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Wallman Compactification and Zero-Dimensionality

Compactaciones de Wallman y Dimensión Cero

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Abstract

In this paper, we give a method to construct a zero-dimensional Wallman compactification for a zero-dimensional T_0 space. This allows us to give new proofs of the following results: every T_0 zero-dimensional space is a Tychonoff space, **C** (the Cantor set) is universal for the class of zero-dimensional separable metrizable spaces and $\mathbb{Q}$ is the only countable perfect metrizable space (first proved by Sierpinski in 1920).
Key words and phrases: Wallman compactification, zero-dimensional, Cantor set, universal space, rational numbers.

Resumen

En este artículo damos un método de construcción de una compactación de Wallman de dimensión cero de un espacio T_0 de dimensión cero. Esto nos permite probar de una forma novedosa los siguientes resultados clásicos: que todo espacio T_0 y de dimensión cero es de Tychonoff, que el conjunto de Cantor es universal para la clase de los espacios metrizables separables y cero-dimensionales y que los racionales son el único espacio metrizable, numerable y perfecto (demostrado por vez primera por Sierpinski en 1920).
Palabras y frases clave: compactaciones de Wallman, dimensión cero, conjunto de Cantor, espacio universal, números racionales.

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The Wallman compactification is a powerful tool that associate a Hausdorff compactification to each normal $\alpha\beta$-lattice of basic closed sets of a topological space. There are many of such lattices to choose, but in a zero-dimensional space the Boolean lattice generated by any clopen base can be considered as a canonical choice. The Wallman compactification constructed in that way is going to be the main tool of this short paper.

This tool is combined with results about homogeneity (see [3]) and universality of the Cantor set (see [6]) in order to obtain a new (and easier) proof of Sierpinski's characterization of the rationals and a new proof of the universality property of the Cantor set, among some results of similar nature showing that this compactification is the natural one for spaces where the zero-dimensionality is viewed as its main property.

For the theory of Wallman compactifications and notations, we refer the reader to [1], Chapter 3. the main construction in the paper is as follows.

Let X be a zero dimensional T_0 space, and let $\mathcal{B} = \{B_i : i \in I\}$ be a base of clopen sets, which we can suppose to be a lattice under $\cap$ and $\cup$. Let $\mathcal{P}(\mathcal{B}) = \{\bigcup_{i \in F} B_i : F$ is finite, and $B_i \in \mathcal{B}$ or $X \setminus B_i \in \mathcal{B}\}$. It is clear that $\mathcal{P}(\mathcal{B})$ is a Boolean lattice and since X is T_0, it is a normal $\alpha\beta$-lattice, and hence the Wallman compactification associated to that lattice is a Hausdorff compactification of X, called $W_0^{\mathcal{B}}X$ hereafter, or called W_0X whenever there is no doubt about the base $\mathcal{B}$ chosen. This gives a new proof of the following result, which has a similar flavor to that obtained by Kakutani in [4] (every T_0 topological group is Tychonoff), and that improves the formerly known (every T_1 zero-dimensional topological space is Tychonoff, see section 6.2 of [2]).

Theorem 1. *Let X be a T_0 zero-dimensional space. Then X is a Tychonoff space.*

Note that if X is zero dimensional, so is W_0X, since $\mathcal{B}_L = W_0X \setminus \mathcal{B}_{X \setminus L}$ is open and closed (where L and $X \setminus L$ are in $\mathcal{P}$, and $\mathcal{B}_L$ is defined in [1] as the set of ultrafilters containing L; the fact that $\mathcal{P}$ is Boolean is essential).

Note that if X is second countable, then $\mathcal{B}$ can be taken to be countable, thus W_0X is also second countable, so X is separable metrizable if and only if W_0X is. As a consequence, every zero dimensional separable metrizable space has a zero dimensional metrizable compactification.

The following Lemma relates the perfectness of a space and its dense subspaces.

Lemma 2. *Let X be a T_1 topological space, and let D be a dense subspace of X. Then X is perfect if and only if D is.*

Proof. If D is not perfect, then there exists a point $x \in D$, open in D, and hence there exists U an open neighborhood of x in X such that $U \cap D = \{x\}$. Suppose that $U \neq \{x\}$. Since $U \setminus \{x\}$ is a nonempty open set in X and D is dense in X, we have that $(U \setminus \{x\}) \cap D \neq \emptyset$, which is a contradiction with the fact that $U \cap D = \{x\}$. Then, we have that $U = \{x\}$ and then $\{x\}$ is open in X, so X is not perfect.

