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An Algorithm for Extending Functions in Hypercubes

Un Algoritmo para Extender Funciones en Hipercubos

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Abstract

Let $Q^n = [0,1]^n$ be the unit cube in $\mathbb{R}^n$ and B^n its border. In this paper an algorithm is described for extending functions $f : B^n \to \mathbb{R}^k$ to the interior of the cube, preserving properties of f such as continuity and polynomial character. The results obtained comprise as special cases linear interpolation and bilinear, ruled and Coons surfaces used in computer graphics.
Key words and phrases: Function extension, polynomials, surfaces, computer graphics, CAD.

Resumen

Sea $Q^n = [0,1]^n$ el cubo unitario en $\mathbb{R}^n$ y B^n su borde. En este artículo se describe un algoritmo para extender funciones $f : B^n \to \mathbb{R}^k$ al interior del cubo, preservando propiedades de f como la continuidad y el carácter polinomial. Los resultados obtenidos comprenden como casos especiales la interpolación lineal y las superficies bilineales, regladas y de Coons usadas en computación gráfica.
Palabras y frases clave: Extensión de funciones, polinomios, superficies, computación gráfica, CAD.

1 Notation and terminology

Let $Q^n = [0,1]^n$ be the unit cube in $\mathbb{R}^n$ and B^n its border, i.e. $B^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_i = 1 \text{ or } x_i = 0 \text{ for some } i\}$. We say that a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is *polynomial in* A if there is a polynomial $P \in \mathbb{R}[x_1,\dots,x_n]$ whose associated function restricted to A is f. A function $f : A \to \mathbb{R}^k$ is *polynomial in* A if all its components are polynomial in A. For $i = 1,2,\dots,n$ and real a we define the projection $p_{i,a}$ from $\mathbb{R}^n$ to the hyperplane $x_i = a$ by

$$(p_{i,a}(x))_i = a \quad \text{and} \quad (p_{i,a}(x))_j = x_j \quad \text{for } j \neq i.$$

2 An extension algorithm

In computer graphics it is often needed to generate surfaces with a given border. For example if $f : B^2 \to \mathbb{R}^3$ is continuous then an extension $f : Q^2 \to \mathbb{R}^3$ of f would be a parameterized surface with the curve $f(B^2)$ as border. Coons surfaces (see [1]) solve this problem. Inspired in this example we look at the general problem of extending functions $f : B^n \to \mathbb{R}^k$ to Q^n, in a simple and effectively computable way. The following algorithm constructs the extension using linear interpolation between opposite faces of the cube, combined with appropriate correction terms.

Algorithm E

Given $f : B^n \to \mathbb{R}^k$ take $f_0 = f$ and define inductively functions $f_i : B^n \to \mathbb{R}^k$ and $g_i : Q^n \to \mathbb{R}^k$ for $i = 1,2,\dots,n$ as follows:

$$\begin{aligned} g_i(x) &= (1-x_i)f_{i-1}(p_{i,0}(x)) + x_i f_{i-1}(p_{i,1}(x)), \quad \forall x \in Q^n, \\ f_i(x) &= f_{i-1}(x) - g_i(x), \quad \forall x \in B^n. \end{aligned}$$

Finally put $F = \sum_{i=1}^n g_i$.

Proposition 1. *With the above notation we have:*

(1) F is an extension of f.

(2) If f is continuous so is F.

(3) If f is polynomial on each face of B^n then F is polynomial on Q^n.

Proof. f_1 is 0 on the faces $x_1 = 0$ and $x_1 = 1$ of Q^n. Inductively it is easily seen that f_i is 0 on the faces $x_j = 0$ and $x_j = 1$ of Q^n for $j = 1, \ldots, i$. Therefore f_n is identically 0 on B^n. Thus for all $x \in B^n$ we have

$$\begin{aligned} f(x) &= f_0(x) = f_1(x) + g_1(x) = f_2(x) + g_2(x) + g_1(x) = \ldots \\ &= f_n(x) + \sum_{i=1}^{n} g_i(x) = F(x). \end{aligned}$$

This proves (1). A look at Algorithm E makes (2) and (3) obvious. □

Examples

For $n = 1$ Algorithm E simply gives:

$$F(x_1) = g_1(x_1) = (1 - x_1)f(0) + x_1 f(1) \qquad \text{(linear interpolation)}$$

For $n = 2$ we have:

$$\begin{aligned} g_1(x_1, x_2) &= (1 - x_1)f(0, x_2) + x_1 f(1, x_2), \\ f_1(x_1, x_2) &= f(x_1, x_2) - g_1(x_1, x_2), \\ g_2(x_1, x_2) &= (1 - x_2)f_1(x_1, 0) + x_2 f_1(x_1, 1) \end{aligned}$$

and finally

$$\begin{aligned} F(x_1, x_2) &= g_1(x_1, x_2) + g_2(x_1, x_2) = (1 - x_1)f(0, x_2) + x_1 f(1, x_2) \\ &\quad + (1 - x_2)[f(x_1, 0) - (1 - x_1)f(0, 0) - x_1 f(1, 0)] \\ &\quad + x_2[f(x_1, 1) - (1 - x_1)f(0, 1) - x_1 f(1, 1)] \end{aligned}$$

For $k = 3$ this is just the Coons surface with border $f(B^2)$.

Algorithm E is suitable for recursive implementation in computer languages like Pascal or C. However F may be also described combinatorially:

Proposition 2. *For all $x \in Q^n$, $F(x)$ is the sum of all the terms of the form*

$$(-1)^{s+1} u_1 u_2 \ldots u_n f(v_1, v_2, \ldots, v_n).$$

where each u_i may be $1 - x_i$, x_i or 1 (but not all of them 1), the corresponding v_i is 0, 1 or x_i respectively and s is the number of u_i's equal to 1. There are a total of $3^n - 1$ terms.

Proof. It is left as an exercise to the reader. □

3 Final comments

Problem E3400 of *The American Mathematical Monthly* (see [2]) asks for a polynomial extension of a real valued continuous function defined on B^2 and polynomial on each edge. Algorithm E gives a solution (with $n = 2$, $k = 1$). We sent our generalization to the editors and it is mentioned in [3], but only a solution for the special case proposed was published. Later we proposed the general problem in this journal (see [4] and [5], Problema 4) but no solutions were received.

References

[1] Rogers, D.F., Adams, J.A. *Mathematical Elements for Computer Graphics*, McGraw-Hill, New York, 1976.

[2] Problem E3400, Amer. Math. Monthly **97** (7) (1990), p. 612.

[3] Polynomials in computer-aided geometric design (Solution to E3400), Amer. Math. Monthly **99** (2) (1992), 170–171.

[4] Problemas y Soluciones, Divulgaciones Matemáticas **1** (1) (1993), p. 106.

[5] Problemas y Soluciones, Divulgaciones Matemáticas **2** (1) (1994), p. 99.