Rectangular modulus, Birkhoff orthogonality and characterizations of inner product spaces

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Abstract. Some characterizations of inner product spaces in terms of Birkhoff orthogonality are given. In this connection we define the rectangular modulus μ_X of the normed space X. The values of the rectangular modulus at some noteworthy points are well-known constants of X. Characterizations (involving μ_X) of inner product spaces of dimension ≥ 2, respectively ≥ 3, are given and the behaviour of μ_X is studied.

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1. Introduction

In the present paper we shall give, at the beginning, natural generalizations of some known characterizations of inner product spaces (i.p.s. for short). By introducing a parameter $\lambda > 0$ we obtain, in the particular case $\lambda = 1$, the known results collected in D. Amir's book [3, p. 79]. The characterizations are expressed in terms of Birkhoff orthogonality and the new conditions will be given in an "anti-symmetric" manner with respect to λ. In this direction one obtains a more general form (depending on λ) of M. Baronti's Lemma 1 in [4] and a generalization of M. del Rio and C. Benitez's Lemma 3 in [15].

These generalizations (especially Lemma 1 in the sequel) suggest to introduce a function $\mu_X : (0, \infty) \to \mathbb{R}$, with the property that $\mu_X(1)$ is the well-known rectangular constant of the normed space X. We call this function the rectangular modulus of X. The rectangular modulus is an increasing convex function and Lipschitz continuous of best Lipschitz constant 2. Moreover, $\mu_X(0+)$ is another well-known constant of the normed space X.

For any fixed $\lambda > 0$ a characterization of i.p.s. in terms of the rectangular modulus is also given. In the limit case when $\lambda \searrow 0$, the analogous characterization of i.p.s. is valid only for normed spaces of dimension ≥ 3.

2. Preliminary results and notation

We denote by $(X, \|\cdot\|)$ a real normed space of dimension ≥ 2. For $x \in X$ and $r > 0$ let $S_X(x, r) = \{y \in X : \|x - y\| = r\}$ and $B_X(x, r) = \{y \in X : \|x - y\| \leq r\}$, be the sphere respectively closed ball with center x and radius r. The unit sphere

$S_X(0,1)$ and the closed unit ball $B_X(0,1)$ of the space X will be denoted by S_X and B_X respectively. The symbol $\perp$ is used for Birkhoff orthogonality in X; namely $x \perp y$ if $\|x\| \leq \|x + t\,y\|$, for all $t \in \mathbb{R}$. Geometrically, this means that the line through x in the $y-$direction supports the ball $B_X(0, \|x\|)$ at x. For $x, y \in X$, $x \neq y$, the closed line segment with vertices x and y is denoted by $[x; y]$. Any two-dimensional subspace of X will be identified with $\mathbb{R}^2$ equipped with an appropriate norm and an orientation ω. The orientation ω of the ordered pair (x, y) of vectors (with $x + y \neq 0$ and $\|x\| = \|y\|$) is recorded by $x \prec y \prec -x$. Denote by $\perp^A$ the area orthogonality ([1], [3, p. 65]) defined for $(\mathbb{R}^2, \|\cdot\|)$ by $x \perp^A y$ if the radius vectors $\pm x$, $\pm y$ divide the unit ball of $\mathbb{R}^2$ into four parts of equal area. The following known lemmas will be used in Section 3.

Lemma A ([2]). *Let $S_{\mathbb{R}^2}$ be the unit sphere of $(\mathbb{R}^2, \|\cdot\|)$ and $s(\alpha)$ be the point of $S_{\mathbb{R}^2}$ which is to a given point $s(0)$ at an angle $0 \leq \alpha < 2\pi$, measured with a given orientation of the plane. Then for every $\lambda > 0$ the real continuous functions*

$$\alpha \in [0, \pi) \to \|s(0) + \lambda s(\alpha)\|,$$

and

$$\alpha \in [0, \pi) \to \|s(0) - \lambda s(\alpha)\|,$$

are decreasing and increasing respectively.

If $(\mathbb{R}^2, \|\cdot\|)$ is strictly convex then the aforementioned functions are strictly monotonic.

In the two-dimensional normed space X let $u^*, v^* \in S_X$ be such that $u^* \perp v^*$ and let us consider the corresponding (u^*, v^*)-coordinate system in which u^*, v^* are versors. For $u, v \in X$ let $A_{u,v}$ be the area of the parallelogram $\{\alpha u + \beta v : \alpha, \beta \in [0,1]\}$ in the (u^*, v^*)-coordinate system. It is clear that if $r, s > 0$ then $A_{ru,sv} = rsA_{u,v}$.

