Comment.Math.Univ.Carolin. 40,1 (1999)71–95

An intrinsic definition of the Colombeau generalized functions

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Abstract. A slight modification of the definition of the Colombeau generalized functions allows to have a canonical embedding of the space of the distributions into the space of the generalized functions on a $\mathcal{C}^\infty$ manifold. The previous attempt in [5] is corrected, several equivalent definitions are presented.

Keywords: Colombeau generalized function, distribution, canonical embedding, manifold

Classification: 46F, 46F05

Introduction

The aim of Colombeau's paper [5] was to avoid the drawback that the embedding of the space $\mathcal{D}'$ of the Schwartz distributions into the algebra (and sheaf) of Colombeau generalized functions is not intrinsic: This canonical embedding (even of the space $\mathcal{C}$ of continuous functions) defined by [4] is not kept under coordinate diffeomorphisms. More precisely: If $\Omega, \widetilde{\Omega}$ are open sets in Euclidean space $\mathbb{R}^d$, T a distribution on Ω, then by [4], T is identified with the generalized function $\langle R\rangle$ having the function $R(\varphi, x) = \langle T, \varphi(\bullet - x)\rangle$ as a representative (provided $\operatorname{supp}\varphi \subset \Omega - x$). For a diffeomorphism $\mu : \widetilde{\Omega} \to \Omega$, the inverse image of the distribution T, denoted by μ^*T or $T\circ\mu$ or $T(\mu(x))$, is defined in the usual way as a distribution on $\widetilde{\Omega}$ ([16]), while by [4], the inverse image $\mu^*\langle R\rangle$ is a generalized function $\langle R\rangle$ having as a representative the function

$$\varphi,\, \widetilde{x} \mapsto R(\varphi, \mu(\widetilde{x})) \qquad (\,\widetilde{x} \in \widetilde{\Omega}\,).$$

The distribution μ^*T turns out to be associated with the generalized function $\mu^*\langle R\rangle$, but in general not identified in the above sense. For this reason, we cannot define an algebra $\mathcal{G}(M)$ of generalized functions on a $\mathcal{C}^\infty$ manifold M in such a way that the space $\mathcal{D}'(M)$ is canonically embedded in $\mathcal{G}(M)$. This inconvenience can be removed by a slight change of the definition of the Colombeau generalized functions and of their inverse image, which is attempted in [5].

Note that there are also simplified definitions of generalized functions of hyperfunction type where a representative is a sequence or a net of $\mathcal{C}^\infty$ functions (see

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Supported by Research Grant GAČR 201/97/1161.

[15], [9]). With these definitions, $\mathcal{C}^\infty$ sheaf morphisms can be easily extended to generalized functions and the generalized functions can be easily defined on a $\mathcal{C}^\infty$ manifold. There are embeddings of $\mathcal{D}'$ into such a space of generalized functions, however no embedding is canonical. One cannot agree completely with a remark in [15] referring to [1] that there is no need for a canonical embedding, since in applications it matters to find a suitable embedding adapted to the problem considered. The existence of an embedding suitable for all applications would simplify the task. For instance in [15] it is proved in a rather complicated way that there is a sheaf morphism (in the category of linear spaces) $\sigma : \mathcal{D}' \to \mathcal{G}$ identical on $\mathcal{C}^\infty$ and such that the image of a distribution is associated with it. Certainly, the constructive proof in [15] gives more, but the only formulation does not ensure even that the product of a continuous function with a Dirac measure is preserved (up to the association).

In [15] it is said: for a sheaf morphism σ one cannot expect that it is compatible with the $\mathcal{C}^\infty$ module structure nor that it commutes with the differentiation in all coordinates. As for the latter, we will see that in our case the canonical embedding is a sheaf morphism commuting with the differentiation and, of course, with coordinate diffeomorphisms. Moreover, thanks to the existence of the canonical embedding, it is possible to define for instance the Colombeau product of distributions on a manifold as it is done on $\mathbb{R}^d$ in [11].

Colombeau's definitions

In the following, Ω will always be an open set in the Euclidean space $\mathbb{R}^d$.

