypothesis on μ-invariants, we prove that the Fitting ideal of the étale cohomology module $e_r \cdot H^2(\mathrm{Spec}(\mathcal{O}_{K,S})_{\text{ét}}, \mathbb{Z}_p(r))$ over the ring $\mathbb{Z}_p[G]e_r$ can be completely described in terms of the values at $1-r$ of the S-truncated Dirichlet L-functions which are associated to K/k. This result improves upon previous results of Cornacchia and Østvær [16, Thm. 1.2], of Kurihara [28, Cor. 12.5] and of Snaith [34, Thm. 1.6, Thm. 2.4, Thm. 5.2] and also implies a natural analogue of the main result of Solomon in [36] concerning relations between Bernoulli numbers and the structure of certain ideal class groups (cf. Remark 8). We finally remark that, under the assumed validity of a particular

case of the Quillen-Lichtenbaum Conjecture, our result verifies a finer and more general version of the well known conjecture formulated by Coates and Sinnott in [14, Conj. 1].

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2. Equivariant Weierstrass Preparation

In this section we discuss a natural generalization of the classical p-adic Weierstrass Preparation Theorem.

We let $\mathfrak{A}$ be a ring, write $\mathrm{rad}(\mathfrak{A})$ for its Jacobson radical and set $\overline{\mathfrak{A}} := \mathfrak{A}/\mathrm{rad}(\mathfrak{A})$. In the sequel we shall say that $\mathfrak{A}$ is *strictly admissible* if it is both separated and complete in the $\mathrm{rad}(\mathfrak{A})$-adic topology and is also such that $\overline{\mathfrak{A}}$ is a skew field. More generally, we shall say that $\mathfrak{A}$ is *admissible* if it is a finite product of strictly admissible rings.

Remark 1. Let G be a finite group and p any prime number. It can be shown that the group ring $\mathfrak{A} = \mathbb{Z}_p[G]$ is admissible if G is the direct product of a p-group and an abelian group (and, in particular therefore, if G is itself abelian). Note also that in any such case the ring $\overline{\mathfrak{A}}$ is a product of finite skew fields and is therefore commutative.

In this manuscript we define the power series ring $\mathfrak{A}[[T]]$ over $\mathfrak{A}$ just as for commutative base rings; in particular, we require that the variable T commutes with all elements of the coefficient ring $\mathfrak{A}$. We observe that, with this definition, an element f of $\mathfrak{A}[[T]]$ is invertible if and only if its constant term $f(0)$ is invertible in $\mathfrak{A}$.

If f is any element of $\mathfrak{A}[[T]]$, then we write $\overline{f}$ for its image under the obvious reduction map $\mathfrak{A}[[T]] \to \overline{\mathfrak{A}}[[T]]$. Assume for the moment that $\mathfrak{A}$ is admissible, with a decomposition $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i$ for strictly admissible rings $\mathfrak{A}_i$. If $f = (f_i)_{i\in I}$ is any element of $\mathfrak{A}[[T]]$, then we define the *degree* $\deg(f)$, respectively *reduced degree* $\mathrm{rdeg}(f)$, of f to be the vector $(\deg(f_i))_{i\in I}$, respectively $(\deg(\overline{f}))_{i\in I}$, where by convention we regard the zero element of each ring $\mathfrak{A}_i[[T]]$ and $\overline{\mathfrak{A}_i}[[T]]$ to be of degree $+\infty$. We observe that if $\mathfrak{A} = \mathbb{Z}_p$, then $\mathrm{rdeg}(f)$ is finite if and only if the μ-invariant of the $\mathbb{Z}_p[[T]]$-module $\mathbb{Z}_p[[T]]/(f)$ is zero. By analogy, if $\mathfrak{A}$ is any admissible ring, then we shall write '$\mu_{\mathfrak{A}}(f) = 0$' to express the fact that (each component of) $\mathrm{rdeg}(f)$ is finite.

If $\mathfrak{A}$ is strictly admissible, respectively admissible, then we shall say that an element of $\mathfrak{A}[[T]]$ is a *distinguished polynomial* if it is a monic polynomial all of whose non-leading coefficients are in $\mathrm{rad}(\mathfrak{A})$, respectively if all of its components are distinguished polynomials (of possibly varying degrees).

Proposition 2.1. *('Equivariant Weierstrass Preparation') Let $\mathfrak{A}$ be an admissible ring. If f is any element of $\mathfrak{A}[[T]]$ for which $\mu_{\mathfrak{A}}(f) = 0$, then there exists* *a unique distinguished polynomial f^* and a unique unit element u_f of $\mathfrak{A}[[T]]$ such that $f = f^* \cdot u_f$.*

Proof. By direct verification one finds that the argument of [4, Chap.VII, §3, no. 8] extends to the present (non-commutative) context to prove the following 'Generalized Division Lemma': if f is any element of $\mathfrak{A}[[T]]$ for which $\mu_{\mathfrak{A}}(f) = 0$, then each element g of $\mathfrak{A}[[T]]$ can be written uniquely in the form $g = fq + r$ where q and r are elements of $\mathfrak{A}[[T]]$ and each component of $\deg(r)$ is strictly less than the corresponding component of $\mathrm{rdeg}(f)$.
The deduction of the claimed result from this Generalized Division Lemma now proceeds exactly as in [31, V.5.3.3-V.5.3.4]. □

Remark 2. i) If $\mathfrak{A}$ is a discrete valuation ring, then it is (strictly) admissible and Proposition 2.1 is equivalent to the classical Weierstrass Preparation Theorem (cf. [38, Thm. 7.1]).
ii) Shortly after the first version of this manuscript was circulated (in December 2001) we learnt of a recent preprint [37] of Venjakob in which a Weierstrass Preparation Theorem is proved under conditions which are considerably more general than those of Proposition 2.1. We remark that if $\mathfrak{A}$ is strictly admissible, then it can be shown that the result of Proposition 2.1 is indeed equivalent to a special case of the main result of loc. cit.
iii) For any prime p, any $\mathbb{Z}_p$-order $\mathfrak{A}$ which is not admissible and any element f of $\mathfrak{A}[[T]]$ a natural interpretation of the equality '$\mu_{\mathfrak{A}}(f) = 0$' would be that, for each primitive central idempotent ϵ of $\mathfrak{A}$, the element $\epsilon \cdot f$ is not divisible by p (indeed, this interpretation recovers that given above in the case that $\mathfrak{A}$ is admissible). However, under this interpretation the product decomposition of Proposition 2.1 is not always possible. For example, if $\mathfrak{A} = \mathrm{M}_2(\mathbb{Z}_p)$, then the constant series $f := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ satisfies $\mu_{\mathfrak{A}}(f) = 0$ (in the above sense) and yet cannot be written in the stated form $f^* \cdot u_f$. Indeed, if it did admit such a decomposition, then the $\mathbb{Z}_p$-module $\mathfrak{A}[[T]]/f \cdot \mathfrak{A}[[T]]$ would be finitely generated and this is not true since $f \cdot \mathfrak{A}[[T]]$ is equal to the subset of $\mathrm{M}_2(\mathbb{Z}_p[[T]])$ consisting of those matrices which have both second row entries divisible by p.
iv) In just the same way as Proposition 2.1, one can prove that if $\mathfrak{A}$ is admissible, then every element f of $\mathfrak{A}[[T]]$ for which $\mu_{\mathfrak{A}}(f) = 0$ can be written uniquely in the form $u^f \cdot f_*$ with f_* a distinguished polynomial and u^f a unit of $\mathfrak{A}[[T]]$. However, as the following example shows, the relation between the elements f^* and f_* (and u^f and u_f) is in general far from clear.

Example 1. Let a and b be elements of $\mathrm{rad}(\mathfrak{A})$, and set $f := (T - a)(1 + bT)$. Then it is clear that $\mu_{\mathfrak{A}}(f) = 0$, $f^* = T - a$ and $u_f = 1 + bT$. On the other hand it may be shown that $f_* = T - c$ where c is the unique element of $\mathfrak{A}$ which satisfies $T - c \in \mathfrak{A} \cdot f$, and that $u^f = (1 - ab + bc) + bT$. Upon calculating c as a power series in the noncommuting variables a and b, one finds that

$$c = a - baa + aba + bbaaa - baaba - abbaa + ababa + \dots.$$

However, describing c completely is tricky. For example, a convenient option is to use the context-free formal language with four productions $S \to Ta$, $S \to TbSS$, $T \to \epsilon, T \to Tab$. Indeed, it may be shown that the degree n part of the series c is equal to the weighted sum of all words w of length n in this language, counted with weight $(-1)^{e(w)}$ where $e(w)$ denotes the number of times that b occurs in w not immediately preceded by a. For background and a similar example, we refer the reader to [32], in particular Chapter VI.

3. p-adic L-functions

In this section we apply Proposition 2.1 in the context of Iwasawa theory. The main result of this section (Theorem 3.1) was first motivated by the observation that the constructions of Deligne and Ribet which are used by Wiles in [39, p. 501f] can be combined with Proposition 2.1 to shed light upon the 'Wiles Unit Conjecture' formulated by Snaith in [34, Conj. 6.3.4]. In particular, by these means we shall prove that the validity in the relative abelian case of a corrected version of Snaith's conjecture is a direct consequence of the (in general conjectural) vanishing of certain natural μ-invariants.

We first introduce some necessary notation. Throughout this section we fix an odd prime p and a finite group G. We recall that $\mathbb{Q}_p^c$ is a fixed algebraic closure of $\mathbb{Q}_p$, and we write $\mathrm{Irr}_p(G)$ for the set of irreducible $\mathbb{Q}_p^c$-characters of G. For each $\rho \in \mathrm{Irr}_p(G)$ we write $\mathbb{Z}_p(\rho)$ for the extension of $\mathbb{Z}_p$ which is generated by the values of ρ.

Following Fröhlich [21, Chap. II], we now define for each element f of $\mathbb{Z}_p[G][[T]]$ a canonical element $\mathrm{Det}(f)$ of $\mathrm{Map}(\mathrm{Irr}_p(G), \mathbb{Q}_p^c[[T]])$. To do this we fix a subfield N of $\mathbb{Q}_p^c$ which is of finite degree over $\mathbb{Q}_p$ and over which all elements of $\mathrm{Irr}_p(G)$ can be realized, and we write $\mathcal{O}_N$ for the valuation ring of N. For each character $\rho \in \mathrm{Irr}_p(G)$ we choose a finitely generated $\mathcal{O}_N[G]$-module L_ρ which is free over $\mathcal{O}_N$ (of rank n say) and is such that the space $L_\rho \otimes_{\mathcal{O}_N} N$ has character ρ, and we write $r_\rho : G \to \mathrm{GL}_n(\mathcal{O}_N)$ for the associated homomorphism. If now $f = \sum_{i\geq 0} c_i T^i$, then $r_\rho(f) := \sum_{i\geq 0} r_\rho(c_i)T^i$ belongs to $M_n(\mathcal{O}_N[[T]])$ and we define $\mathrm{Det}(f)(\rho) := \det(r_\rho(f)) \in \mathcal{O}_N[[T]]$ (which is indeed independent of the choices of field N and lattice L_ρ). We observe in particular that if ρ is any element of $\mathrm{Irr}_p(G)$ which is of dimension 1, then one has $\mathrm{Det}(f)(\rho) = \rho(f)$.

In the remainder of this manuscript we assume given a finite Galois extension of number fields K/k for which $\mathrm{Gal}(K/k) = G$. We write k_∞ (or k_∞^p if we need to be more precise) for the cyclotomic $\mathbb{Z}_p$-extension of k, and K_∞ (or K_∞^p) for the compositum of K and k_∞.

In the rest of this section we assume that k is totally real and that K is either totally real or a CM field. We also fix a finite set S of non-archimedean places of k which contains all non-archimedean places which ramify in K/k. We write $\mathrm{Irr}_p^+(G)$ for the subset of $\mathrm{Irr}_p(G)$ consisting of those characters which are even (that is, factor through characters of the Galois group of the maximal totally real extension K^+ of k in K). We fix a topological generator γ of $\mathrm{Gal}(k_\infty/k)$ and, with χ_{cyclo} denoting the cyclotomic character, we set $u := \chi_{\mathrm{cyclo}}(\gamma) \in \mathbb{Z}_p^\times$.

We recall that for each $\rho \in \mathrm{Irr}_p^+(G)$ there exists a p-adic L-function $L_{p,S}(-,\rho)$ and an associated element $f_{S,\rho}$ of the quotient field of $\mathbb{Z}_p(\rho)[[T]]$ such that

$$L_{p,S}(1-s,\rho) = f_{S,\rho}(u^s - 1) \tag{1}$$

for almost all $s \in \mathbb{Z}_p$. To be more precise about the denominator of $f_{S,\rho}$ we set $H_\rho := 1$ unless ρ is induced by a multiplicative character of $\mathrm{Gal}(k_\infty/k)$ (that is, ρ is a character of 'type W' in the terminology of Wiles [39]) in which case we set $H_\rho := \rho(\gamma)(1+T) - 1 \in \mathbb{Z}_p(\rho)[[T]]$. Then there exists an element $G_{S,\rho}$ of $\mathbb{Z}_p(\rho)[[T]][\frac{1}{p}]$ such that $f_{S,\rho} = G_{S,\rho} \cdot H_\rho^{-1}$ (cf. [39, Thm. 1.1]). We hope that the reader will not in the sequel be confused by our notation: whenever G occurs without a subscript it denotes a Galois group; whenever G is adorned with a subscript it denotes a power series.