Lemma B ([3, p. 78]). *Let X be a two-dimensional normed space in which orthogonality is symmetric. Then $A_{u,v} = A_{u^*,v^*} = 1$, $\forall\, u, v \in S_X$, $u \perp v$.*

3. Characterizations of inner product spaces and Birkhoff orthogonality

For $u, v \in S_X$, $u \neq \pm v$ and $\lambda > 0$ we define the function $\varphi_{\lambda,u,v} : (0, \infty) \to (0, \infty)$ by

$$\varphi_{\lambda,u,v}(t) = \frac{\lambda^2 + t}{\|\lambda u + tv\|},\ \forall\, t > 0.$$

With the above notation we have the following generalization of Lemma 1 in [4].

Lemma 1. *Let $u, v \in S_X$, $u \neq \pm v$ and $\lambda, t_0 > 0$ be fixed. The following are equivalent:*

(a) $(\lambda u + t_0 v) \perp (u - \lambda v)$.

(b) $\varphi_{\lambda,u,v}(t_0) \geq \varphi_{\lambda,u,v}(t)$, $\forall\, t > 0$.

Proof: If we suppose that (a) holds then we have

$$\left(u - \frac{t_0}{\lambda^2 + t_0}\,(u - \lambda v)\right) \perp (u - \lambda v),$$

which implies

$$\text{(1)}\qquad \left\| u - \frac{t_0}{\lambda^2 + t_0}\,(u - \lambda v)\right\| \le \left\| u - \frac{t}{\lambda^2 + t}\,(u - \lambda v)\right\|, \ \forall\, t > 0.$$

and hence

$$\frac{\lambda^2 + t_0}{\|\lambda u + t_0 v\|} \ge \frac{\lambda^2 + t}{\|\lambda u + t v\|}\,, \ \forall\, t > 0.$$

Now, if (b) is satisfied then (1) holds and this shows that in the two-dimensional space X_2 generated by u and v the straight line containing the open line segment

$$l = \left\{ u - \frac{t}{\lambda^2 + t}\,(u - \lambda v) : t > 0 \right\}$$

supports the ball $B_X(0, \|w_0\|)$ at w_0, where $w_0 = u - t_0(u - \lambda v)/(\lambda^2 + t_0)$. Then $w_0 \perp (u - \lambda v)$ or equivalently $(\lambda u + t_0 v) \perp (u - \lambda v)$. □

Remark. If we consider the function $\psi_{\lambda,u,v} : (0,\infty) \to (0,\infty)$ defined by

$$\psi_{\lambda,u,v}(t') = \lambda \varphi_{1/\lambda,u,v}\left(\frac{1}{t'}\right) = \frac{\lambda^2 + t'}{\|t' u + \lambda v\|}\,,$$

then we easily deduce:

Lemma 1′. *With the previous notation, let $t'_0 > 0$ be fixed. The following are equivalent:*

(a′) $(t'_0 u + \lambda v) \perp (\lambda u - v)$.
(b′) $\psi_{\lambda,u,v}(t'_0) \ge \psi_{\lambda,u,v}(t')$, $\forall\, t' > 0$.

The next theorem is known for $\lambda = 1$, see Propositions 10.1–10.3, 10.3′ and 10.4 in [3] (see also [4] and [15]).

Theorem 2. *Let $\lambda > 0$ be fixed. The following are equivalent:*

1) $\forall\, u, v \in S_X,\ u \perp v \ \Rightarrow\ (\lambda u + v) \perp (u - \lambda v)$;
2) $\forall\, u, v \in S_X,\ u \perp v \ \Rightarrow\ \|\lambda u + v\| = \|u - \lambda v\|$;
3) $\forall\, u, v \in S_X,\ u \perp v \ \Rightarrow\ \|\lambda u + v\| \le \sqrt{1 + \lambda^2}$;
4) $\forall\, u, v \in S_X,\ u \perp v \ \Rightarrow\ \|\lambda u + v\| \ge \sqrt{1 + \lambda^2}$;
5) $\forall\, u, v \in S_X,\ u \perp v \ \Rightarrow\ \|\lambda u + v\| = \sqrt{1 + \lambda^2}$;
6) *the normed space X is an i.p.s.*

Remarks. As we can see a little later the equivalences 3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5) are simple consequences of a result in [12]. The implication 5) $\Rightarrow$ 6) is a strong result recently obtained (among other results) by C. Benitez, K. Przeslawski and D. Yost in [6]. We note that the weaker result 5′) $\Rightarrow$ 6) was also proved and used in [18, pp. 388–389], where 5′) is given by

$$\forall\, u, v \in S_X,\ u \perp v \Rightarrow\ \|\lambda u + v\| = \sqrt{1+\lambda^2},\ \ \|u + \lambda v\| = \sqrt{1+\lambda^2},$$

$\lambda > 0$ being fixed.