Notation 1 (by [4]).

$$\mathcal{A}_q(\mathbb{R}^d) := \{\varphi \in \mathcal{D}(\mathbb{R}^d); \textstyle\int\varphi = 1,\ \int\varphi(x)x^\beta\,\mathrm{d}x = 0 \ \text{ for } \ \beta \in \mathbb{N}_0^d, 1 \le |\beta| \le q\},$$
$$\mathcal{A}_q(M) := \mathcal{A}_q \cap \mathcal{D}(M) \ \text{ for } \ M \subset \mathbb{R}^d.$$

If there is no danger of misunderstanding, we write $\mathcal{A}_q$ instead of $\mathcal{A}_q(M)$. We denote by $\mathcal{A} := \mathcal{A}_0 - \mathcal{A}_0$ and we do not introduce any special symbol for $\mathcal{A}_q - \mathcal{A}_q \quad (q \neq 0)$.

Originally, the notation φ_ε was used for the function

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\Big(\frac{x}{\varepsilon}\Big) \quad (\varepsilon \in \left]0,1\right]). \tag{1}$$

In [5], this notion is replaced with $\mathcal{C}^\infty$ bounded paths of functions $(\varphi^\varepsilon)_{\varepsilon\in\left]0,1\right]}$, $(\varphi^\varepsilon \in \mathcal{A}_0)$, and with the unbounded paths $(\varphi_\varepsilon)_{\varepsilon\in\left]0,1\right]}$, developed from it by $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi^\varepsilon\big(\frac{x}{\varepsilon}\big)$. We will accept this notation. There is another change in [5]: $\mathcal{A}_q$ are no more sets of functions as above but sets of bounded paths satisfying

$$\int x^\alpha\varphi^\varepsilon(x)\,\mathrm{d}x = O(\varepsilon^q) \quad \text{if} \ \ \alpha \in \mathbb{N}_0^d,\ 1 < |\alpha| \le q,\ \varepsilon \searrow 0.$$

Since we need both meanings of $\mathcal{A}_q$, we keep Notation 1 above, used in [4], and unlike in [5] we introduce semi-norms a_q as follows.

Notation 2. For $\varphi \in \mathcal{A}_0$, we define

$$a_q(\varphi) = \sup \left\{ \left| \int x^\alpha \varphi(x)\, \mathrm{d}x \right| ;\ \alpha \in \mathbb{N}_0^d,\ 1 < |\alpha| \le q \right\}.$$

So we have $\mathcal{A}_q = \{\varphi \in \mathcal{A}_0;\ a_q(\varphi) = 0\}$.

A similar change is done in [5] with the definition of $\mathcal{E}[\Omega]$, too: the set $\mathcal{E}[\Omega]$ containing the set of representatives $\mathcal{E}_M[\Omega]$ is no more a set of functions $R(\varphi, x)$ but the set of all $\mathcal{C}^\infty$ maps $\mathcal{R}(\Phi, x)$ into $\mathbb{C}^{]0,1]}$ where $\Phi = (\varphi^\varepsilon)_{\varepsilon \in]0,1]}$ is a bounded path, $x \in \Omega$ and

$$\mathcal{R}(\Phi, x) = \big(R(\varphi_\varepsilon, x)\big)_{0<\varepsilon\le 1}.$$

If it is the case (and if there is not a misunderstanding), then the formula define a one-to-one mapping $\mathcal{R} \leftrightarrow R$, and there is no reason for accepting this change here: $\mathcal{E}[\Omega]$ will stand for the space of functions $R(\varphi, x)$ like in [4] and paths will only be used to define the moderate growth and other similar notions. However, unlike in [4] and as in [5], $R(\varphi, x)$ are $\mathcal{C}^\infty$ complex valued functions in both variables $\varphi \in \mathcal{A}_0$, $x \in \Omega$ simultaneously. Other notions defined in [5], like the set of the moderate functions $\mathcal{E}_M[\Omega] \subset \mathcal{E}[\Omega]$, will be introduced or recalled later.