By the classical Weierstrass Preparation Theorem, each series $G_{S,\rho}$ can be decomposed as a product

$$G_{S,\rho} = \pi(\rho)^{\mu(S,\rho)} \cdot G^*_{S,\rho} \cdot U_{S,\rho} \tag{2}$$

where $\pi(\rho)$ is a uniformising parameter of $\mathbb{Z}_p(\rho)$, $\mu(S,\rho)$ is an integer, $G^*_{S,\rho}$ is a distinguished polynomial and $U_{S,\rho}$ is a unit of $\mathbb{Z}_p(\rho)[[T]]$.
We now proceed to describe four natural hypotheses relating to the Weierstrass decompositions (2). The main result of this section will then describe certain relations that exist between these hypotheses.

Hypothesis (μ_p): *For each $\rho \in \mathrm{Irr}_p^+(G)$ one has $\mu(S,\rho) = 0$.*

Remark 3. i) It is a standard conjecture that Hypothesis (μ_p) is always valid (the first statement of this was due to Iwasawa [23]). However, at present the only general result one has in this direction is that Hypothesis (μ_p) is valid for K/k when $k = \mathbb{Q}$ and G is abelian. Indeed, this is proved by Ferrero and Washington in [18].
ii) In this remark we describe a natural Iwasawa-theoretical reinterpretation of Hypothesis (μ_p) in the case that K is totally real. To do this we write S_p for the union of S and the set of places of k which lie above p, and we let $Y(S_p)$ denote the Galois group of the maximal abelian pro-p-extension of K_∞ which is unramified outside the set of places which lie above any element of S_p. (We note that, since p is odd, any pro-p-extension of K_∞ is automatically unramified at all archimedean places.)

LEMMA 1. *If K is totally real, then Hypothesis (μ_p) is valid for K/k if and only if the μ-invariant of the $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/K)]$-module $Y(S_p)$ is* 0.

Proof. We set $L := K(\zeta_p)$ and $\Delta := \mathrm{Gal}(L/K)$ and let $\omega : \Delta \to \mathbb{Z}_p^\times$ denote the Teichmüller character. For each $\mathbb{Z}_p[\Delta]$-module M and integer i we write $M^{(i)}$ for the submodule consisting of those elements m which satisfy $\delta(m) = \omega^i(\delta)\cdot m$ for all $\delta \in \Delta$. (Since $p \nmid |\Delta|$, each such functor $M \mapsto M^{(i)}$ is exact.)
We write Y, respectively Y_L, for the Galois group of the maximal abelian pro-p-extension of K_∞, respectively of L_∞, which is unramified outside all places above p. For each $\rho \in \mathrm{Irr}_p(\mathrm{Gal}(K/k))$ we also write G_ρ for the element of

$\mathbb{Z}_p(\rho)[[T]][\frac{1}{p}]$ which is defined just as $G_{S,\rho}$ but with S taken to be the empty set.

We first observe that $\mu(Y(S_p)) = 0$ if and only if $\mu(Y) = 0$ and that for each $\rho \in \mathrm{Irr}_p(\mathrm{Gal}(K/k))$ one has $\mu(S,\rho) = 0$ if and only if $\mu(G_\rho) = 0$.

We also note that $\mu(Y) = 0$ if and only if $\mu(X_L^{(1)}) = 0$, where X_L denotes the Galois group of the maximal unramified abelian pro-p-extension of L_∞. For the reader's convenience, we briefly sketch the argument. By Kummer duality, for each even integer i the module $Y_L^{(i)}$ is isomorphic to $\mathrm{Hom}(\mathrm{Cl}(L_\infty)^{(1-i)}, \mathbb{Q}_p/\mathbb{Z}_p)(1)$ and hence in turn pseudo-isomorphic to $X_L^{(1-i),\#}(1)$ where $\#$ indicates contragredient action of $\mathrm{Gal}(L_\infty/k)$ [31, (11.1.8), (11.4.3)]. This implies, in particular, that $Y = Y_L^{(0)}$ is pseudo-isomorphic to $X_L^{(1),\#}(1)$, and this in turn implies the claimed result.

We next recall that, as a consequence of [39, Thm. 1.4], one has $\mu(X_L^{(1)}) = 0$ if and only if $\mu(G_{\eta_0}) = 0$, where η_0 denote the trivial character of Δ.

To finish the proof of the lemma, we now need only invoke the inductive property of L-functions and Iwasawa series. Indeed, one has $G_{\eta_0} = G_{\chi_{\mathrm{reg}}}$, where χ_{reg} is the character of the regular representation of $\mathrm{Gal}(K/k)$ (note that this is equal to the induction of η_0 from K to k), and by its very definition, one has $G_{\chi_{\mathrm{reg}}} = \prod_\rho G_\rho^{\deg(\rho)}$, where ρ runs over all elements of $\mathrm{Irr}_p(\mathrm{Gal}(K/k))$. It is therefore clear that $\mu(G_{\eta_0}) = 0$ if and only if for all $\rho \in \mathrm{Irr}_p(\mathrm{Gal}(K/k))$ one has $\mu(G_\rho) = 0$ (or equivalently $\mu(S,\rho) = 0$), as required. □

We continue to introduce further natural hypotheses relating to the decompositions (2).

If f and f' are elements of $\mathbb{Z}_p[G][[T]]$, then we say that f is *right associated*, respectively *left associated*, to f' if there exists a unit element u of $\mathbb{Z}_p[G][[T]]$ such that $f = f' \cdot u$, respectively $f = u \cdot f'$.

Hypothesis (EPS) ('Equivariant Power Series') *Assume that Hypothesis (μ_p) is valid for K/k. Then there exist elements $G_S = G_S(T)$ and $H = H(T)$ of $\mathbb{Z}_p[G][[T]]$ which are each right associated to distinguished polynomials and are such that for all $\rho \in \mathrm{Irr}_p^+(G)$ the quotient* $\mathrm{Det}(G_S)(\rho)/\mathrm{Det}(H)(\rho)$ *is defined and equal to $G_{S,\rho}/H_\rho = f_{S,\rho}$.*

Remark 4. i) If $\mathbb{Z}_p[G]$ is admissible, then Proposition 2.1 (and Remark 2iv)) implies that an element f of $\mathbb{Z}_p[G][[T]]$ is right associated to a distinguished polynomial if and only if it is left associated to a distinguished polynomial and that these conditions are in turn equivalent to an equality $\mu_{\mathbb{Z}_p[G]}(f) = 0$.

ii) The completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ is naturally isomorphic to the power series ring $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/k_\infty)][[T]]$. If $K \cap k_\infty = k$, then this ring can be identified with $\mathbb{Z}_p[G][[T]]$ but in general no such identification is possible.

We write c_ρ for the leading coefficient of H_ρ (so, explicitly, one has $c_\rho = 1$ unless ρ is a non-trivial character of 'type W' in which case $c_\rho = \rho(\gamma)$). We observe that the polynomial $H_\rho^* := c_\rho^{-1} \cdot H_\rho$ is distinguished.

The following hypothesis is directly motivated by the 'Wiles Unit Conjecture' which is formulated by Snaith in [34, Conj. 6.3.4]. (We shall explain the precise connection at the end of this section.)

Hypothesis (EUS) ('Equivariant Unit Series') *There exists a unit element* U_S *of* $\mathbb{Z}_p[G][[T]]$ *which is such that for all* $\rho \in \mathrm{Irr}_p^+(G)$ *one has* $\mathrm{Det}(U_S)(\rho) = c_\rho^{-1} U_{S,\rho}$.

For each character $\rho \in \mathrm{Irr}_p^+(G)$ we next consider the vector space over N which is given by $H^0(\mathrm{Gal}(K_\infty/k_\infty), \mathrm{Hom}_N(L_\rho \otimes_{\mathcal{O}_N} N, Y(S_p) \otimes_{\mathbb{Z}_p} N))$. This space is finite-dimensional (over N) and also equipped with a canonical action of the quotient group $\mathrm{Gal}(K_\infty/k)/\mathrm{Gal}(K_\infty/k_\infty) \cong \mathrm{Gal}(k_\infty/k)$ and hence, in particular, of the automorphism γ. We write $h_{S,\rho}$ for the characteristic polynomial of the endomorphism of the above space which is induced by the action of $\gamma - 1$.

Hypothesis (ECP) ('Equivariant Characteristic Polynomials') *Assume that Hypothesis* (μ_p) *is valid for* K/k. *Then there exist distinguished polynomials* $G_S^* = G_S^*(T)$ *and* $H^* = H^*(T)$ *in* $\mathbb{Z}_p[G][T]$ *which are such that for all* $\rho \in \mathrm{Irr}_p^+(G)$ *the quotient* $\mathrm{Det}(G_S^*)(\rho)/\mathrm{Det}(H^*)(\rho)$ *is defined and equal to* $h_{S,\rho}/H_\rho^*$.

We can now state the main result of this section.

Theorem 3.1. *Assume that Hypothesis* (μ_p) *is valid for* K/k.

i) *If Hypotheses (ECP) and (EUS) are both valid for* K/k, *then Hypothesis (EPS) is valid for* K/k *with* $G_S = G_S^* \cdot U_S$ *and* $H = H^*$.

ii) *If* $\mathbb{Z}_p[G]$ *is admissible and Hypothesis (EPS) is valid for* K/k, *then Hypotheses (EUS) and (ECP) are both valid for* K/k.

iii) *If* K/k *is abelian, then Hypotheses (EPS), (ECP) and (EUS) are all valid for* K/k.

Proof. i) We suppose that Hypotheses (μ_p), (ECP) and (EUS) are all valid for K/k. Under these hypotheses we claim that the series $G_S := G_S^* \cdot U_S \in \mathbb{Z}_p[G][[T]]$ and $H := H^* \in \mathbb{Z}_p[G][T]$ are as described in Hypothesis (EPS). To show this we first observe that for every character $\rho \in \mathrm{Irr}_p^+(G)$ one has

$$\begin{aligned}\mathrm{Det}(G_S)(\rho) \cdot \mathrm{Det}(H)(\rho)^{-1} &= h_{S,\rho} c_\rho^{-1} U_{S,\rho} \cdot (H_\rho^*)^{-1} \\ &= h_{S,\rho} \cdot U_{S,\rho} \cdot H_\rho^{-1} \\ &= (h_{S,\rho} \cdot (G_{S,\rho}^*)^{-1}) \cdot (G_{S,\rho} \cdot H_\rho^{-1}),\end{aligned}$$

where the last equality is a consequence of the decomposition (2) and our assumption that $\mu(S,\rho) = 0$. It is therefore enough to show that for each $\rho \in \mathrm{Irr}_p^+(G)$ one has an equality

$$h_{S,\rho} = G_{S,\rho}^*. \tag{3}$$

Now if ρ is a one-dimensional even character which is of 'type S', then this equality is equivalent to the Main Conjecture of Iwasawa theory as proved by Wiles [39, Thm. 1.3]. In the general case the equality has been verified by

Snaith [34, Thm. 6.2.5] and, for the reader's convenience, we briefly sketch the argument (for details see loc. cit.). One proceeds by reduction to the result of Wiles by means of Brauer's Induction Theorem and the fact that characteristic polynomials and Iwasawa series enjoy the same inflation and induction properties. The only complication in this reduction is caused by the need to twist with characters which are of 'type W' and by the denominator polynomials H_ρ, and this is resolved by using the fact, first observed by Greenberg [22], that the denominator of the Iwasawa series which is attached to each irreducible character ρ of dimension greater than 1 is trivial (this justifies our setting $H_\rho = 1$ for these ρ). We would like to point out that there is actually a very simple argument for this absence of denominator in the case that $K \cap k_\infty = k$. Indeed, in this case, if ρ is any irreducible character of G which has degree greater than 1, then Brauer induction implies that $\rho = \sum_i n_i \operatorname{Ind}_{k_i}^k(\rho_i)$ where the k_i are suitable intermediate fields and ρ_i is a one-dimensional character of $\operatorname{Gal}(K/k_i)$ which is of 'type S'. It follows that the denominator of each series f_{ρ_i} is trivial unless ρ_i is itself the trivial character. Further, if χ_0 denotes the one-dimensional trivial representation of G, then one has $0 = \langle \chi_0, \rho \rangle = \sum_i n_i \langle \chi_0, \operatorname{Ind}_{k_i}^k(\rho_i)\rangle$. Since the latter sum is equal to the sum of the multiplicities n_i for which ρ_i is trivial, it follows that the denominator of f_ρ is indeed trivial.