Proof of Theorem 2: We show that 1) $\Rightarrow$ 2). Suppose that 1) is verified and let $u, v \in S_X$, $u \perp v$, and $\lambda > 0$ be fixed. It follows that

$$\left(\lambda\frac{\lambda u + v}{\|\lambda u + v\|} + \frac{u - \lambda v}{\|u - \lambda v\|}\right) \perp \left(\frac{\lambda u + v}{\|\lambda u + v\|} - \lambda\frac{u - \lambda v}{\|u - \lambda v\|}\right).$$

If we put $t = \|u - \lambda v\|/\|\lambda u + v\|$ then, by Lemma 1, we have:

$$\frac{\lambda^2 + 1}{\|\ \lambda(\lambda u + v)/\|\lambda u + v\| + (u - \lambda v)/\|u - \lambda v\|\ \|} \geq \frac{\lambda^2 + t}{\|\ \lambda(\lambda u + v)/\|\lambda u + v\| + t(u - \lambda v)/\|u - \lambda v\|\ \|},$$

and consequently

$$\frac{\lambda^2 + 1}{\|\lambda^2 u + \lambda v + (1/t)(u - \lambda v)\|} \geq \frac{\lambda^2 + t}{\|\lambda^2 u + \lambda v + u - \lambda v\|}.$$

From $u \perp v$ one obtains

$$(\lambda^2 + 1)^2 \geq (\lambda^2 + t)\cdot\left\|(\lambda^2 + \frac{1}{t})u + \lambda(1 - \frac{1}{t})v\right\| \geq (\lambda^2 + t)\left(\lambda^2 + \frac{1}{t}\right),$$

yielding

$$\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 \leq 0 \ \ \Leftrightarrow\ \ t = 1.$$

This implies that $\|\lambda u + v\| = \|u - \lambda v\|$.

Now we show that 2) implies the strict convexity of X. Suppose that 2) is satisfied and, on the contrary, there exists a support line l of S_X such that $l \cap S_X = [u_1, u_2]$, $u_1 \neq u_2$. Then any $u \in [u_1, u_2]$ can be written as $u = u_t = u_1 + t(u_2 - u_1)$, $t \in [0,1]$ and $\|u_t\| = 1$. The function $t \to \|u_1 + t(u_2 - u_1)\|$, $t \in \mathbb{R}$ is 1 on $[0,1]$, strictly increasing for $t > 1$ and strictly decreasing for $t < 0$. Denoting by $v = (u_2 - u_1)/\|u_2 - u_1\|$ we have that $u_t \perp v$, $\forall\, t \in [0,1]$, and the application

$$t \to \|\lambda u_t + v\| = \lambda\left\|u_1 + t(u_2 - u_1) + \frac{u_2 - u_1}{\lambda\|u_2 - u_1\|}\right\|,\ t \in (1 - \varepsilon_1, 1]$$

with sufficiently small $\varepsilon_1 > 0$ is strictly increasing. On the other hand, the application

$$t \to \|u_t - \lambda v\| = \left\| u_1 + t(u_2 - u_1) - \lambda \frac{u_2 - u_1}{\|u_2 - u_1\|} \right\|, \ \forall t \in (1 - \varepsilon_2, 1],$$

with small enough $\varepsilon_2 > 0$ is constant or strictly decreasing. But from 2) we have that $\|\lambda u_t + v\| = \|u_t - \lambda v\|$, $\forall t \in (1 - \min\{\varepsilon_1, \varepsilon_2\}, 1]$, a contradiction.

We prove that if 2) is satisfied then

$$\text{(2)} \qquad u, v \in S_X \ \text{ and } \ \|\lambda u + v\| = \|u - \lambda v\| \ \Rightarrow \ u \perp v.$$

Suppose that 2) holds and, on the contrary, there exist $u, v' \in S_X$ such that $\|\lambda u + v'\| = \|u - \lambda v'\|$ and u is not orthogonal to v'. In the space X_2' generated by u and v' (understood as $(\mathbb{R}^2, \|\cdot\|)$) we choose the orientation such that $u \prec v' \prec -u$, $(v' \neq \pm u)$. Let $v \in S_{X_2'}$ be such that $u \perp v$ and $u \prec v \prec -u$. Then $v \neq v'$. Supposing that $u \prec v' \prec v \prec -u$, by Lemma A and the strict convexity of X we have

$$\|u - \lambda v'\| < \|u - \lambda v\|$$

respectively

$$\|\lambda u + v'\| = \lambda \|u + \frac{1}{\lambda} v'\| > \lambda \|u + \frac{1}{\lambda} v\| = \|\lambda u + v\|,$$

implying $\|\lambda u + v\| < \|u - \lambda v\|$, a contradiction. The case $u \prec v \prec v' \prec -u$ can be treated in a similar way.