3. Now, if $\mu : \widetilde{\Omega} \to \Omega$ is a diffeomorphism, a representative $\widetilde{R}$ of the composition $\langle R\rangle \circ \mu$ (i.e. of the inverse image $\mu^*\langle R\rangle$) is defined in [5] by the formula

$$\widetilde{R}(\varphi_\varepsilon, \widetilde{x}) = R\big(\widetilde{\varphi_\varepsilon}, \mu(\widetilde{x})\big) \qquad (\widetilde{x} \in \widetilde{\Omega}),$$

where $\widetilde{\varphi_\varepsilon}$ is defined by a rather complicated formula in order to obtain a composition for the generalized functions equal to the classical one for the distributions. There is however an apparent inconsistency: $\widetilde{R}$ seems to depend on ε. In our new notation the formulas will be simpler and will not contain ε. Unfortunately there is a true inconsistency, too: $\widetilde{\varphi_\varepsilon}$ depends on x and the definition of $\mathcal{E}_M[\Omega]$ does not deal with test functions depending on x (i.e. on the second variable of R). As a consequence, it may happen that $\widetilde{R}$ is not moderate even if R is. For instance, if $\langle R\rangle$ is a constant generalized function on $\mathbb{R}$ with a representative $R(\varphi, x) = \exp\big(i \exp \int |\varphi(x)|^2\, \mathrm{d}x\big)$, then $R \in \mathcal{E}_M[\Omega]$, and one can check using formulas in [5] (see also (42) later) that, for arbitrary non-linear coordinate diffeomorphism μ, the first derivative of $\widetilde{R}$ does not have a moderate growth. In order to correct it, we have to modify the definition of $\mathcal{E}_M[\Omega]$ and, as consequence, to restrict the set of generalized functions only accepting those one which have moderate growth in all coordinate systems.

Change 4 in notation**.** The representative which is denoted by $R(\varphi, x)$ in [4] will be denoted by $R(\varphi(\bullet - x), x)$ here. In other words, our notation $R(\varphi, x)$ means what was denoted by $R(\varphi(x + \bullet), x)$ in [4].

According to the definition of the null ideal $\mathcal{N}$ in [4], only the values $R(\varphi, x)$ matter for determining the generalized function $\langle R\rangle$, where $\operatorname{supp}\varphi$ is in an arbitrarily chosen neighborhood of 0. In our notation, only the values $R(\varphi, x)$ matter where $\operatorname{supp}\varphi$ is in a neighborhood of the point x. So the values for $\operatorname{supp}\varphi \subset \Omega$ suffice and we can formulate the definition of $\mathcal{E}[\Omega]$ as follows.

Definition 5. Ω being an open set in $\mathbb{R}^d$, we define $\mathcal{E}[\Omega]$ to be the set of all $\mathcal{C}^\infty$ maps

$$\begin{aligned} R : \mathcal{A}_0(\Omega) \times \Omega &\to \mathbb{C} \\ \varphi,\, x \;&\mapsto R(\varphi, x). \end{aligned}$$

Thus the test functions have their supports in Ω. This is more natural and will simplify the definition of generalized functions on a $\mathcal{C}^\infty$ manifolds: in this case φ will be defined on this manifold. With this change the embedding of $\mathcal{D}'$ into $\mathcal{G}$ becomes simpler: if f is a distribution, then the function $\varphi \mapsto \langle f, \varphi\rangle$ is a representative of f as a generalized function. However, some other notions become more complicated, the formula (1) for φ_ε is even useless in this simple form. Also, the notion of a constant generalized function becomes less natural (anyway, on a manifold this notion has no sense) and the definition of the derivative becomes more complicated. For this reason, we are introducing the notation $(R)_\varepsilon$ replacing the notation (1).

Notation 6. If $R \in \mathcal{E}[\Omega]$, we denote by $(R)_\varepsilon$ or simpler R_ε, if there is no danger of misunderstanding, the function defined on a part of $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$ by

$$R_\varepsilon(\varphi, x) = R(\varphi_{x,\varepsilon}, x) \quad \text{with} \quad \varphi_{x,\varepsilon}(\xi) = \varepsilon^{-d}\, \varphi(\tfrac{\xi - x}{\varepsilon})$$
$$(\text{provided} \;\; \operatorname{supp} \varphi_{x,\varepsilon} \subset \Omega\,).\ \text{Equivalently,}\ \ R(\psi, x) = R_\varepsilon\big(\varepsilon^d \psi(x + \bullet\varepsilon),\, x\big).$$

By Change 4, for $\varepsilon = 1$ we get the original notion of representative introduced in [4]. Only the values $R_\varepsilon(\varphi, x)$ with $\operatorname{supp}\varphi$ in a neighborhood of zero matter for determining the generalized function $\langle R\rangle$. Note that $\operatorname{supp}\varphi_{x,\varepsilon} \longrightarrow \{x\}$ for $\varepsilon \searrow 0$ (uniformly when φ runs over a set of functions with uniformly bounded supports).