ii) We now suppose that $\mathbb{Z}_p[G]$ is admissible and that Hypotheses (μ_p) and (EPS) are both valid for K/k. Recalling Remark 4i) (and Proposition 2.1), we find that the series G_S and H (as given by Hypothesis (EPS)) admit canonical decompositions $G_S = G_S^* \cdot U_S'$ and $H = H^* \cdot V$, where G_S^* and H^* are distinguished polynomials in $\mathbb{Z}_p[G][T]$ and U_S' and V are units of the ring $\mathbb{Z}_p[G][[T]]$. It follows that for each character $\rho \in \operatorname{Irr}_p^+(G)$ one has an equality

$$\frac{G_{S,\rho}}{H_\rho} = \frac{\operatorname{Det}(G_S^*)(\rho)\ \operatorname{Det}(U_S')(\rho)}{\operatorname{Det}(H^*)(\rho)\ \operatorname{Det}(V)(\rho)}.$$

We now recall that $G_{S,\rho} = G_{S,\rho}^* \cdot U_{S,\rho} = h_{S,\rho} \cdot U_{S,\rho}$ (by (2) and (3)) and we set $U_S := U_S' \cdot V^{-1} \in \mathbb{Z}_p[G][[T]]^\times$. Upon clearing denominators in the last displayed formula, we therefore obtain equalities

$$h_{S,\rho} \cdot \operatorname{Det}(H^*)(\rho) \cdot U_{S,\rho} = \operatorname{Det}(G_S^*)(\rho) \cdot H_\rho \cdot \operatorname{Det}(U_S)(\rho),$$

or equivalently

$$h_{S,\rho} \cdot \operatorname{Det}(H^*)(\rho) \cdot c_\rho^{-1} U_{S,\rho} = \operatorname{Det}(G_S^*)(\rho) \cdot H_\rho^* \cdot \operatorname{Det}(U_S)(\rho). \tag{4}$$

Lemma 2. *Let f be a distinguished polynomial in $\mathbb{Z}_p[G][[T]]$. Then, for each $\rho \in \operatorname{Irr}_p(G)$, the series $\operatorname{Det}(f)(\rho)$ is a distinguished polynomial in $\mathcal{O}_N[[T]]$.*

Proof. It is clear that the series $\operatorname{Det}(f)(\rho)$ is a polynomial in $\mathcal{O}_N[[T]]$ which is distinguished if and only if the polynomial $\operatorname{Det}(f)(\rho)^{p^j}$ is distinguished for any natural number j. In addition, since f is a distinguished polynomial, of degree d say, there exists a natural number j such that $f^{p^j} \equiv T^{dp^j}$ (modulo $p \cdot \mathbb{Z}_p[G][[T]]$). Since $\operatorname{Det}(f^{p^j})(\rho) = \operatorname{Det}(f)(\rho)^{p^j}$ we may therefore assume in the sequel that f is a monic polynomial which satisfies $f \equiv T^d$ (modulo $p \cdot \mathbb{Z}_p[G][[T]]$).

We now use the notation introduced at the beginning of §3 (when defining the map $\mathrm{Det}(f)$). We write $f = T^d + \sum_{i=0}^{d-1} \alpha_i T^i$ where $\alpha_i \in p \cdot \mathbb{Z}_p[G]$ for each integer i with $0 \le i < d$, and for each such i we set $R_i := r_\rho(\alpha_i) \in \mathrm{M}_n(\mathcal{O}_N)$. Upon denoting the $n \times n$ identity matrix by R_d, we obtain an equality $r_\rho(f) = \sum_{i=0}^{d} R_i T^i$ in $\mathrm{M}_n(\mathcal{O}_N[T])$. We observe that each off-diagonal entry of $r_\rho(f)$ belongs to $p \cdot \mathcal{O}_N[T]$ and is of degree strictly less than d, and that each diagonal entry of $r_\rho(f)$ is a monic polynomial which is congruent to T^d modulo $p \cdot \mathcal{O}_N[T]$. From this description it is immediately clear that $\mathrm{Det}(f)(\rho) := \det(r_\rho(f))$ is a monic polynomial which is congruent to T^{nd} modulo $p \cdot \mathcal{O}_N[T]$, and hence that it is distinguished, as claimed. □

Upon applying this lemma with f equal to G_S^* and H^* we deduce that the polynomials $\mathrm{Det}(G_S^*)(\rho)$ and $\mathrm{Det}(H^*)(\rho)$, and hence also $\mathrm{Det}(G_S^*)(\rho) \cdot H_\rho^*$ and $h_{S,\rho} \cdot \mathrm{Det}(H^*)(\rho)$, are distinguished. When combined with the equality (4) and the uniqueness of Weierstrass product decompositions in the ring $\mathcal{O}_N[[T]]$, this observation implies that $h_{S,\rho} \cdot \mathrm{Det}(H^*)(\rho) = \mathrm{Det}(G_S^*)(\rho) \cdot H_\rho^*$, as required by Hypothesis (ECP), and also that $c_\rho^{-1} U_{S,\rho} = \mathrm{Det}(U_S)(\rho)$, as required by Hypothesis (EUS).

iii) We now assume that G is abelian, so that $\mathbb{Z}_p[G]$ is admissible. Following claim ii), it is therefore enough for us to assume that Hypothesis (μ_p) is valid for K/k, and then to prove that Hypothesis (EPS) is valid for K/k.

To verify Hypothesis (EPS) for K/k we first assume that $K \cap k_\infty = k$. In this case, we may use the constructions of Deligne and Ribet which are used by Wiles in [39, p.501f.]. To be explicit, we obtain the elements G_S and H as required by Hypothesis (EPS) by combining in the obvious way the elements $G_{m,c,S}$ and $H_{m,c}$ of loc. cit., where m runs over the 'components' of $\mathbb{Z}_p[G]$ (and for each non-trivial component m we set $H_{m,c} := 1$). The required equalities $\rho(G_S)/\rho(H) = f_{S,\rho}$ (for each $\rho \in \mathrm{Irr}_p^+(G)$) and the fact that $\mu_{\mathbb{Z}_p[G]}(H) = 0$ then follow as direct consequences of the properties of the series $G_{m,c,S}$ and $H_{m,c}$ described by Wiles in loc. cit., and the fact that $\mu_{\mathbb{Z}_p[G]}(G_S) = 0$ follows from the assumed validity of Hypothesis (μ_p) for K/k. From Remark 4i) we therefore deduce that G_S and H are both right associated to distinguished polynomials, as required.

In general one has $K \cap k_\infty \neq k$, and in this case we proceed as follows. There exists an extension K' of k such that $K' \cap k_\infty = k$ and K_∞ is the compositum of K' and k_∞. We observe that K'/k is a finite abelian extension and we set $G' := \mathrm{Gal}(K'/k)$ and $\Gamma := \mathrm{Gal}(k_\infty/k)$. Each character $\rho \in \mathrm{Irr}_p(G)$ can be lifted to a character of $\mathrm{Gal}(K_\infty/k) \cong G' \times \Gamma$ (which we again denote by ρ), and as such it has a unique factorisation $\rho = \psi\kappa$ where $\psi \in \mathrm{Irr}_p(G')$ and κ is a character of Γ which has finite order. As a consequence of the formula [39, (1.4)] one has an equality $G_{S,\psi\kappa}(T) = G_{S,\psi}(\kappa(\gamma)(1+T)-1)$. By the very definition of the polynomials $H_{\psi\kappa}$ and H_ψ one also has an equality $H_{\psi\kappa}(T) = H_\psi(\kappa(\gamma)(1+T)-1)$.

We now write $G_{K',S}(T)$ and $H_{K'}(T)$ for the elements of $\mathbb{Z}_p[G'][[T]]$ which are afforded by Hypothesis (EPS) for the extension K'/k (which we know to be valid by the above argument since K', which sits between k and some K_n, is again either totally real or a CM field). Let $\tilde{G}_S(T) = G_{K',S}(\gamma(1+T)-1)$ and $\tilde{H}(T) = H_{K'}(\gamma(1+T)-1)$; these series lie in $\mathbb{Z}_p[[G' \times \Gamma]][[T]]$. Then applying the character ρ of G (considered as a character of $G' \times \Gamma$ by inflation) to $\tilde{G}_S(T)$ yields $\psi(G_{K',S})(\kappa(\gamma)(1+T)-1)$, similarly for $\tilde{H}(T)$, so applying ρ to the quotient $\tilde{G}_S(T)/\tilde{H}(T)$ gives $f_\psi(\kappa(\gamma)(1+T)-1)$ by Hypothesis (EPS) for K'/k. By the aforementioned formulas, we therefore have $f_\psi(\kappa(\gamma)(1+T)-1) = f_\rho(T)$ and so we are almost done: indeed, it simply suffices to define $G_S(T)$, respectively $H(T)$, to be equal to the image of $\tilde{G}_S(T)$, respectively $\tilde{H}(T)$, under the map $\mathbb{Z}_p[[G' \times \Gamma]][[T]] \to \mathbb{Z}_p[G][[T]]$ which is induced by the epimorphism $G' \times \Gamma \to G$. □

Remark 5. To end this section we now explain the precise connection between Hypothesis (EUS) and the 'Wiles Unit Conjecture' [34, Conj. 6.3.4] of Snaith. To do this we fix an integer n with $n > 1$ and, assuming Hypothesis (EUS) to be valid for K/k, we set $\alpha_{S,n} := U_S(u^n - 1) \in \mathbb{Z}_p[G]^\times$. Then for each $\rho \in \mathrm{Irr}_p^+(G)$ one has an equality

$$\mathrm{Det}(\alpha_{S,n})(\rho) = c_\rho^{-1} U_{S,\rho}(u^n - 1).$$

After taking account of the equalities (2) and (3) one finds that this property of $\alpha_{S,n}$ is closely related to that which should be satisfied by the element $\alpha_{n,K^+/k}$ of $\mathbb{Z}_p[\mathrm{Gal}(K^+/k)]^\times$ whose existence is predicted by [34, Conj. 6.3.4]. However, there are two important differences: in loc. cit. the p-adic L-functions are untruncated and the factors c_ρ are omitted. Whilst, a priori, Hypothesis (EUS) and [34, Conj. 6.3.4] could be simultaneously valid, we now present an explicit example which shows that [34, Conj. 6.3.4] is not valid (because the relevant p-adic L-functions are untruncated). We remark that similarly explicit examples exist to show that [34, Conj. 6.3.4] must also be corrected by the introduction of the factors c_ρ.

Example 2. We set $p := 3$ and $k := \mathbb{Q}$ and we let K denote the composite of the cyclic cubic extension K_1 of $\mathbb{Q}$ which has conductor 7 and the field $K_2 := \mathbb{Q}(\sqrt{5})$. We set $G := \mathrm{Gal}(K/\mathbb{Q})$ (which is cyclic), we write ρ_0 for the nontrivial character of $\mathrm{Gal}(K_2/\mathbb{Q})$ (considered as a character of G) and ρ_1 for any faithful character of G, and we set $S := \{5, 7\}$. Then ρ_0 and ρ_1 have conductors 5 and 35 respectively and Theorem 3.1iii) implies that there exists a unit element U_S of $\mathbb{Z}_p[G][[T]]$ such that, for $i \in \{1, 2\}$, the unit part U_{ρ_i} of the Iwasawa series f_{S,ρ_i} which is associated to $L_{p,S}(-, \rho_i)$ is equal to $\rho_i(U_S)$. We now let $w(T)$ be the power series such that $w(u^n - 1)L_{p,\{5\}}(1-n, \rho_0) = L_{p,S}(-, \rho_0)$ for all natural numbers n. Then $w(u^n - 1) = 1 - \rho_0(7)\omega^{-n}(7)7^{n-1}$ where ω is the 3-adic Teichmüller character. Since $\omega(7) = 1$ it follows that $w(T) = 1 - \rho_0(7) \cdot 7^{-1}(T+1)^a$ with $u^a = 7$. Now $\rho_0(7) = -1$ and so $w(0) = \frac{8}{7} \equiv -1 \pmod 3$; in particular $w(T) \in \mathbb{Z}_p[[T]]^\times$ and so the unit part U'_{ρ_0} of the Iwasawa series $f_{\{5\},\rho_0}$ is equal to $w(T)^{-1}U_{\rho_0}$. In this setting [34, Conj. 6.3.4] predicts the existence (for any given n) of an element α'_n of $\mathbb{Z}_3[G]^\times$ which satisfies both $\rho_0(\alpha'_n) = U'_{\rho_0}(u^n - 1)$ and $\rho_1(\alpha'_n) = U_{\rho_1}(u^n - 1)$. If such an element existed, then the element $q := U_S(u^n-1)(\alpha'_n)^{-1}$ of $\mathbb{Z}_3[G]^\times$ would satisfy both $\rho_0(q) = w(u^n-1)$ and $\rho_1(q) = 1$. However, the above calculation shows that $w(u^n-1) \equiv -1 \pmod 3$ and so no such element q can exist (indeed, ρ_0 and ρ_1 differ by a character of order 3 and so, for any $q' \in \mathbb{Z}_3[G]$, the elements $\rho_0(q')$ and $\rho_1(q')$ must be congruent modulo the maximal ideal of $\mathbb{Z}_3[\zeta_3]$).