Suppose now that 2) holds. Then 1) holds as well. Indeed, if $u, v \in S_X$, $u \perp v$ and $\lambda > 0$ is fixed then

$$\left\| \lambda \frac{\lambda u + v}{\|\lambda u + v\|} + \frac{u - \lambda v}{\|u - \lambda v\|} \right\| = \frac{\lambda^2 + 1}{\|\lambda u + v\|} = \left\| \frac{\lambda u + v}{\|\lambda u + v\|} - \lambda \frac{u - \lambda v}{\|u - \lambda v\|} \right\|.$$

From (2) we have

$$\frac{\lambda u + v}{\|\lambda u + v\|} \perp \frac{u - \lambda v}{\|u - \lambda v\|},$$

which yields $(\lambda u + v) \perp (u - \lambda v)$.

Observe now that 2) implies the symmetry of orthogonality. Indeed, if $u, v \in S_X$ and $\lambda > 0$ then from 2) and (2) one obtains:

$$u \perp v \ \Leftrightarrow \ u \perp -v \ \Leftrightarrow \ \|\lambda u - v\| = \|u + \lambda v\| \ \Leftrightarrow$$

$$\Leftrightarrow \ \|\lambda v + u\| = \|v - \lambda u\| \ \Leftrightarrow \ v \perp u.$$

Moreover, since X is strictly convex, it follows that X is also smooth (see [3, p. 78]).

In order to prove 3) $\Rightarrow$ 4), it is sufficient to consider the case of two-dimensional spaces, i.e. X may be considered $\mathbb{R}^2$ with the norm $\|\cdot\|$. It follows that S_X is a rectifiable simple closed Jordan curve. Denoting

$$S_\lambda = \{\lambda u + v : \ u, v \in S_X, \ u \perp v\},$$

it follows that S_λ is also a closed rectifiable Jordan curve. A parametrization of S_λ may be given as in J. Joly [12, p. 304]. More precisely, let $u = u(\theta) = (u_1(\theta), u_2(\theta))$, $\theta \in [0, 2\pi)$ be the parametrization of S_X in a rectangular system of axes with $u(0) \prec u(\theta) \prec -u(0)$, for all $\theta \in [0, \pi)$. Now, consider the vectors $u, v \in S_X$, $u \perp v$ such that $u \prec v \prec -u$. We have

$$\begin{aligned} u &= u(\theta(\sigma)) = (u_1(\theta(\sigma)), u_2(\theta(\sigma))), \\ v &= v(\nu(\sigma)) = (v_1(\nu(\sigma)), v_2(\nu(\sigma))), \end{aligned}$$

where $\theta, \nu : [0, 4\pi) \to [0, 2\pi)$, are continuous increasing and surjective functions and u_1, u_2, v_1, v_2 are continuous functions with bounded variation. Moreover, $\sigma = \theta(\sigma) + \nu(\sigma)$ and the decomposition is unique. Then S_λ can be rewritten

$$S_\lambda = \{\lambda u(\theta(\sigma)) + v(\nu(\sigma)) : \sigma \in [0, 4\pi)\}.$$

Let A be the area of the unit ball of X and let A_λ be the area enclosed by S_λ. Then with a similar computation as in [12], we have:

$$A_\lambda = \lambda^2 \int_{S_X} u_1 \, du_2 + \int_{S_X} v_1 \, dv_2 = (\lambda^2 + 1)A. \tag{3}$$

Now, from 3) and the continuity of the functions u_1, u_2, v_1, v_2, θ and ν we have:

$$\|\lambda u + v\| \geq \sqrt{1 + \lambda^2},$$

for all $u, v \in S_X$, $u \perp v$ proving that 3) $\Rightarrow$ 4). Analogously 4) $\Rightarrow$ 3) and finally we have 3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5).