On the other hand, if X is not perfect, then there exists $x \in X$ such that $\{x\}$ is open in X; since D is dense, $\{x\} \cap D = \{x\}$, therefore $x \in D$, and hence $\{x\}$ is open in D □

Therefore we have that W_0X is perfect whenever X is.

Corollary 3. *Let X be a perfect zero-dimensional T_0 topological space. Then W_0X is a perfect zero-dimensional Hausdorff compactification of X.*

The above results give the following.

Proposition 4. *Let X be a zero dimensional perfect separable metrizable space. Then W_0X is homeomorphic to the Cantor set.*

Proof. It is clear from the above, since then W_0X is a perfect compact zero-dimensional metrizable space, and the Cantor set is the only perfect compact zero-dimensional metrizable space. □

So we have a characterization of zero-dimensional perfect separable metrizable spaces.

Corollary 5. *A topological space X is a zero-dimensional perfect separable metrizable space if and only if it can be densely embedded into the Cantor set.*

Proof. It is a consequence of the above Proposition and Lemma 2. □

The following example shows that perfectness is essential in the above result and that some classical compactifications can be obtained from W_0X.

Proposition 6. *Let X be an infinite discrete space. Then $W_0^{\mathcal{B}}X$ is the one point compactification of X (taking $\mathcal{B}$ to be the base built from the finite subsets of X).*

Proof. First, we describe the base of $W_0^{\mathcal{B}}X$. If L is finite, then $\mathcal{B}_L = L$. If $L = X \setminus F$, with F finite, then $W_0^{\mathcal{B}}X \setminus \mathcal{B}_L = \mathcal{B}_{X\setminus L} = \mathcal{B}_F = F$, hence $\mathcal{B}_L = W_0^{\mathcal{B}}X \setminus F$.

Now suppose there are two distinct points $x \neq y \in W_0^{\mathcal{B}}X \setminus X$. Since $W_0^{\mathcal{B}}X$ is Hausdorff, then there exist F_1, F_2 finite subsets of X such that $x \in \mathcal{B}_{X\setminus F_1}$,

$y \in \mathcal{B}_{X \setminus F_2}$ and $\mathcal{B}_{X \setminus F_1} \cap \mathcal{B}_{X \setminus F_2} = \emptyset$ (note that $\mathcal{B}_F = F \subseteq X$ if F is finite, and $x, y \notin X$). Then $W_0^{\mathcal{B}} X \setminus (F_1 \cup F_2) = \emptyset$, and hence $W_0^{\mathcal{B}} X = F_1 \cup F_2$ is finite, which is a contradiction with the fact that X is infinite. Therefore $W_0^{\mathcal{B}} X \setminus X$ is one point (note that X is not compact, since it is discrete and infinite), and hence $W_0^{\mathcal{B}} X$ is the one point compactification of X. □

We strengthen the fact that the Cantor set is universal for the class of zero dimensional compact metrizable spaces to separable zero dimensional metrizable spaces.

Theorem 7. *The Cantor set is universal for the class of zero dimensional separable metrizable spaces.*

Proof. It is known that it is universal for the class of zero dimensional compact metrizable spaces (for a short proof, see [6]). But if X is a zero dimensional separable metrizable space, then $W_0 X$ is a zero dimensional compact metrizable space, and hence it can be embedded into the Cantor set. □

Now we give a new proof of Sierpinski's characterization of the set of rational numbers (see [5]).

Theorem 8. *All countable perfect metrizable spaces are homeomorphic to the rationals.*

Proof. Let X be a countable perfect metrizable space, then $W_0 X$ is homeomorphic to the Cantor set, or in other words, X can be densely embedded into the Cantor set, and since the Cantor set is countable dense homogeneous (see [3], Example 2), then X is homeomorphic to $\mathbb{Q} \cap \mathbf{C}$ and hence to $\mathbb{Q}$. On the other hand it is clear that the set of rational numbers is a countable perfect metrizable space. □

Countable dense subsets in perfect metrizable spaces are now characterized.

Corollary 9. *Let X be a separable perfect metrizable space. Then every countable dense subset D of X is homeomorphic to the rationals.*

Proof. It is clear from the above Corollary, and Lemma 2. □

Thus, in separable metrizable spaces, countable dense subsets are the rationals together with the (possible) isolated points.

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