4. Algebraic Preliminaries

In the remainder of this manuscript our aim is to describe certain explicit consequences of Theorem 3.1iii) concerning the values of Dirichlet L-functions at strictly negative integers. However, before doing so, in this section we describe some necessary algebraic preliminaries.

We now let K/k be any finite Galois extension of number fields of group G (which is not necessarily abelian). We fix any rational prime p and a finite set of places T of k which contains all archimedean places, all places which ramify in K/k and all places of residue characteristic p. For any extension E of k we write $\mathcal{O}_{E,T}$ for the ring of T_E-integers in E, where T_E denotes the set of places of E which lie above those in T. We set $U := \mathrm{Spec}(\mathcal{O}_{k,T})$ and we write $G_{k,T}$ for the Galois group of the maximal algebraic extension of k which is unramified outside T. For each non-negative integer n we write K_n for the subextension of K^p_∞ which is of degree p^n over K, and $\pi_n : \mathrm{Spec}(\mathcal{O}_{K_n,T}) \to U$ for the morphism of schemes which is induced by the inclusion $\mathcal{O}_{k,T} \subseteq \mathcal{O}_{K_n,T}$. If $\mathcal{F}$ is any finite $G_{k,T}$-module, then we use the same symbol to denote the associated locally-constant sheaf on the étale site $U_{\text{ét}}$. If $\mathcal{F}$ is any continuous $G_{k,T}$-module which is finitely generated over $\mathbb{Z}_p$, then we let $\mathcal{F}_\infty$ denote the associated pro-sheaf $(\mathcal{F}_n, t_n)_{n\geq 0}$ on $U_{\text{ét}}$ where, for each non-negative integer n, we set $\mathcal{F}_n := \pi_{n,*} \circ \pi_n^*(\mathcal{F}/p^{n+1})$ and the transition morphism t_n is induced by the composite of the trace map $\pi_{n+1,*} \circ \pi_{n+1}^*(\mathcal{F}/p^{n+2}) \to \pi_{n,*} \circ \pi_n^*(\mathcal{F}/p^{n+2})$ and the natural projection $\mathcal{F}/p^{n+2} \to \mathcal{F}/p^{n+1}$.

Let Λ be a pro-p ring. (Thus we depart here from the usual convention that Λ has the fixed meaning $\mathbb{Z}_p[[T]]$.) We write $\mathcal{D}(\Lambda)$ for the derived category of bounded complexes of Λ-modules and $\mathcal{D}^{\mathrm{p}}(\Lambda)$, respectively $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$, for the full triangulated subcategory of $\mathcal{D}(\Lambda)$ consisting of those complexes which are perfect, respectively are perfect and have finite cohomology groups.

If $\mathcal{F}$ is any (p-adic) étale sheaf of Λ-modules on U, then we follow the approach of [9, §3.2] to define the complex of compactly supported cohomology $R\Gamma_c(U_{\text{ét}}, \mathcal{F})$ so as to lie in a canonical distinguished triangle in $\mathcal{D}(\Lambda)$

$$R\Gamma_c(U_{\text{ét}}, \mathcal{F}) \longrightarrow R\Gamma(U_{\text{ét}}, \mathcal{F}) \longrightarrow \bigoplus_{v\in T} R\Gamma(\mathrm{Spec}(k_v)_{\text{ét}}, \mathcal{F}). \tag{5}$$

We recall that the approach developed by Kato in [25, §3.1] and by Nekovář in [30] (cf. also [11, Rem. 4.1] in this regard) allows one to extend the definitions of each of these complexes in a natural manner to the case of pro-sheaves of Λ-modules of the form $\mathcal{F}_\infty$ discussed above, and that in this case there is again a canonical distinguished triangle of the form (5). If $p = 2$, then we set $R\Gamma_*(U_{\text{ét}}, -) := R\Gamma_c(U_{\text{ét}}, -)$. If p is odd, then we let $R\Gamma_*(U_{\text{ét}}, -)$ denote either $R\Gamma_c(U_{\text{ét}}, -)$ or $R\Gamma(U_{\text{ét}}, -)$. In each degree i we then set $H^i_*(U_{\text{ét}}, -) := H^i R\Gamma_*(U_{\text{ét}}, -)$.

We next recall that if Λ is any $\mathbb{Z}_p$-order which spans a finite dimensional semisimple $\mathbb{Q}_p$-algebra $\Lambda_{\mathbb{Q}_p}$, then to each object C of $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$ one can associate a canonical element $\chi^{\mathrm{rel}}_\Lambda C$ of the relative algebraic K-group $K_0(\Lambda, \mathbb{Q}_p)$ (cf. [5, Prop. 1.2.1] or [9, §2.8, Rem. 4]). We recall further that the Whitehead group $K_1(\Lambda_{\mathbb{Q}_p})$ of $\Lambda_{\mathbb{Q}_p}$ is generated by elements of the form $[\alpha]$ where α is an automorphism of a finitely generated $\Lambda_{\mathbb{Q}_p}$-module, and we write $\delta_\Lambda : K_1(\Lambda_{\mathbb{Q}_p}) \to K_0(\Lambda, \mathbb{Q}_p)$ for the homomorphism which occurs in the long exact sequence of relative K-theory (as described explicitly in, for example, [5, §1.1]).

Proposition 4.1. *Let $\mathcal{F}$ be a continuous $\mathbb{Z}_p[G_{k,T}]$-module which is both finitely generated and free over $\mathbb{Z}_p$. Then $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_\infty)$ is an object of $\mathcal{D}^{\mathrm{p}}(\mathbb{Z}_p[\mathrm{Gal}(K^p_\infty/k)])$.*

Assume now that $K \cap k^p_\infty = k$. Let ϵ be a central idempotent of $\mathbb{Z}_p[G]$, set $\Lambda := \mathbb{Z}_p[G]\epsilon$, and let θ_∞ be an injective $\mathbb{Z}_p[\mathrm{Gal}(K^p_\infty/k)]$-equivariant endomorphism of $\epsilon \cdot \mathcal{F}_\infty$. If both

- ci) *in each degree i the $\mathbb{Z}_p$-module $H^i_*(U_{\text{ét}}, \epsilon \cdot \mathcal{F}_\infty)$ is finitely generated, and*
- cii) *in each degree i the endomorphism $H^i_*(U_{\text{ét}}, \theta_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is bijective,*

then $R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty))$ is an object of $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$, and in $K_0(\Lambda, \mathbb{Q}_p)$ one has an equality

$$\chi^{\mathrm{rel}}_\Lambda R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty)) = \sum_{i \in \mathbb{Z}} (-1)^i \delta_\Lambda([H^i_*(U_{\text{ét}}, \theta_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p]).$$

Proof. For each non-negative integer n we set $\Lambda_n := (\mathbb{Z}/p^{n+1})[\mathrm{Gal}(K_n/k)]$. We also set $\Lambda_\infty := \varprojlim_n \Lambda_n$ where the limit is taken with respect to the natural projection morphisms $\rho_n : \Lambda_{n+1} \to \Lambda_n$. In the sequel we identify Λ_∞ with $\mathbb{Z}_p[\mathrm{Gal}(K^p_\infty/k)]$ in the natural way.

We first note that, for each non-negative integer n, $\mathcal{F}_n$ is the sheaf which is associated to the free Λ_n-module $\Lambda_n \otimes_{\mathbb{Z}_p} \mathcal{F}$ and that t_n is the morphism which is associated to the natural morphism of Λ_{n+1}-modules

$$\Lambda_{n+1} \otimes_{\mathbb{Z}_p} \mathcal{F} \to \Lambda_n \otimes_{\Lambda_{n+1}, \rho_n} (\Lambda_{n+1} \otimes_{\mathbb{Z}_p} \mathcal{F}) \cong \Lambda_n \otimes_{\mathbb{Z}_p} \mathcal{F}.$$

By using results of Flach [19, Thm. 5.1, Prop. 4.2] we may therefore deduce that, for each such n, $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_n)$ is an object of $\mathcal{D}^{\mathrm{p}}(\Lambda_n)$ which is acyclic outside degrees $0, 1, 2, 3$ and is also such that there exists an isomorphism ψ_n in $\mathcal{D}^{\mathrm{p}}(\Lambda_n)$ between $\Lambda_n \otimes^{\mathbb{L}}_{\Lambda_{n+1}, \rho_n} R\Gamma_*(U_{\text{ét}}, \mathcal{F}_{n+1})$ and $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_n)$.

We observe next that Λ_{n+1} is Artinian and that $\ker(\rho_n)$ is a two sided nilpotent ideal. By using the structure theory of [17, Prop. (6.17)] we may thus deduce that for any morphism of finitely generated projective Λ_n-modules $\phi_n : M_n \to N_n$ there exists a morphism of finitely generated projective Λ_{n+1}-modules $\phi_{n+1} : M_{n+1} \to N_{n+1}$ for which one has $M_n = \Lambda_n \otimes_{\Lambda_{n+1}, \rho_n} M_{n+1}$,

$N_n = \Lambda_n \otimes_{\Lambda_{n+1},\rho_n} N_{n+1}$ and $\phi_n = \Lambda_n \otimes_{\Lambda_{n+1},\rho_n} \phi_{n+1}$. This fact allows one to adapt the constructions of Milne in [29, p.264-265] and hence to prove that, for each non-negative integer n, there exists a complex of finitely generated projective Λ_n-modules $C_n^\cdot$ with the following properties: $C_n^i = 0$ for $i \notin \{0,1,2,3\}$; $C_n^\cdot$ is isomorphic in $\mathcal{D}^{\mathrm{p}}(\Lambda_n)$ to $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_n)$; there exists a Λ_{n+1}-equivariant homomorphism of complexes $\psi'_n : C_{n+1}^\cdot \to C_n^\cdot$ which is such that the morphism $\Lambda_n \otimes_{\Lambda_{n+1},\rho_n} \psi'_n : \Lambda_n \otimes_{\Lambda_{n+1},\rho_n} C_{n+1}^\cdot \to C_n^\cdot$ is bijective in each degree and induces ψ_n. In this way we obtain a bounded complex of finitely generated projective Λ_∞-modules $C_\infty^\cdot := \varprojlim_{\psi'_n} C_n^\cdot$ which represents $R\Gamma_*(U_{\text{ét}}, \mathcal{F}_\infty)$. This proves the first claim of the proposition.

We now assume that $K \cap k_\infty^p = k$ and that θ_∞ is an injective Λ_∞-equivariant endomorphism of the pro-sheaf $\epsilon \cdot \mathcal{F}_\infty$. By adapting the constructions of [29, Chap. VI, Lem. 8.17, Lem. 13.10] (but note that [loc. cit., Lem. 8.17] is incorrect as stated since the morphism ψ need not be a quasi-isomorphism) it may be shown that there exists a Λ_∞-equivariant endomorphism $\theta_\infty^\cdot$ of the complex $D_\infty^\cdot := \epsilon \cdot C_\infty^\cdot$ which induces the morphism $R\Gamma_*(U_{\text{ét}}, \theta_\infty)$. In this way one obtains a canonical short exact sequence of complexes

$$0 \to D_\infty^\cdot \to \mathrm{Cone}(\theta_\infty^\cdot) \to D_\infty^\cdot[1] \to 0 \tag{6}$$

and also an isomorphism in $\mathcal{D}^{\mathrm{p}}(\Lambda_\infty)$ between $\mathrm{Cone}(\theta_\infty^\cdot)$ and $R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty))$.

Now Λ_∞ is a free $\mathbb{Z}_p[G]$-module and so $D_\infty^\cdot$ is a bounded complex of projective Λ-modules. If also each $\mathbb{Z}_p$-module $H^i(D_\infty^\cdot)$ is finitely generated, as is implied by condition ci), then by a standard argument (see, for example, the proof of [12, Thm. 1.1, p.447]) it follows that $D_\infty^\cdot$ belongs to $\mathcal{D}^{\mathrm{p}}(\Lambda)$. The exact sequence (6) then implies that $\mathrm{Cone}(\theta_\infty^\cdot)$ also belongs to $\mathcal{D}^{\mathrm{p}}(\Lambda)$. Further, condition cii) now combines with the long exact sequence of cohomology which is associated to (6) to imply that each module $H_*^i(\mathrm{Cone}(\theta_\infty^\cdot))$ is finite and hence that $\mathrm{Cone}(\theta_\infty^\cdot)$ belongs to $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$, as claimed.

It only remains to prove the explicit formula for $\chi_\Lambda^{\mathrm{rel}} R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty))$. To do this we let $P^\cdot$ be a bounded complex of finitely generated projective Λ-modules which is quasi-isomorphic to $D_\infty^\cdot$ and $\hat{\theta}^\cdot : P^\cdot \to P^\cdot$ a morphism of complexes which induces $\theta_\infty^\cdot$. Condition cii) combines with the argument of [13, Lem. 7.10] to imply we may assume that in each degree i the map $\hat{\theta}^i$ is injective (and so has finite cokernel). It follows that $R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty))$ is isomorphic in $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$ to the complex $\mathrm{coker}(\hat{\theta}^\cdot)$ which is equal to $\mathrm{coker}(\hat{\theta}^i)$ in each degree i and for which the differentials are induced by those of $P^\cdot$, and hence that $\chi_\Lambda^{\mathrm{rel}} R\Gamma_*(U_{\text{ét}}, \mathrm{coker}(\theta_\infty)) = \chi_\Lambda^{\mathrm{rel}} \mathrm{coker}(\hat{\theta}^\cdot)$.