We shall show that 2) $\Rightarrow$ 5). Since the Birkhoff orthogonality in X is symmetric, as it is well known, $\dim(X) \geq 3$ implies that X is an i.p.s. ([11], [3, p. 143]), and in this case the result follows. Suppose X is two-dimensional and for fixed $u^*, v^* \in S_X$, $u^* \perp v^*$, consider the (u^*, v^*)-coordinate system of X. Let $u, v \in S_X$, $u \perp v$ be given. Then the area $A_{\lambda u+v, u-\lambda v}$ can be computed by $A_{\lambda u+v, u-\lambda v} = |\Delta| \cdot A_{u,v}$, where

$$\Delta = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\lambda^2 - 1.$$

Now, from Lemma B, $A_{\lambda u+v,u-\lambda v} = \lambda^2 + 1$ in the (u^*, v^*)-coordinate system. Since by 2) $\Leftrightarrow$ 1), $\lambda u + v \perp u - \lambda v$, we have

$$A_{(\lambda u+v)/\|\lambda u+v\|,\,(u-\lambda v)/\|u-\lambda v\|} = 1 = \frac{A_{\lambda u+v,u-\lambda v}}{\|\lambda u+v\|\cdot\|u-\lambda v\|} = \frac{\lambda^2+1}{\|\lambda u+v\|\cdot\|u-\lambda v\|},$$

and again by 2) $\|\lambda u + v\| = \|u - \lambda v\| = \sqrt{\lambda^2+1}$, $\forall\, u, v \in S_X$, $u \perp v$. From $u \perp v \Leftrightarrow u \perp -v$ we obtain the desired result.

Now, by the quoted result in [6], we have 5) $\Rightarrow$ 6). In fact in [6] it was proved that 5) implies the symmetry of Birkhoff orthogonality and that the Birkhoff orthogonality $\perp$ implies the area orthogonality $\perp^A$. By [15] it follows that X is an i.p.s. Since the implications 6) $\Rightarrow$ 5) and 5) $\Rightarrow$ 2) are trivial the theorem is completely proved. □

4. The rectangular modulus of a normed space

For the normed space X the *rectangular constant* $\mu(X)$ was defined in [12] by

$$\mu(X) = \sup\{\mu[x,y] : x, y \in X \setminus \{0\},\ x \perp y\},$$

where

$$\mu[x,y] = \sup_{s\in\mathbb{R}} \frac{\|x\| + \|sy\|}{\|x+sy\|}\,,\quad \forall\, x,y \in X \setminus \{0\},\ x \perp y.$$

Since $x \perp y \Leftrightarrow x \perp -y$ we easily deduce that

$$\mu(X) = \sup\left\{ \frac{1+|s|\,\|y\|/\|x\|}{\left\| \frac{x}{\|x\|} \pm |s|\,\|y\|/\|x\| \cdot \frac{y}{\|y\|} \right\|} : s \neq 0,\ x,y \in X\setminus\{0\}, x \perp y \right\}$$
$$= \sup\left\{ \frac{1+t}{\|u+tv\|} : t > 0,\ u, v \in S_X, u \perp v \right\}.$$

We define the *rectangular modulus* of X as the function $\mu_X : (0,\infty) \to \mathbb{R}$

$$\mu_X(\lambda) = \sup\{\max\{\varphi_{\lambda,u,v}(t), \lambda\varphi_{1/\lambda,u,v}(t)\} : t > 0,\ u, v \in S_X,\ u \perp v\}$$
$$= \sup\left\{ \max\left\{ \frac{\lambda^2+t}{\|\lambda u+tv\|}, \frac{1+\lambda^2 t}{\|u+\lambda t v\|} \right\} : t > 0,\ u, v \in S_X,\ u \perp v \right\},$$

for all $\lambda > 0$. From the definition it is clear that $\mu_X(1) = \mu(X)$. As it is well known the *modulus of convexity* of X ([7]), denoted by δ_X and the *modulus of smoothness* of X ([13]), denoted by ρ_X satisfy Nordlander's type inequalities, i.e.

$$\delta_X(\varepsilon) \le \delta_H(\varepsilon) = 1 - \sqrt{1-\varepsilon^2/4},\ \forall\, \varepsilon \in [0,2]$$

and

$$\rho_X(\tau) \geq \rho_H(\tau) = \sqrt{\tau^2+1} - 1, \ \forall \tau \geq 0,$$

where H is an i.p.s.

G. Nordlander [14] has conjectured that if $\delta_X(\varepsilon) = 1 - \sqrt{1-\varepsilon^2/4}$ for a fixed $\varepsilon \in (0,2)$ then X is an i.p.s. J. Alonso and C. Benitez [2] proved that this assertion is true exactly for $\varepsilon \in (0,2) \setminus D$ where $D = \{2\cos(k\pi/(2n)) : k = 1, \dots, n-1; n = 2, 3, \dots\}$. Analogous results were obtained for the modulus of smoothness and for other known moduli. Generally, if γ_X denotes such a modulus and t is fixed then from $\gamma_X(t) = \gamma_H(t)$ it follows that X is an i.p.s. except for a countable set of points t in the domain of γ_X ([21]).