7. As we have already noticed, the definition of moderate growth of the representatives $R(\varphi, x)$ of the generalized functions must be modified, taking into account the dependence of φ on x. Thus the definition becomes more complicated. On the other hand, we simplify this definition, requiring the moderate growth of $R(\varphi, x)$ for all paths $(\varphi^\varepsilon)_\varepsilon$, unlike Definition 3 in [5], where this was required only for $(\varphi^\varepsilon)_\varepsilon \in \mathcal{A}_N$ (using the notation in [5]). We can see later, using Theorem 21, that this restriction does not restrict the set of generalized functions.

It does not matter that the paths $(\varphi^\varepsilon)_\varepsilon$ in [5] are $\mathcal{C}^\infty$ in the variable ε. So we replace them simply with bounded sets of test functions.

Notation. If $\mathcal{F}$ is a locally convex space, denote by $\mathcal{E}(\Omega{\to}\mathcal{F})$ the locally convex space of all $\mathcal{C}^\infty$ maps (vector valued functions)

$$\begin{aligned} \Phi = (\varphi_x)_{x\in\Omega} : \Omega &\to \mathcal{F} \\ x &\mapsto \varphi_x \end{aligned}$$

with the usual topology of locally uniform convergence of every derivative with respect to x. By $\mathcal{E}(\Omega{\to}\mathcal{A}_q)$ we mean the topological (affine) subspace of $\mathcal{E}(\Omega{\to}\mathcal{D})$ consisting of the $\mathcal{A}_q$-valued functions.

It is useful to consider the convergence

$$\lim_{\varepsilon\searrow 0}\Phi^\varepsilon=\Phi\quad(\,\Phi\in\mathcal{E}(\Omega{\to}\mathcal{F})\,),$$

even in the case when the maps $\Phi^\varepsilon\in\mathcal{E}(\Omega_\varepsilon{\to}\mathcal{F})$ are not defined on the same set. We only need that every compact $K\Subset\Omega$ is contained in Ω_ε for all $\varepsilon>0$ sufficiently small.

Definition 8. $\mathcal{E}_M[\Omega]$ is the set of all $R\in\mathcal{E}[\Omega]$ such that $\forall\,K\Subset\Omega$ (compact), $\alpha\in\mathbb{N}_0^d\ \ \exists\,N\in\mathbb{N}$ such that $\big(\frac{\partial}{\partial x}\big)^\alpha R_\varepsilon(\varphi_x,x)=O(\varepsilon^{-N})\quad(\varepsilon\searrow 0)$ uniformly when $x\in K$ and $(\varphi_x)_{x\in\Omega}$ runs over any bounded subset of $\mathcal{E}\big(\Omega{\to}\mathcal{A}_0(\mathbb{R}^d)\big)$.

If the variable $(\varphi_x)_{x\in\Omega}$ runs over a bounded subset of $\mathcal{E}\big(\Omega{\to}\mathcal{A}_0(\mathbb{R}^d)\big)$, then the values φ_x, for $x\in K$, remain bounded in $\mathcal{A}_0(\mathbb{R}^d)$. Hence their supports are uniformly bounded in $\mathbb{R}^d$. It is easy to check from the definition of R_ε that $R_\varepsilon(\varphi_x,x)$ is always defined (and $\mathcal{C}^\infty$ with respect to x) for all ε sufficiently small independently on these φ_x and $x\in K$.