We next recall that $\chi_\Lambda^{\mathrm{rel}}$ is additive on exact triangles in $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$ [5, Prop. 1.2.2]. From the short exact sequences of complexes

$$0 \to \mathrm{coker}(\hat{\theta}^i)[-i] \to \tau_i \,\mathrm{coker}(\hat{\theta}^\cdot) \to \tau_{i-1} \,\mathrm{coker}(\hat{\theta}^\cdot) \to 0$$

(where, for each integer j, τ_j denotes the naive truncation in degree j) we may therefore deduce that $\chi_\Lambda^{\mathrm{rel}} \mathrm{coker}(\hat{\theta}^\cdot) = \sum_{i\in\mathbb{Z}} \chi_\Lambda^{\mathrm{rel}}(\mathrm{coker}(\hat{\theta}^i)[-i])$. The claimed formula now follows directly from the fact that for each integer i one has $\chi^{\mathrm{rel}}_\Lambda(\mathrm{coker}(\hat\theta^i)[-i]) = (-1)^i\delta_\Lambda([\hat\theta^i\otimes_{\mathbb{Z}_p}\mathbb{Q}_p])$ and in $K_1(\Lambda_{\mathbb{Q}_p})$ there is an equality

$$[\hat\theta^i\otimes_{\mathbb{Z}_p}\mathbb{Q}_p] = [\hat\theta^{i+1}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\mid_{B^{i+1}}] + [H^i_*(U_{\acute{e}t},\theta_\infty)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p] + [\hat\theta^i\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\mid_{B^i}].$$

Here we write B^i for the submodule of coboundaries of $P^\cdot\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ in degree i, and the displayed equality is a consequence of the natural filtration of $P^i\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ which has graded pieces isomorphic to B^{i+1}, $H^i_*(U_{\acute{e}t},\epsilon\cdot\mathcal{F}_\infty)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and B^i. □

Remark 6. In this remark we assume that G is abelian, but otherwise use the same notation and hypotheses as in the second part of Proposition 4.1. We write Det_Λ for the determinant functor introduced by Knudsen and Mumford in [27], and (both here and in the sequel) we identify any graded invertible Λ-module of the form $(I,0)$ with the underlying invertible Λ-module I.
We recall that the assignment $\chi^{\mathrm{rel}}_\Lambda C\mapsto \mathrm{Det}_\Lambda C$ (where C ranges over all objects of $\mathcal{D}^{\mathrm{p,f}}(\Lambda)$) induces a well-defined isomorphism between $K_0(\Lambda,\mathbb{Q}_p)$ and the multiplicative group of invertible Λ-lattices in $\Lambda_{\mathbb{Q}_p}$ (cf. [1, Lem. 2.6]). In particular, in this case the equality at the end of Proposition 4.1 is equivalent to the following equality in $\Lambda_{\mathbb{Q}_p}$

$$\mathrm{Det}_\Lambda R\Gamma_*(U_{\acute{e}t},\mathrm{coker}(\theta_\infty)) = \prod_{i\in\mathbb{Z}}\mathrm{det}_{\Lambda_{\mathbb{Q}_p}}(H^i_*(U_{\acute{e}t},\theta_\infty)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{(-1)^{i+1}}\cdot\Lambda.$$

(We remark that the exponent $(-1)^{i+1}$ on the right hand side of this formula is not a misprint!)

5. Values of Dirichlet L-functions

In this section we derive certain explicit consequences of Theorem 3.1iii) and Proposition 4.1 concerning the values of Dirichlet L-functions at strictly negative integers.
To this end we continue to use the notation introduced in §3. In particular, we now assume that k is totally real and that K is a CM *abelian* extension of k and we set $G := \mathrm{Gal}(K/k)$. We also fix an odd prime p, algebraic closures $\mathbb{Q}^c$ of $\mathbb{Q}$ and $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$, and we set $G^\wedge := \mathrm{Hom}(G,\mathbb{Q}^{c\times})$ and $G^{\wedge,p} := \mathrm{Hom}(G,\mathbb{Q}_p^{c\times})$. We let τ denote the complex conjugation in G, and for each integer a we write e_a for the idempotent $\frac{1}{2}(1+(-1)^a\tau)$ of $\mathbb{Z}[\frac{1}{2}][G]$, and $G^\wedge_{(a)}$ and $G^{\wedge,p}_{(a)}$ for the subsets of $G^\wedge$ and $G^{\wedge,p}$ respectively which consist of those characters ψ satisfying $\psi(\tau) = (-1)^a$. For each element ψ of $G^\wedge$, respectively of $G^{\wedge,p}$, we write e_ψ for the associated idempotent $\frac{1}{|G|}\sum_{g\in G}\psi(g)g^{-1}$ of $\mathbb{Q}^c[G]$, respectively of $\mathbb{Q}_p^c[G]$.
We fix a finite set S of non-archimedean places of k which contains all non-archimedean places which ramify in K/k and, for each $\psi\in G^\wedge$, we write $L_S(s,\psi)$ for the Dirichlet L-function of ψ which is truncated by removing the Euler factors at all places in S.

If r is any integer with $r > 1$, then each function $L_S(s, \psi)$ is holomorphic at $s = 1 - r$ and so we may set

$$L_S(1-r) := \sum_{\psi \in G^\wedge} L_S(1-r, \psi) e_\psi \in \mathbb{C}[G].$$

If $k = \mathbb{Q}$, then this element can be interpreted in terms of higher Bernoulli numbers and is therefore a natural analogue of the classical Stickelberger element. In general, by a result of Siegel [33], one knows that $L_S(1-r)$ belongs to the unit group of the ring $\mathbb{Q}[G]e_r$.

In order to state our next result we assume that $K \cap k^p_\infty = k$. Under this hypothesis we set

$$h_S := \sum_{\rho \in G^{\wedge,p}_{(0)}} h_{S,\rho} e_\rho \in \mathbb{Z}_p[G][[T]][\frac{1}{p}].$$

We also write e for the idempotent $\frac{1}{|G|} \sum_{g \in G} g$ of $\mathbb{Q}_p[G]$ and then set

$$H' := \sum_{\rho \in G^{\wedge,p}_{(0)}} H_\rho e_\rho = Te + (e_0 - e) \in \mathbb{Q}_p[G][T],$$

where the second equality is a consequence of our assumption that $K \cap k^p_\infty = k$. For the purposes of the next result we also assume that K contains a primitive p-th root of unity, and we write ω for the Teichmüller character of G. For each integer b we then let tw_b denote the $\mathbb{Z}_p$-linear automorphism of $\mathbb{Z}_p[G]$ which sends each element g of G to $\omega^b(g) \cdot g$.

THEOREM 5.1. *Assume that Hypothesis (μ_p) is valid for K/k, that $K \cap k^p_\infty = k$ and that K contains a primitive p-th root of unity. Then for each integer $r > 1$ one has an equality*

$$L_S(1-r) \cdot \mathbb{Z}_p[G] = \mathrm{tw}_r(H'(u^r - 1)^{-1} h_S(u^r - 1)) \cdot \mathbb{Z}_p[G]e_r.$$

Proof. At the outset we fix an integer $r > 1$ and an embeddding $j : \mathbb{Q}^c \to \mathbb{Q}^c_p$ and, for each $\chi \in G^\wedge$, we set $\rho_\chi := (j \circ \chi) \cdot \omega^r \in G^{\wedge,p}$.

We observe that ω belongs to $G^{\wedge,p}_{(1)}$ and hence that χ belongs to $G^\wedge_{(r)}$ if and only if ρ_χ belongs to $G^{\wedge,p}_{(0)}$. In addition, for all characters $\chi \in G^\wedge_{(r)}$ one has an equality

$$(j \circ \chi)(L_{S_p}(1-r)) = j(L_{S_p}(1-r, \chi)) = L_{p,S}(1-r, \rho_\chi). \tag{7}$$

We now set $Z_S := (H')^{-1} \cdot h_S$. Upon comparing images under each character $\rho \in G^{\wedge,p}$, recalling the equality (3) and noting that in the present case $c_\rho = 1$ for all such ρ, we may deduce that Z_S is equal to the quotient G^*_S/H^* which occurs in Hypothesis (ECP). With U_S denoting the unit element which occurs in Hypothesis (EUS) for K/k, and setting $G_S := G^*_S \cdot U_S$ and $H := H^*$ it therefore follows that

$$\begin{aligned} Z_S \cdot U_S &= (G^*_S \cdot U_S) \cdot (H^*)^{-1} \\ &= G_S \cdot H^{-1}. \end{aligned}$$

From Theorem 3.1i),iii) we know that the series G_S and H satisfy the conditions specified in Hypothesis (EPS). Hence the last displayed formula implies that, for each character $\chi \in G^{\wedge}_{(r)}$, one has

$$\begin{aligned}\rho_\chi(Z_S \cdot U_S) &= \rho_\chi(G_S)\rho_\chi(H)^{-1}\\ &= f_{S,\rho_\chi}.\end{aligned}$$

Upon combining this formula with the equalities (1) and (7) we deduce that

$$\begin{aligned}(j\circ\chi)(L_{S_p}(1-r)) &= \rho_\chi(Z_S(u^r-1)\cdot U_S(u^r-1))\\ &= (j\circ\chi)(\mathrm{tw}_r(Z_S(u^r-1)\cdot U_S(u^r-1)))\\ &= (j\circ\chi)(\mathrm{tw}_r(Z_S(u^r-1))\cdot \mathrm{tw}_r(U_S(u^r-1))).\end{aligned}$$

Since this equality is valid for every character χ in $G^{\wedge}_{(r)}$ it implies that the elements $L_{S_p}(1-r)$ and $\mathrm{tw}_r(Z_S(u^r-1))$ of $\mathbb{Q}_p[G]e_r$ differ by the factor $\mathrm{tw}_r(U_S(u^r-1))$ which is a unit of the ring $\mathbb{Z}_p[G]e_r$.
It now only remains for us to show that the elements $L_{S_p}(1-r)$ and $L_S(1-r)$ differ by a unit of $\mathbb{Z}_p[G]e_r$. But $L_{S_p}(1-r) = L_S(1-r)x$ where x is a product of Euler factors of the form $1-\mathrm{N}v^{r-1}\cdot f_v$ where v is a place of k which divides p and does not belong to S, $\mathrm{N}v$ is the absolute norm of v and f_v is the Frobenius automorphism of v in G. Further, since $r>1$, it is clear that each such element $1-\mathrm{N}v^{r-1}\cdot f_v$ is a unit of $\mathbb{Z}_p[G]$. □

Our next result concerns a special case of Kato's Generalized Iwasawa Main Conjecture. However, before stating this result, it will be convenient to introduce some further notation.
For the remainder of this section we let Σ denote the (finite) set of rational primes ℓ which satisfy either $\ell = 2$ or $K \cap k^{\ell}_{\infty} \neq k$. We also write $\mathbb{Z}_\Sigma$ for the subring of $\mathbb{Q}$ which is generated by the inverses of each element of Σ.
For any extension E of k and any finite set of places V of k we let $\mathcal{O}_{E,V}$ denote the ring of V_E-integers in E, where V_E denotes the set of all places of E which are either archimedean or lie above a place in V. We set $U_k := \mathrm{Spec}(\mathcal{O}_{k,S_p})$ and for each p-adic étale sheaf $\mathcal{F}$ on U_k and each finite Galois extension E/k which is unramified at all non-archimedean places outside S_p we write $\mathcal{F}_E$ for the étale sheaf of $\mathbb{Z}_p[\mathrm{Gal}(E/k)]$-modules $\pi_*\pi^*\mathcal{F}$ on U_k where π denotes the morphism $\mathrm{Spec}(\mathcal{O}_{E,S_p}) \to U_k$ which is induced by the inclusion $\mathcal{O}_{k,S_p} \subseteq \mathcal{O}_{E,S_p}$. We recall that, since π_* is exact, the complexes $R\Gamma(U_{k,\text{ét}}, \mathcal{F}_E)$ and $R\Gamma_c(U_{k,\text{ét}}, \mathcal{F}_E)$ are canonically isomorphic in $\mathcal{D}(\mathbb{Z}_p[\mathrm{Gal}(E/k)])$ to $R\Gamma(\mathrm{Spec}(\mathcal{O}_{E,S_p})_{\text{ét}}, \pi^*\mathcal{F})$ and $R\Gamma_c(\mathrm{Spec}(\mathcal{O}_{E,S_p})_{\text{ét}}, \pi^*\mathcal{F})$ respectively, and in the sequel we shall often use such identifications without explicit comment.
For each integer $r>1$ we set

$$C_{K,1-r} := R\Gamma_c(U_{k,\text{ét}}, e_r\mathbb{Z}_p(1-r)_K)$$

and we recall that (since $r>1$) this complex is an object of $\mathcal{D}^{\mathrm{p,f}}(\mathbb{Z}_p[G]e_r)$ (see the upcoming proof of Lemma 3 for further details in this regard). From the equalities of [8, (11),(12)] (with r replaced by $1-r$), it therefore follows that the 'Equivariant Tamagawa Number Conjecture' of [9, Conj. 4(iv)] is for the pair $(h^0(\mathrm{Spec}(K))(1-r), \mathbb{Z}_\Sigma[G]e_r)$ equivalent to asserting that if p does not belong to Σ, then in $\mathbb{Q}_p[G]e_r$ one has an equality

$$\mathrm{Det}^{-1}_{\mathbb{Z}_p[G]e_r} C_{K,1-r} = L_S(1-r)\cdot \mathbb{Z}_p[G] \tag{8}$$

(cf. Remark 6).