The *modulus of squareness* ξ_X studied in [6], [16], [17], [18] satisfies also the inequality

$$\xi_X(\beta) \geq \xi_H(\beta) = 1/\sqrt{1-\beta^2}, \ \forall \beta \in [0,1).$$

Moreover, if $\xi_X(\beta) = 1/\sqrt{1-\beta^2}$, for a fixed $\beta \in (0,1)$ then X is an i.p.s.

For the rectangular modulus we have:

Theorem 3. (a) *If H is an i.p.s. then $\mu_H(\lambda) = \sqrt{1+\lambda^2}$, $\forall \lambda > 0$.*

(b) *If X is a normed space and H is an i.p.s. then*

$$\mu_X(\lambda) \geq \mu_H(\lambda), \ \forall \lambda > 0.$$

(c) *If $\mu_X(\lambda) = \sqrt{1+\lambda^2}$ for a fixed $\lambda > 0$ then X is an i.p.s.*

Proof: (a) $\mu_H(\lambda) =$

$$= \sup\left\{\max\left\{\frac{\lambda^2+t}{\|\lambda u + tv\|}, \frac{1+\lambda^2 t}{\|u+\lambda t v\|}\right\} : t > 0, \, u, v \in S_H, \, u \perp v\right\}$$
$$= \sup\left\{\max\left\{\frac{\lambda^2+t}{\sqrt{\lambda^2+t^2}}, \frac{1+\lambda^2 t}{\sqrt{1+\lambda^2 t^2}}\right\} : t > 0\right\}.$$

It is easily seen that the function $f_\lambda : (0,\infty) \to \mathbb{R}$

$$f_\lambda(t) = \frac{\lambda^2+t}{\sqrt{\lambda^2+t^2}} - \frac{1+\lambda^2 t}{\sqrt{1+\lambda^2 t^2}}, \ t > 0$$

satisfies the condition $\operatorname{sign} f_\lambda'(t) = \operatorname{sign}(1-\lambda)$ and from $f_\lambda(1) = 0$, $\forall \lambda > 0$ we deduce that $\mu_H(\lambda) = \sqrt{1+\lambda^2}$, $\forall \lambda > 0$.

(b) Let $\lambda \in (0,\infty)$ be a fixed number. We can suppose that X is a two-dimensional normed space. By using formula (3) we conclude that

$$\inf\{\|\lambda u + v\| : u, v \in S_X, \, u \perp v\} \leq \sqrt{\lambda^2+1}$$

and this implies

$$\begin{aligned}\mu_X(\lambda) &\geq \sup\left\{\frac{\lambda^2+t}{\|\lambda u+tv\|} : t>0,\ u,v\in S_X,\ u\perp v\right\}\\ &\geq \sup\left\{\frac{\lambda^2+1}{\|\lambda u+v\|} : u,v\in S_X,\ u\perp v\right\}\\ &= \frac{\lambda^2+1}{\inf\{\|\lambda u+v\| : u,v\in S_X,\ u\perp v\}} \geq \frac{\lambda^2+1}{\sqrt{\lambda^2+1}} = \sqrt{\lambda^2+1}.\end{aligned}$$

In particular $\mu(X)=\mu_X(1)\geq\sqrt{2}$, as in [12].

(c)
$$\begin{aligned}\mu_X(\lambda) &= \sqrt{1+\lambda^2}\\ &\geq \sup\left\{\max\left\{\frac{\lambda^2+1}{\|\lambda u+v\|},\ \frac{1+\lambda^2}{\|u+\lambda v\|}\right\} : u,v\in S_X,\ u\perp v\right\}\\ &\geq \frac{\lambda^2+1}{\|\lambda u+v\|},\ \forall\, u,v\in S_X, u\perp v,\end{aligned}$$

$\lambda>0$ being fixed. Hence $\|\lambda u+v\|\geq\sqrt{\lambda^2+1}$, $\forall\, u,v\in S_X$, $u\perp v$. By Theorem 2, 4) $\Leftrightarrow$ 6), we have that X is an i.p.s. □

Remark. Let us define the **-rectangular modulus* by the simpler formula

$$\begin{aligned}\mu^*_X(\lambda) &= \sup\{\varphi_{\lambda,u,v}(t) : t>0,\ u,v\in S_X,\ u\perp v\}\\ &= \sup\left\{\frac{\lambda^2+t}{\|\lambda u+tv\|} : t>0,\ u,v\in S_X,\ u\perp v\right\},\ \forall\,\lambda>0.\end{aligned}$$

It is clear (with similar proofs) that:

(a′) $\mu^*_H(\lambda)=\sqrt{\lambda^2+1}$, $\forall\,\lambda>0$, H being an i.p.s.;
(b′) for each normed space X, $\mu^*_X(\lambda)\geq\mu^*_H(\lambda)=\sqrt{\lambda^2+1}$, $\forall\,\lambda>0$;
(c′) if $\mu^*_X(\lambda)=\sqrt{1+\lambda^2}$, for a fixed $\lambda>0$ then X is an i.p.s.