Remark. Evidently, the moderate growth condition in this definition can be equivalently formulated as follows. $\forall\,K\Subset\Omega$ (compact), $\alpha\in\mathbb{N}_0^d\ \ \exists\,N\in\mathbb{N}$ such that, for every bounded path $\big\{(\varphi_x^\varepsilon)_{x\in\Omega};\ \varepsilon\in\,]0,1]\big\}\subset\mathcal{E}\big(\Omega{\to}\mathcal{A}_0(\mathbb{R}^d)\big)$, we have $\big(\frac{\partial}{\partial x}\big)^\alpha R_\varepsilon(\varphi_x^\varepsilon,x)=O(\varepsilon^{-N})\quad(\varepsilon\searrow 0)$ uniformly with respect to $x\in K$. Here "bounded path" means simply a bounded set of elements depending on $\varepsilon\in\,]0,1]$. The smoothness with respect to ε is not required. However, if the smoothness is required, it can be easily shown that the above formulation remains equivalent. We will do a similar thing in details in the proof of Equivalent definitions 18.

Differential calculus

9. We recall some theorems from differential calculus ([2], [17]) which we will need later. Theorems are usually formulated for vector valued functions defined on an open subset of a locally convex space; however they can be evidently generalized for functions defined on an open subset of an affine space, for instance $\mathcal{A}_0$ provided the derivatives are taken with respect to vectors belonging to $\mathcal{A}=\mathcal{A}_0-\mathcal{A}_0$. While applying differential calculus, we consider a complex linear structure to be a real one, the differential means the Fréchet differential.

Notation. Let X,Y be locally convex spaces, U an open subset of X, $R:U\to Y$ a mapping. We denote the value of the k-th Fréchet differential of R at the point $u\in U$ with respect to the vectors $x_1,\dots,x_k\in X$ by $\mathrm{d}^kR(u)[x_1,\dots,x_k]$. Different brackets, used for clarity, are not obligatory. Another notation $\mathrm{d}^k_{x_1,\dots,x_k}R(u)$ is used mainly for the first differential.

Theorem 10 ([2, 1.2.5], [17, 1.8.2]). *The k-th differential $\mathrm{d}^k R(u)$, if it exists, belongs to the space $L_s({}^k X\to Y)$ of all hypo-continuous symmetric poly-linear (here: k-linear) maps of X^k into Y, endowed with the topology of the uniform convergence on the cartesian products of k bounded subsets of X.*

Note that if X is a Fréchet spaces (and this is always here), any hypo-continuous poly-linear map is continuous.

If the map $u \mapsto \mathrm{d}^k R(u)$ is continuous, then R is said to be $\mathcal{C}^k$ (or of the class $\mathcal{C}^k$). If it is so for all $k \in \mathbb{N}_0$, then R is said to be of the class $\mathcal{C}^\infty$ ($\mathrm{d}^0 R$ means R).

Theorem 11 (Mean value theorem [17, 1.3.3.4°]). *If R is $\mathcal{C}^1$on an open neighborhood U of a segment $[u,\, u+x] \subset X$ then*

$$R(u+x) - R(u) \in \overline{\operatorname{conv}}\,\{\mathrm{d}R(u+tx)[x];\, t \in [0,1]\}$$

(*a closed convex hull*).

12. For the theorem on the differentiation of a composition ([17, 1.5.3]), we introduce the following notations. For a finite set $I \subset \mathbb{N}$, we denote by $\#I$ its cardinality and by $I = \{i_1, \dots, i_{\#I}\}$ its elements in the increasing order. If we have elements $x_1, x_2, \dots$, then we denote the finite sequence $x_{i_1}, \dots, x_{i_{\#I}}$ by x_I. By a decomposition of I we mean a subset $\mathcal{I} = \{I_1, \dots, I_k\}$ of $\exp I \setminus \{\emptyset\}$ such that the sets $I_1, \dots, I_k$ are non-empty, pairwise disjoint and $\bigcup I_j = I$.

Theorem. *Let X, Y, Z be locally convex spaces, U, V open sets in X, Y respectively, $R : U \to Y$, $S : V \to Z$ maps of the class $\mathcal{C}^n$, ($n \in \mathbb{N}$), $R(U) \subset V$. Then $S\circ R$ is a map of the class $\mathcal{C}^n$ and, for $u \in U$ and $x_1, \dots, x_n \in X$, we have*

$$\mathrm{d}^n(T\circ S)(u)[x_1, \dots, x_n]$$
$$= \sum_{k=1}^{n} \sum_{\substack{\{I_1,\dots,I_k\} \\ \text{pairwise disjoint,} \\ \bigcup I_j = \{1,2,\dots,n\}}} \mathrm{d}^k T(S(u)) \Big[\mathrm{d}^{\#I_1} S(u)[x_{I_1}], \dots, \mathrm{d}^{\#I_k} S(u)[x_{I_k}]\Big],$$

where the summation is extended over all decompositions $\mathcal{I} = \{I_1, \dots, I_k\}$ of the multi-index $I = \{1, \dots, n\}$.