Before proceeding, we remark that the above equality is in general strictly finer than the corresponding case of the Generalized Iwasawa Main Conjecture which Kato formulates in [25, Conj. 3.2.2 and 3.4.14]. Indeed, since graded determinants are not used in [25] the central conjecture of loc. cit. is in this case only well defined to within multiplication by elements of $(\mathbb{Q}_p[G]e_r)^\times$ of square 1 which reflect possible re-ordering of the factors in tensor products. For more details in this regard we refer the reader to [loc cit., Rem. 3.2.3(3) and 3.2.6(3),(5)] and [9, Rem. 9]. We recall also that a direct comparison of [9, Conj. 4(iv)] with the central conjecture formulated by Kato in [24, Conj. (4.9)] can be found in [10, §2].

THEOREM 5.2. *Assume that p does not belong to Σ and that Hypothesis (μ_p) is valid for K/k. Then for each integer $r > 1$ the equality (8) is valid. In particular, the Generalized Iwasawa Main Conjecture of Kato is valid for each such pair $(h^0(\mathrm{Spec}(K))(1-r), \mathbb{Z}_p[G]e_r)$.*

Remark 7. i) It is straightforward to describe explicit conditions on K/k which ensure that $\Sigma = \{2\}$. For example, if $[K : k]$ is coprime to the class number of k, then $\Sigma = \{2\}$ whenever the conductor of K/k is not divisible by the square of any prime ideal which divides $[K : k]$.

ii) If $K/\mathbb{Q}$ is abelian, then Hypothesis (μ_p) is known to be valid for all p (Remark 3i)) and so Theorem 5.2 gives an alternative proof of parts of the main result (Cor. 8.1) of [11]. The reader will find that the approach of loc. cit. is considerably more involved than that used here. We remark that, nevertheless, the approach of loc. cit. can be extended to improve upon Theorem 5.2 by showing that [9, Conj. 4(iv)] is valid for the pair $(h^0(\mathrm{Spec}(K))(1-r), \mathbb{Z}[\frac{1}{2}][G]e_r)$ under the assumption that Hypothesis (μ_p) is valid for K/k at all odd p.

Proof of Theorem 5.2. For the purposes of this argument we set $\mathfrak{A} := \mathbb{Z}_p[G]e_r$ and $A := \mathbb{Q}_p[G]e_r$.

We first remark that, when verifying the equality (8), the functorial behaviour of compactly supported étale cohomology and of Dirichlet L-functions under Galois descent allows us to replace K by the extension of K which is generated by a primitive p-th root of unity (cf. [9, Prop. 4.1b)]). We may therefore henceforth assume that K contains a primitive p-th root of unity and is such that $K \cap k^p_\infty = k$. After taking into account the result of Theorem 5.1 it is therefore enough for us to prove that in $\mathbb{Q}_p[G]e_r$ one has an equality

$$\mathrm{Det}^{-1}_{\mathfrak{A}} C_{K,1-r} = \mathrm{tw}_r(H'(u^r-1)^{-1}h_S(u^r-1))\cdot\mathfrak{A}. \tag{9}$$

We now use the notation of Proposition 4.1. We regard γ as a topological generator of $\mathrm{Gal}(K^p_\infty/K)\cong \mathrm{Gal}(k^p_\infty/k)$, we set $\hat\gamma := 1-\gamma \in \mathbb{Z}_p[\mathrm{Gal}(K^p_\infty/k)]$ and we observe that the action of $\hat\gamma$ induces an injective $\mathbb{Z}_p[\mathrm{Gal}(K^p_\infty/k)]$-equivariant endomorphism $\hat\gamma_{1-r}$ of the pro-sheaf $e_r\cdot\mathbb{Z}_p(1-r)_\infty$ on $U_{k,\text{ét}}$.

Lemma 3. *Let T denote the union of S_p and the set of archimedean places of k, and set $U := U_k, \mathcal{F} := \mathbb{Z}_p(1-r), \epsilon := e_r$ and $\theta_\infty := \hat\gamma_{1-r}$. If Hypothesis ($\mu_p$) is valid for K/k, then this data satisfies the conditions ci) and cii) of Proposition 4.1, and in A one has an equality*

$$\prod_{i\in\mathbb{Z}}\det{}_A(H^i_c(U_{k,\text{ét}},\hat\gamma_{1-r})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{(-1)^i}$$
$$=\det{}_A(\hat\gamma\mid e_0Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-r))\cdot\det{}_A(\hat\gamma\mid\mathbb{Q}_p(-r))^{-1}$$

where $\hat\gamma$ acts diagonally on $e_0Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-r)$.

Proof. We assume that Hypothesis (μ_p) is valid for K/k, and we recall (from Remark 3ii)) that this is equivalent to asserting that $e_0Y(S_p)$ is a finitely generated $\mathbb{Z}_p$-module.

For each integer i we set $H^i_c(1-r) := H^i_c(U_{\text{ét}}, e_r\cdot\mathbb{Z}_p(1-r)_\infty)$. To verify that condition ci) of Proposition 4.1 is satisfied by the given data and also to prove the claimed equality, it is clearly enough to show that $H^i_c(1-r)$ vanishes if $i\notin\{2,3\}$ and that $H^2_c(1-r)$ and $H^3_c(1-r)$ are canonically isomorphic to $e_0Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(-r)$ (endowed with the natural diagonal action of γ) and $\mathbb{Z}_p(-r)$ respectively.

To show this we first observe that, since p is odd and k is totally real, for each archimedean place v of k and any non-negative integer n the complex $R\Gamma(\mathrm{Spec}(k_v)_{\text{ét}}, e_r\cdot\mathbb{Z}_p(1-r)_n)$ is acyclic. This implies that our definition of compactly supported cohomology (as in (5)) coincides with that used by Nekovář in [30, (5.3)], and hence that the complex $R\Gamma_c(U_{\text{ét}}, e_r\cdot\mathbb{Z}_p(1-r)_\infty)$ coincides with the complex $R\Gamma_{c,\mathrm{Iw}}(K_\infty/k,\mathbb{Z}_p(1-r))$ which is defined in [loc. cit., (8.5.4)]. To compute $H^i_c(1-r)$ we may therefore use the fact that there are natural isomorphisms of $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/k)]$-modules

$$\begin{aligned} H^i_c(1-r) &\cong e_r(H^i_c(U_{\text{ét}},\mathbb{Z}_p(1)_\infty\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(-r))) \\ &\cong (\varprojlim_n H^{i,+}_{c,n}(1))\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(-r) \\ &\cong (\varprojlim_n H^{i,+}_{c,n,n}(1))\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(-r) \end{aligned} \tag{10}$$

where, for each non-negative integer n, we set $H^{i,+}_{c,n}(1) := e_0\cdot H^i_c(U_{k,\text{ét}},\mathbb{Z}_p(1)_{K_n})$ and $H^{i,+}_{c,n,n}(1) := e_0\cdot H^i_c(U_{k,\text{ét}},(\mu_{p^{n+1}})_{K_n})$, each limit over the integers $n\geq 0$ is taken with respect to the natural projection maps, $\mathrm{Gal}(K_\infty/k)$ acts diagonally on each tensor product, and the second and third isomorphisms follow as a consequence of [loc. cit., Prop 8.5.5(ii), respectively Lem. (4.2.2)].

Now to compute explicitly each group $H^{i,+}_{c,n}(1)$ for $i\neq 2$ it is enough to combine the long exact sequence of cohomology of the triangle (5) (with $U=U_k$ and

$\mathcal{F} = \mathbb{Z}_p(1)_{K_n}$) together with certain standard results of Kummer theory and class field theory. To describe the result we write

$$\lambda_n : \mathcal{O}^\times_{K_n,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \prod_{w_n} \varprojlim_{m\geq 1} K^\times_{n,w_n}/(K^\times_{n,w_n})^{p^m}$$

for the natural 'diagonal' morphism where on the right hand side w_n runs over all places of K_n which lie above places in T and the limit over m is taken with respect to the natural projection morphisms. Then one finds that $H^{i,+}_{c,n}(1)$ vanishes if $i \notin \{1,2,3\}$, that $H^{3,+}_{c,n}(1)$ identifies with $\mathbb{Z}_p$ and that $H^{1,+}_{c,n}(1)$ is isomorphic to $e_0 \cdot \ker(\lambda_n)$. Upon passing to the inverse limit over n (and by using (10)) one finds that $H^i_c(1-r)$ vanishes if $i \notin \{2,3\}$ and that $H^3_c(1-r)$ is canonically isomorphic to $\mathbb{Z}_p(-r)$.

To proceed we next recall that, for each pair of non-negative integers m and n, the Artin-Verdier Duality Theorem induces a canonical isomorphism in $\mathcal{D}(\mathbb{Z}/p^m\mathbb{Z}[\mathrm{Gal}(K_n/k)])$

$$R\Gamma_c(U_{\text{ét}}, e_r(\mu_{p^m}^{\otimes(1-r)})_{K_n}) \cong \mathrm{Hom}_{\mathbb{Z}/p^m\mathbb{Z}}(R\Gamma(U_{\text{ét}}, e_r(\mu_{p^m}^{\otimes r})_{K_n}), \mathbb{Z}/p^m\mathbb{Z}[-3])$$

where the linear dual is endowed with the contragredient action of $\mathrm{Gal}(K_n/k)$ (cf. [30, Prop. (5.4.3)(i), (2.11)] with $R = \mathbb{Z}/p^m\mathbb{Z}[\mathrm{Gal}(K_n/k)], J = R[0], K = k$, $S = T$ and $X = e_r(\mu_{p^m}^{\otimes(1-r)})_{K_n}[0]$). Now if $K^{\mathrm{ab},n}_{n,T}$ denotes the maximal abelian extension of K_n which is unramified outside T and of exponent dividing p^{n+1}, then the above isomorphism (with $r=0$ and $m = n+1$) implies that $H^{2,+}_{c,n,n}(1)$ is canonically isomorphic to $e_0 \cdot \mathrm{Gal}(K^{\mathrm{ab},n}_{n,T}/K_n)$. These isomorphisms are compatible with the natural transition morphisms as n varies and hence upon passing to the inverse limit (and using (10)) we obtain a canonical isomorphism between $H^2_c(1-r)$ and $e_0 Y(S_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-r)$ (endowed with the diagonal action of $\mathrm{Gal}(K_\infty/k)$), as required.

At this stage we need only verify that condition cii) of Proposition 4.1 is satisfied by the specified data. However, this is so because $\mathrm{coker}(\hat{\gamma}_{1-r})$ is isomorphic to the constant pro-sheaf $e_r\mathbb{Z}_p(1-r)_K$ and all cohomology groups of the complex $C_{K,1-r}$ are finite. To explain the latter fact we recall that, for any given n and r, the above displayed duality isomorphisms are compatible with the natural transition morphisms as m varies and hence (in the case $n=0$) induce upon passing to the inverse limit a canonical isomorphism in $\mathcal{D}(\mathfrak{A})$

$$C_{K,1-r} \cong R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma(U_{\text{ét}}, e_r\mathbb{Z}_p(r)_K), \mathbb{Z}_p[-3]). \tag{11}$$

where the linear dual is endowed with the action of $\mathfrak{A}$ which is induced by the contragredient action of G. (The existence of such an isomorphism also follows from the exactness of the central column of [8, diagram (114)] (where L corresponds to our field K) and the fact that each complex $R\Gamma_\Delta(L_w, \mathbb{Z}_p(1-r))^*$ which occurs in that diagram becomes acyclic upon multiplication by e_r.) Now, after taking (11) into account, it is enough for us to prove that all of the groups $H^i(U_{\text{ét}}, e_r\mathbb{Z}_p(r)_K)$ are finite and this follows, for example, as a consequence of the description of [11, Lem. 3.2ii)] and the fact that $e_r(K_{2r-i}(\mathcal{O}_{K,T}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ is finite for both $i \in \{1,2\}$. □

We next observe that (as can be verified by explicit computation)

$$\begin{aligned}\det_A(\hat{\gamma} \mid \mathbb{Q}_p(-r)) &= (1-u^{-r})e_{\omega^{-r}} + (e_r - e_{\omega^{-r}})\\ &= \mathrm{tw}_r(v_r \cdot H'(u^r-1))\end{aligned}$$

where $v_r := u^{-r}e + (e_0 - e)$, and also

$$\begin{aligned}&\det_A(\hat{\gamma} \mid e_0Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-r))\\ =& \sum_{\rho\in G^{\wedge,p}_{(r)}} \det_{\mathbb{Q}_p^c}(1-\gamma \mid e_\rho(Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c(-r)))e_\rho\\ =& \sum_{\rho\in G^{\wedge,p}_{(r)}} \det_{\mathbb{Q}_p^c}(1-u^{-r}\gamma \mid e_{\rho\omega^r}(Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c))e_\rho\\ =& \ \mathrm{tw}_r(v'_r\cdot h_S(u^r-1)),\end{aligned}$$

where $v'_r := \sum_{\rho\in G^{\wedge,p}_{(0)}} u^{-rd_\rho}e_\rho$ with $d_\rho := \dim_{\mathbb{Q}_p^c}(e_\rho(Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c))$ for each $\rho\in G^{\wedge,p}_{(0)}$. We remark that in proving the last displayed equality one uses the fact that for each $\kappa\in G^{\wedge,p}$ the $\mathbb{Q}_p^c[\gamma]$-module $e_\kappa(Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c)$ is isomorphic to $H^0(\mathrm{Gal}(K_\infty/k_\infty), \mathrm{Hom}_{\mathbb{Q}_p^c}(\mathbb{Q}_p^c\cdot e_\kappa, Y(S_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^c))$.
Upon combining the last two displayed formulas with the result of Lemma 3, the quasi-isomorphism $C_{K,1-r}\cong R\Gamma_c(U_{k,\text{ét}}, \mathrm{coker}(\hat{\gamma}_{1-r}))$ and the equality of Remark 6 we find that

$$\mathrm{Det}_{\mathfrak{A}}^{-1} C_{K,1-r} = \mathrm{tw}_r(v_r^{-1}v'_r)\,\mathrm{tw}_r(H'(u^r-1)^{-1}h_S(u^r-1))\cdot\mathfrak{A}.$$

The required equality (9) is thus a consequence of the following observation.