Some properties of the rectangular modulus are collected in

Theorem 4. (a) *For each* $\lambda>0$

$$\mu_X(\lambda)=\max\{\mu^*_X(\lambda),\lambda\mu^*_X(1/\lambda)\}\ \textit{and}\ \mu_X(\lambda)=\lambda\mu_X(1/\lambda).$$

(b) *The rectangular modulus (*-rectangular modulus) is an increasing and convex function on* $(0,\infty)$.
(c) *We have*

(4)
$$\mu_X(\lambda)\leq\max\{\lambda+2,1+2\lambda\},\ \forall\,\lambda>0.$$

Proof: (a) The first part of (a) easily follows from the definitions of μ_X and μ_X^*. The second part of (a) follows from the first part.

(b) The modulus μ_X^* can be rewritten as

$$\mu_X^*(\lambda) = \sup\left\{\frac{\lambda + t/\lambda}{\|u + (t/\lambda)v\|} : t > 0,\, u, v \in S_X,\, u \perp v\right\}$$
$$= \sup\left\{\frac{\lambda + t'}{\|u + t'v\|} : t' > 0,\, u, v \in S_X,\; u \perp v\right\},\; \lambda > 0.$$

Consequently, μ_X^* and, by analogy, μ_X are increasing and convex functions as suprema of families of increasing and convex functions of variable λ.

(c) For $t \leq 2$, by $u \perp v$ we have:

$$\frac{\lambda + t}{\|u + tv\|} \leq \frac{\lambda + 2}{\|u\|} = \lambda + 2,\; \forall\, \lambda > 0.$$

For $t > 2$, by the triangle inequality one obtains

$$\frac{\lambda + t}{\|u + tv\|} \leq \frac{\lambda + t}{t - 1} < \lambda + 2,\; \forall\, \lambda > 0.$$

It follows that $\mu_X^*(\lambda) \leq \lambda + 2$, $\forall\, \lambda > 0$,

$$\lambda \mu_X^*(1/\lambda) \leq \lambda(1/\lambda + 2) = 1 + 2\lambda,\; \forall\, \lambda > 0,$$

and

$$\mu_X(\lambda) \leq \max\{\lambda + 2, 1 + 2\lambda\}.$$

In particular, the rectangular constant $\mu(X)$ satisfies the inequality:
$\mu(X) = \mu_X(1) \leq 3$ ([12]). □

Remark. The inequality (4) is sharp. Indeed, let X be the two-dimensional l^1-space and let $u_1 = (1, 0)$ and $v_1 = (-1/2, 1/2)$ be in S_X. We have

$$\|u_1 + tv_1\| = |1 - \frac{t}{2}| + |\frac{t}{2}| \geq 1 = \|u_1\|,\; \forall\, t \in \mathbb{R},$$

implying $u_1 \perp v_1$. It follows that

$$\mu_X^*(\lambda) = \sup\left\{\frac{\lambda + t}{\|u + tv\|} : t > 0,\; u, v \in S_X,\, u \perp v\right\}$$
$$\geq \frac{\lambda + 2}{\|u_1 + 2v_1\|} = \frac{\lambda + 2}{|1 - 1| + 1} = \lambda + 2,\; \forall\, \lambda > 0.$$

Then $\mu_X^*(\lambda) = \lambda + 2$, $\forall\, \lambda > 0$, and consequently $\mu_X(\lambda) = \max\{\lambda + 2, 1 + 2\lambda\}$, $\forall\, \lambda > 0$.

Now, by Theorem 4 (b), (c) and Theorem 3 (b) it follows that there exists

$$\mu_X(0+) := \lim_{\lambda \searrow 0} \mu_X(\lambda) \in [1,2].$$

The extension (by continuity) of μ_X in origin (denoted by $\overline{\mu}_X$) remains an increasing and convex function on $[0,\infty)$. The function

$$\lambda \to \overline{\mu}_X(\lambda) - \mu_X(0+),\ \forall\, \lambda \geq 0$$

is convex, zero in origin and, consequently, the function

$$\lambda \to \frac{\mu_X(\lambda) - \mu_X(0+)}{\lambda},\ \lambda > 0,$$

is increasing on $(0,\infty)$.