Lemma 4. *$v_r^{-1}v'_r$ is a unit of $\mathbb{Z}_p[G]e_0$.*

Proof. We start by making a general observation. For this we set $\mathfrak{B} := \mathbb{Z}_p[G]e_0$, and we let $f_1(T)$ and $f_2(T)$ denote any elements of $\mathfrak{B}[[T]]$ which satisfy $\mu_{\mathfrak{B}}(f_1(T)) = \mu_{\mathfrak{B}}(f_2(T)) = 0$. For $i=1,2$ we write $f_i^*(T)$ and $U_i(T)$ for the distinguished polynomial and unit series which occur in the product decomposition of $f_i(T)$ afforded by Proposition 2.1 (with $\mathfrak{A}=\mathfrak{B}$). We also set $f_{i,r}(T) := f_i(u^r(1+T)-1)\in\mathfrak{B}[[T]]$ and, observing that $\mu_{\mathfrak{B}}(f_{i,r}(T))=0$, we write $U_{i,r}(T)$ for the unit series which occurs in the product decomposition of $f_{i,r}(T)$ afforded by Proposition 2.1. Then, by explicit computation, one verifies that the element $(U_1(u^r-1)U_{2,r}(0))(U_2(u^r-1)U_{1,r}(0))^{-1}$ of $\mathfrak{B}^\times$ is equal to $\sum_{\rho\in G^{\wedge,p}_{(0)}} u^{-r\delta_\rho}e_\rho$ where $\delta_\rho := \deg(\rho(f_1^*(T))) - \deg(\rho(f_2^*(T)))$.
We now apply this observation with $f_1(T)$ and $f_2(T)$ equal to the series G_S and H which occur in Hypothesis (EPS). We observe that, in this case, the equality (3) implies that for each $\rho\in G^{\wedge,p}_{(0)}$ one has $\deg(\rho(G_S^*)) - \deg(\rho(H)) = d'_\rho$ where here $d'_\rho := d_\rho - 1$ if ρ is trivial, and $d'_\rho := d_\rho$ otherwise. From the general observation made above we may therefore deduce that the element $v_r^{-1}v'_r = \sum_{\rho\in G^{\wedge,p}_{(0)}} u^{-rd'_\rho}e_\rho$ belongs to $\mathfrak{B}^\times$, as claimed. □

This completes our proof of Theorem 5.2. □

We next use Theorem 5.2 to prove a result concerning the element $\Omega_{r-1}(K/k)$ of $\mathrm{Pic}(\mathbb{Z}[G])$ which is defined by Chinburg, Kolster, Pappas and Snaith in [13, §3]. We recall that it has been conjectured by the authors of loc. cit. that $\Omega_{r-1}(K/k) = 0$.
We write $\rho_{\Sigma,r}$ for the natural scalar extension morphism $\mathrm{Pic}(\mathbb{Z}[G]) \to \mathrm{Pic}(\mathbb{Z}_\Sigma[G]e_r)$. With R denoting either $\mathbb{Z}$ or $\mathbb{Q}$ we also write $\rho_\#$ for the R-linear involution of $R[G]$ which is induced by sending each element of G to its inverse.

COROLLARY 1. *Assume that (if $k \neq \mathbb{Q}$, then) Hypothesis (μ_p) is valid for K/k at each prime $p \notin \Sigma$. Then for each integer $r > 1$ one has an equality $\rho_{\Sigma,r}(\Omega_{r-1}(K/k)) = 0$.*

Proof. The key point we use here is a result of Flach and the first named author. Indeed, the result of [8, Thm. 4.1] implies that $\rho_{\Sigma,r}(\Omega_{r-1}(K/k))$ is equal to the class of the invertible $\mathbb{Z}_\Sigma[G]e_r$-submodule of $\mathbb{Q}[G]e_r$ which is defined by means of the intersection

$$\Big(\bigcap_{p\notin\Sigma} \mathrm{Det}^{-1}_{\mathbb{Z}_p[G]e_r} R\Gamma_c(U_{k,\text{ét}}, e_r\mathbb{Z}_p(1-r)_K)\Big) \otimes_{\mathbb{Z}[G],\rho_\#} \mathbb{Z}[G].$$

(To see this one must recall that the normalisation of the determinant functor which is used in [8] is the inverse of that used here.)
On the other hand, Theorem 5.2 implies that the above intersection is equal to the free $\mathbb{Z}_\Sigma[G]e_r$-module which is generated by the element $\rho_\#(L_S(1-r))$. Hence one has $\rho_{\Sigma,r}(\Omega_{r-1}(K/k)) = 0$, as required. □

Before stating our final result we introduce a little more notation. If V is any finite set of places of k, then for each rational prime ℓ we let V_ℓ denote the union of V and the set of places of k which are either archimedean or of residue characteristic ℓ. We set $\mathbb{Z}' := \mathbb{Z}[\frac{1}{2}]$ and we define a $\mathbb{Z}'[G]$-module by setting

$$H^2(\mathcal{O}_{K,V}, \mathbb{Z}'(r)) := \bigoplus_{\ell\neq 2} H^2(\mathrm{Spec}(\mathcal{O}_{K,V_\ell})_{\text{ét}}, \mathbb{Z}_\ell(r)).$$

We let Σ' denote the set $\{2\}$, respectively Σ, if $k = \mathbb{Q}$, respectively $k \neq \mathbb{Q}$, and we write $\mathbb{Z}_{\Sigma'}$ for the subring of $\mathbb{Q}$ which is generated by the inverses of each element of Σ'. We also write $\mu_{\mathbb{Q}^c}$ for the torsion subgroup of $\mathbb{Q}^{c\times}$.

COROLLARY 2. *Assume that (if $k \neq \mathbb{Q}$, then) Hypothesis (μ_p) is valid for K/k at each prime $p \notin \Sigma$. Then for each integer $r > 1$ one has an equality*

$$\rho_\#(L_S(1-r)) \cdot \mathrm{Ann}_{\mathbb{Z}[G]}(H^0(K, \mu_{\mathbb{Q}^c}^{\otimes r})) \otimes_{\mathbb{Z}} \mathbb{Z}_{\Sigma'} = e_r \cdot \mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,S}, \mathbb{Z}'(r))) \otimes_{\mathbb{Z}'} \mathbb{Z}_{\Sigma'}.$$

Remark 8. i) If T is any subset of S, then the localisation sequence of étale cohomology induces a natural inclusion of $\mathbb{Z}'[G]$-modules $H^2(\mathcal{O}_{K,T}, \mathbb{Z}'(r)) \subseteq H^2(\mathcal{O}_{K,S}, \mathbb{Z}'(r))$. From this we may deduce that $\mathrm{Ann}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,S}, \mathbb{Z}'(r))) \subseteq$

$\mathrm{Ann}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,T},\mathbb{Z}'(r)))$ and also if, for example, G is cyclic, that $\mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,S},\mathbb{Z}'(r))) \subseteq \mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_{K,T},\mathbb{Z}'(r)))$. In particular, if $\Sigma' = \{2\}$ (cf. Remark 7i)), and we write $\mathcal{O}_K$ in place of $\mathcal{O}_{K,\emptyset}$ (which, in terms of our current notation, denotes the ring of algebraic integers in K), then the equality of Corollary 2 implies that

$$(12)\quad \rho_{\#}(L_S(1-r)) \cdot \mathrm{Ann}_{\mathbb{Z}[G]}(H^0(K,\mu_{\mathbb{Q}^c}^{\otimes r})) \otimes_{\mathbb{Z}} \mathbb{Z}' \subseteq \mathrm{Ann}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_K,\mathbb{Z}'(r))),$$

and also if, for example, G is cyclic, that

$$\rho_{\#}(L_S(1-r)) \cdot \mathrm{Ann}_{\mathbb{Z}[G]}(H^0(K,\mu_{\mathbb{Q}^c}^{\otimes r})) \otimes_{\mathbb{Z}} \mathbb{Z}' \subseteq \mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_K,\mathbb{Z}'(r))).$$

We observe that if (as has been famously conjectured by Quillen and Lichtenbaum) the $\mathbb{Z}'[G]$-module $H^2(\mathcal{O}_K,\mathbb{Z}'(r))$ is isomorphic to $K_{2r-2}(\mathcal{O}_K)\otimes_{\mathbb{Z}}\mathbb{Z}'$, then the inclusion (12) is finer than (the image under $-\otimes_{\mathbb{Z}}\mathbb{Z}'$ of) the inclusion

$$\#H^0(\mathbb{Q},\mu_{\mathbb{Q}^c}^{\otimes r})\rho_{\#}(L_S(1-r)) \cdot \mathrm{Ann}_{\mathbb{Z}[G]}(H^0(K,\mu_{\mathbb{Q}^c}^{\otimes r})) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(K_{2r-2}(\mathcal{O}_K))$$

which was conjectured in the case $k=\mathbb{Q}$ by Coates and Sinnott in [14, Conj. 1]. We also recall that if $k=\mathbb{Q}$, r is even and K is any abelian extension of $\mathbb{Q}$, then Kurihara has recently used different methods to explicitly compute $\mathrm{Fit}_{\mathbb{Z}'[G]}(H^2(\mathcal{O}_K,\mathbb{Z}'(r)))$ in terms of Stickelberger elements [28, Cor. 12.5 and Rem. 12.6].

ii) In the case that $k=\mathbb{Q}$ and the conductor of K/k is a prime power, the image under multiplication by $e_r - e_{\omega^r}$ of the equality of Corollary 2 has already been proved by Cornacchia and Østvær in [16, Thm. 1.2].

iii) If r is even, then the equality of Corollary 2 can be re-expressed as an equality in which K is replaced by K^+ and the idempotent factor e_r is omitted. In a recent preprint [35], Snaith uses results from [11] to prove a weaker version of the equality of Corollary 2 in this context. More precisely, Snaith's results [loc. cit., Th. 1.6, Th. 5.2] assume that K is totally real, that r is even and that Hypothesis (μ_p) is valid for K/k at all odd primes p, and involve chains of inclusions rather than a precise specification of Fitting ideals (see also Remark 9 in this regard).

iv) In this remark we fix an odd prime p, an embedding $j:\mathbb{Q}^c\to\mathbb{Q}_p^c$ and a character $\chi\in G^{\wedge}_{(r)}$. We set $\mathcal{O}:=\mathbb{Z}_p(j\circ\chi)$ and we write $\mathrm{lth}_{\mathcal{O}}(M)$ for the length of any finite $\mathcal{O}$-module M. We let K_χ denote the (cyclic) extension of $\mathbb{Q}$ which corresponds to $\ker(\chi)$. Then the image under the functor $-\otimes_{\mathbb{Z}'[G]}\mathcal{O}$ of the equality of Corollary 2 with $k=\mathbb{Q}$ and $K=K_\chi$ is equivalent to an equality

$$\mathrm{val}_{\mathcal{O}}(j(L_S(1-r,\chi^{-1}))) = \mathrm{lth}_{\mathcal{O}}(H^2(\mathcal{O}_{K,S},\mathbb{Z}'(r))\otimes_{\mathbb{Z}'[G]}\mathcal{O}) - \mathrm{lth}_{\mathcal{O}}(H^0(K,\mu_{\mathbb{Q}^c}^{\otimes r})\otimes_{\mathbb{Z}[G]}\mathcal{O}).$$

In addition, since in this case G is cyclic, for any finite $\mathbb{Z}'[G]$-module N one has $\mathrm{lth}_{\mathcal{O}}(N\otimes_{\mathbb{Z}'[G]}\mathcal{O}) = \mathrm{lth}_{\mathcal{O}}(\mathrm{Hom}_{\mathbb{Z}'[G]}(\mathcal{O},N))$ and so the previous displayed equality provides a natural analogue of the main result (Thm. II.1) of Solomon in [36] concerning the relation between generalised Bernoulli numbers and the structure of certain ideal class groups. (The first named author is very grateful to Masato Kurihara for a most helpful conversation in this regard.)

Proof of Corollary 2. We now fix a prime $p \notin \Sigma'$ and we assume that Hypothesis (μ_p) is valid for K/k (as is known in the case $k = \mathbb{Q}$). We set $\mathfrak{A} := \mathbb{Z}_p[G]e_r$, $\mu(r) := H^0(K, \mu_{\mathbb{Q}^c}^{\otimes r}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $C := R\Gamma(U_{k,\text{ét}}, e_r\mathbb{Z}_p(r)_K)$.
In the sequel we shall say that a commutative $\mathbb{Z}_p$-algebra Λ is 'relatively Gorenstein over $\mathbb{Z}_p$' if $\mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Z}_p)$ (endowed with the natural action of Λ) is a free Λ-module of rank one.
For each bounded object X of $\mathcal{D}(\mathfrak{A})$ we set $X^* := R\,\mathrm{Hom}_{\mathbb{Z}_p}(X, \mathbb{Z}_p)$ which we endow with the action of $\mathfrak{A}$ which is induced by the contragredient action of G. We observe that $\mathfrak{A}$ is relatively Gorenstein over $\mathbb{Z}_p$ and hence that X^* belongs to $\mathcal{D}^{\mathrm{p}}(\mathfrak{A})$, respectively $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$, if and only if X belongs to $\mathcal{D}^{\mathrm{p}}(\mathfrak{A})$, respectively $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$. Now for any $\mathfrak{A}$-module X there exists a canonical isomorphism between the $\mathfrak{A}$-modules $\mathrm{Hom}_{\mathfrak{A}}(X, \mathfrak{A}) \otimes_{\mathbb{Z}_p[G],\rho_\#} \mathbb{Z}_p[G]$ and $\mathrm{Hom}_{\mathbb{Z}_p}(X, \mathbb{Z}_p)$. This in turn implies that for any object X of $\mathcal{D}^{\mathrm{p}}(\mathfrak{A})$ the lattice $\mathrm{Det}_{\mathfrak{A}}^{-1} X^*[-3]$ identifies canonically with $(\mathrm{Det}_{\mathfrak{A}}^{-1} X) \otimes_{\mathbb{Z}_p[G],\rho_\#} \mathbb{Z}_p[G]$. Upon noting that (11) induces an isomorphism in $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$ of the form $C \cong R\Gamma_c(U_{k,\text{ét}}, e_r\mathbb{Z}_p(1-r)_K)^*[-3]$, and recalling that the equality (8) is known to be valid as a consequence of Theorem 5.2 in the case $k \neq \mathbb{Q}$ and as a consequence of [11, Cor. 8.1] in the case $k = \mathbb{Q}$, we deduce that $\mathrm{Det}_{\mathfrak{A}}^{-1} C = \rho_\#(L_S(1-r)) \cdot \mathfrak{A}$. The equality of Corollary 2 will therefore follow if we can show that

$$\mathrm{Det}_{\mathfrak{A}}^{-1} C \cdot \mathrm{Ann}_{\mathfrak{A}}(\mu(r)) = \mathrm{Fit}_{\mathfrak{A}}(H^2(C)). \tag{13}$$

We next observe that, since C belongs to $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$ and is acyclic outside degrees 1 and 2, there exists an exact sequence of $\mathfrak{A}$-modules

$$0 \to H^1(C) \to Q \xrightarrow{d} Q' \to H^2(C) \to 0 \tag{14}$$

which is such that both Q and Q' are finite and of projective dimension at most 1 and there exists an isomorphism ι in $\mathcal{D}^{\mathrm{p,f}}(\mathfrak{A})$ between C and the complex $Q \xrightarrow{d} Q'$ (where the modules are placed in degrees 1 and 2, and the cohomology is identified with $H^1(C)$ and $H^2(C)$ by using the maps in (14)) for which $H^i(\iota)$ is the identity map in each degree i. This implies that $\mathrm{Fit}_{\mathfrak{A}}(Q)$ and $\mathrm{Fit}_{\mathfrak{A}}(Q')$ are invertible ideals of $\mathfrak{A}$ and that $\mathrm{Det}_{\mathfrak{A}}^{-1} C = \mathrm{Fit}_{\mathfrak{A}}(Q)^{-1}\,\mathrm{Fit}_{\mathfrak{A}}(Q')$.

LEMMA 5. *Let R be any reduced commutative $\mathbb{Z}_p$-algebra which is finitely generated, free and relatively Gorenstein over $\mathbb{Z}_p$. If*

$$0 \to A \to P \to P' \to A' \to 0$$

is any exact sequence of finite R-modules in which P and P' are both of projective dimension at most 1 over R, then $\mathrm{Fit}_R(P)$ and $\mathrm{Fit}_R(P')$ are principal ideals of R and one has an equality

$$\mathrm{Fit}_R(A^\vee)\,\mathrm{Fit}_R(P') = \mathrm{Fit}_R(P)\,\mathrm{Fit}_R(A'),$$

where $A^\vee$ denotes the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ (endowed with the natural action of R).

Proof. This is almost covered by the result of [15, Prop. 6]; however, the latter is stated only for rings R of the form $\mathcal{O}[G]$ with G a finite abelian p-group, and involves $\mathrm{Fit}_R(A)$ rather than $\mathrm{Fit}_R(A^\vee)$. In addition, the second named author would like to take this opportunity to point out that the argument in loc. cit. makes the *assumption* that G (which is written as P in loc. cit.) is *cyclic*, which is unfortunately not stated explicitly at the appropriate place, and that the equality of [15, Prop. 6] does not appear to hold in general. We will therefore now quickly adapt the arguments of [15, Prop. 6] to better suit our present purpose.

We first observe that, since R is semilocal, the Fitting ideal of any finite R-module P which is of projective dimension at most 1 is principal, being generated by the determinant of α in any presentation $R^n \xrightarrow{\alpha} R^n \to P \to 0$. We may find the following data, proceeding exactly as in [15, p.456f.]: a nonzerodivisor f of R (it is in fact always possible to take f to be a large enough power of p); a natural number n; short exact sequences $0 \to Q \to \tilde{A} \to A \to 0$ and $0 \to A' \to \tilde{A}' \to Q' \to 0$ in which Q and Q' are both finite and of projective dimension at most 1, and a four term exact sequence

$$0 \to \tilde{A} \to (R/fR)^n \to (R/fR)^n \to \tilde{A}' \to 0.$$

In a similar way one obtains the equalities

$$\mathrm{Fit}_R(P)\,\mathrm{Fit}_R(Q) = f^n R = \mathrm{Fit}_R(P')\,\mathrm{Fit}_R(Q').$$

Now R/fR is Gorenstein of dimension zero, that is: $(R/fR)^\vee \cong R/fR$ as R-modules. Therefore the argument in loc. cit. starting with equation (4) applies to give an equality

$$\mathrm{Fit}_R(\tilde{A}') = \mathrm{Fit}_R(\tilde{A}^\vee).$$

(Note that we do not claim that $\mathrm{Fit}_R(\tilde{A}^\vee) = \mathrm{Fit}_R(\tilde{A})$, as was done in loc. cit.) Applying the result of [15, Lem. 3] to the sequence $0 \to A' \to \tilde{A}' \to Q' \to 0$, respectively to the Pontryagin dual of the sequence $0 \to Q \to \tilde{A} \to A \to 0$, we obtain an equality

$$\mathrm{Fit}_R(\tilde{A}') = \mathrm{Fit}_R(A')\,\mathrm{Fit}_R(Q'),$$

respectively

$$\mathrm{Fit}_R(\tilde{A}^\vee) = \mathrm{Fit}_R(A^\vee)\,\mathrm{Fit}_R(Q^\vee).$$

Lemma 6. $\mathrm{Fit}_R(Q^\vee) = \mathrm{Fit}_R(Q)$.

Proof. Since Q is finite and of projective dimension at most 1 there exists an exact sequence of R-modules of the form

$$0 \to R^n \xrightarrow{\alpha} R^n \to Q \to 0.$$

Now $\mathrm{Hom}_{\mathbb{Z}_p}(Q, \mathbb{Z}_p) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}_p}(Q, \mathbb{Z}_p)$ is isomorphic to $Q^\vee$ (as Q is finite), $\mathrm{Ext}^1_{\mathbb{Z}_p}(R^n, \mathbb{Z}_p) = 0$ (as R is $\mathbb{Z}_p$-free) and $\mathrm{Hom}_{\mathbb{Z}_p}(R^n, \mathbb{Z}_p)$ is isomorphic to R^n (as R is relatively Gorenstein over $\mathbb{Z}_p$). From the long exact sequence of $\mathrm{Ext}^i_{\mathbb{Z}_p}(-, \mathbb{Z}_p)$-groups which is associated to the above sequence we therefore obtain a further exact sequence of R-modules

$$0 \to R^n \xrightarrow{\alpha^t} R^n \to Q^\vee \to 0,$$

where α^t denotes the transpose of α. By using the two displayed sequences we now compute that $\mathrm{Fit}_R(Q) = \det(\alpha)R = \det(\alpha^t)R = \mathrm{Fit}_R(Q^\vee)$, as claimed. $\square$

The equality of Lemma 5 now follows directly upon combining the equality of Lemma 6 with the four displayed equalities which immediately precede it. $\square$

Upon applying Lemma 5 with $R = \mathfrak{A}$ (which is both reduced and relatively Gorenstein over $\mathbb{Z}_p$) to the exact sequence (14) we obtain an equality

$$\begin{aligned}\mathrm{Det}_{\mathfrak{A}}^{-1} C \cdot \mathrm{Fit}_{\mathfrak{A}}(H^1(C)^\vee) &= \mathrm{Fit}_{\mathfrak{A}}(Q)^{-1}\,\mathrm{Fit}_{\mathfrak{A}}(Q')\,\mathrm{Fit}_{\mathfrak{A}}(H^1(C)^\vee)\\ &= \mathrm{Fit}_{\mathfrak{A}}(H^2(C)).\end{aligned}$$

To deduce the required equality (13) from this equality we now simply observe that the $\mathfrak{A}$-module $H^1(C)^\vee$ is isomorphic to the cyclic $\mathfrak{A}$-module $\mu(r)^\vee$, and hence that $\mathrm{Fit}_{\mathfrak{A}}(H^1(C)^\vee) = \mathrm{Fit}_{\mathfrak{A}}(\mu(r)^\vee) = \mathrm{Ann}_{\mathfrak{A}}(\mu(r)^\vee) = \mathrm{Ann}_{\mathfrak{A}}(\mu(r))$. This completes our proof of Corollary 2. $\square$

Remark 9. Let Γ be any finite abelian group and ℓ any rational prime. If M is any finite $\mathbb{Z}_\ell[\Gamma]$-module, then $\mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(M) = \mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(M^\vee)$ and $\mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(M)^{n(M)} \subseteq \mathrm{Fit}_{\mathbb{Z}_\ell[\Gamma]}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(M)$ where $n(M)$ denotes the minimal number of elements required to generate M over $\mathbb{Z}_\ell[\Gamma]$. Hence, if X is any object of $\mathcal{D}^{\mathrm{p,f}}(\mathbb{Z}_\ell[\Gamma])$ which is acyclic outside degrees 0 and 1 and $t_i \in \mathrm{Ann}_{\mathbb{Z}_\ell[\Gamma]}(H^i(X))$ for $i \in \{0,1\}$, then (since $\mathbb{Z}_\ell[\Gamma]$ is both reduced and relatively Gorenstein over $\mathbb{Z}_\ell$) the equality of Lemma 5 implies that

$$t_0^{n(H^0(X)^\vee)} \cdot \mathrm{Det}_{\mathbb{Z}_\ell[\Gamma]} X \subseteq \mathrm{Fit}_{\mathbb{Z}_\ell[\Gamma]}(H^1(X))$$

and also

$$t_1^{n(H^1(X))} \cdot \mathrm{Det}_{\mathbb{Z}_\ell[\Gamma]}^{-1} X \subseteq \mathrm{Fit}_{\mathbb{Z}_\ell[\Gamma]}(H^0(X)^\vee).$$

In particular, if $n(H^0(X)^\vee) = n(H^0(X))$ (which, for example, is the case when $H^0(X) = H^0(K, \mu_{\mathbb{Q}^c}^{\otimes r}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$), then Lemma 5 refines the main algebraic result (Thm. 2.4) of Snaith in [35].

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