By Theorem 4. (b), μ_X is locally Lipschitz on $(0,\infty)$. Moreover it is Lipschitz continuous as it will be shown by the following theorem:

Theorem 5. *The rectangular modulus verifies the inequality*

$$\mu_X(\lambda_2) - \mu_X(\lambda_1) \leq \mu_X(0+)(\lambda_2 - \lambda_1) \leq 2(\lambda_2 - \lambda_1),$$

for all $\lambda_1, \lambda_2 > 0$, $\lambda_1 \leq \lambda_2$, *and the absolute constant* 2 *is the best possible.*

Proof: We have

$$\begin{aligned}\mu_X(\lambda) - \mu_X(0+) &= \lambda\mu_X(\frac{1}{\lambda}) - \mu_X(0+)\\ &= \frac{\mu_X(1/\lambda) - \mu_X(0+)}{1/\lambda} + \mu_X(0+)(\lambda - 1),\end{aligned}$$

and

$$\begin{aligned}\mu_X(\lambda_2) - \mu_X(\lambda_1) &= \mu_X(\lambda_2) - \mu_X(0+) - (\mu_X(\lambda_1) - \mu_X(0+))\\ &= \frac{\mu_X(1/\lambda_2) - \mu_X(0+)}{1/\lambda_2} - \frac{\mu_X(1/\lambda_1) - \mu_X(0+)}{1/\lambda_1} + \mu_X(0+)(\lambda_2 - \lambda_1)\\ &\leq \mu_X(0+)(\lambda_2 - \lambda_1) \leq 2(\lambda_2 - \lambda_1).\end{aligned}$$

The constant 2 is attained for instance when X is the two-dimensional l^1-space. □

In the following, we are interested to know the properties of the constant $\mu_X(0+) \in [1,2]$. At the beginning let us recall some notions:

The *radial projection constant* ([20]) of the space X is the best Lipschitz constant $k(X)$ for the radial projection $r : X \to B_X$ defined by

$$r(x) = \begin{cases} x, & \text{for } \|x\| \leq 1 \\ x/\|x\|, & \text{for } \|x\| > 1. \end{cases}$$

One of the representations of $k(X)$ is given in [4, p. 1075] by:

$$k(X) = \sup\left\{\frac{1}{\|tu+v\|} : t \in \mathbb{R}, v \in S_X, u \perp v\right\}.$$

The radial projection constant is equal to other four constants of X, denoted by $MPB(X)$, $MPB'(X)$, $\overline{MPB}(X)$, $\beta(X)$ respectively. For more information on this subject see [4], [5] and [8]–[10].

Recall that by Theorem 3, for a fixed $\lambda > 0$ and for a normed space X, with $\dim(X) \geq 2$ we have

$$\mu_X(\lambda) = \sqrt{1+\lambda^2} \;\Leftrightarrow\; X \text{ is an i.p.s.}$$

In the limit case when $\lambda \searrow 0$ we are interested to see the relevance of the equality $\mu_X(0+) = 1$ to the geometry of X.

Theorem 6. (a) *For any normed space X we have:*

$$\mu_X(0+) = k(X).$$

(b) *The equality $\mu_X(0+) = 1$ is equivalent to the symmetry of Birkhoff orthogonality.*

PROOF: (a) A continuity argument and the equivalence $x \perp y \;\Leftrightarrow\; -x \perp y$ show that

$$\mu^*_X(0+) = \sup\left\{\frac{t}{\|u+tv\|} : t > 0,\ u, v \in S_X,\ u \perp v\right\}$$
$$= \sup\left\{\frac{1}{\|t'u+v\|} : t' \in \mathbb{R},\ v \in S_X,\ u \perp v\right\} = k(X).$$

But from $\lambda\mu^*_X(1/\lambda) \leq 1 + 2\lambda$, $\forall\, \lambda > 0$ it follows that:

$$\mu^*_X(0+) \leq \mu_X(0+) = \max\left\{\mu^*_X(0+),\ \lim_{\lambda\searrow 0} \lambda\mu^*_X(1/\lambda)\right\}$$
$$\leq \max\{\mu^*_X(0+), 1\} = \mu^*_X(0+) = k(X).$$

(b) The equality $\mu_X(0+) = 1$ is equivalent to $BMP(X) = 1$, which in its turn is equivalent to the symmetry of Birkhoff orthogonality ([19]). □

Remarks. If $\dim(X) \geq 3$ then $\mu_X(0+) = 1$ implies that X is an i.p.s. On the other hand, by a result of M.A. Smith [19], $1 \leq MPB(X) < 2$, $\Leftrightarrow$ X is uniformly non-square. It follows that X is uniformly non-square $\Leftrightarrow$ $1 \leq \mu_X(0+) < 2$, and we expect that the rectangular modulus characterizes new geometric properties of X. Such geometric considerations will be given elsewhere.

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