As a special case, we have for the first differential

$$\mathrm{d}(T\circ S)(u)[x] = \mathrm{d}T(S(u))\,[\mathrm{d}S(u)[x]]\,.$$

13. The following theorems concern mappings of two variables. According to our needs we will formulate them for a mapping of an open subset of $\mathcal{A} \times \mathbb{R}^d$ (or $\mathcal{A}_q \times \mathbb{R}^d$) with values in a locally convex space Z. In order to avoid the use of indexes in the notation of partial differentials, we will denote the total differential by the letter $\mathbf{d}$, the partial differential with respect to the variable $\varphi \in \mathcal{A}$ resp. $x \in \mathbb{R}^d$ by d resp. ∂. For the latter we also use the symbol ∂^α ($\alpha \in \mathbb{N}_0^d$), which denotes the α-th derivative. Thus $\partial^\alpha R(\varphi, x) = \big(\frac{\partial}{\partial x}\big)^\alpha R(\varphi, x)$, provided φ does not depend on x.

Theorem 14 ([17, 1.11.2]). *If the first differential* $\mathbf{d}R$ *exists in a point* $(\varphi, x) \in \mathcal{A}_0(\Omega) \times \mathbb{R}^d$*, then* $\mathrm{d}R$ *and* ∂R *exist in* (φ, x) *and*

$$\mathbf{d}R(\varphi, x)[\psi, h] = \mathrm{d}_\psi R(\varphi, x) + \partial_h R(\varphi, x) \qquad (\, \psi \in \mathcal{A},\ h \in \mathbb{R}^d\,).$$

It follows for the differentials of higher degree

$$\begin{aligned}\mathbf{d}^n R(\varphi, x)[(\psi_1, h_1), \dots, (\psi_n, h_n)] &= (\mathrm{d}_{\psi_1} + \partial_{h_1}) \dots (\mathrm{d}_{\psi_n} + \partial_{h_n}) R(\varphi, x)\\ &= \sum_{I \subset (1,2,\dots,n)} \mathrm{d}_{\psi_I}^{\#I} \partial_{h_{(1,\dots,n)\setminus I}}^{n-\#I} R(\varphi, x)\end{aligned}$$

(using the notation for Theorem 12).

Theorem 15 ([17, 1.11.3]). *A map* R *is of the class* $\mathcal{C}^1$ *iff the partial differentials* $\mathrm{d}R$ *and* ∂R *exist and are continuous.*

Theorem 16 (Schwartz, [17, 1.11.5.2°]). *If* $\mathrm{d}R$ *and* ∂R *exist and if* $\mathrm{d}_\psi \partial_h R$ *or* $\partial_h \mathrm{d}_\psi R$ *is continuous on a neighborhood of a point* (φ, x)*, then* $\mathrm{d}_\psi \partial_h R(\varphi, x) = \partial_h \mathrm{d}_\psi R(\varphi, x)$.

Remark. If $\mathbf{d}^2 R(\varphi, x)$ exists, then $\mathrm{d}_\psi \partial_h R(\varphi, x) = \partial_h \mathrm{d}_\psi R(\varphi, x)$.
Indeed, by Theorem 14,

$$\begin{aligned}\mathbf{d}R(\varphi, x)[(\psi, 0)] &= \mathrm{d}_\psi R(\varphi, x),\\ \mathbf{d}^2 R(\varphi, x)[(\psi, 0), (0, h)] &= \partial_h \mathrm{d}_\psi R(\varphi, x)\end{aligned}$$

and the bilinear mapping $\mathbf{d}^2 R(\varphi, x)$ on the left hand side is symmetric by Theorem 10.

Note that we deal only with $\mathcal{C}^\infty$ maps in this paper, hence the order of taking derivatives does not matter.

Example (The differential of the product). If

$$\begin{aligned}F : \mathbb{R}^2 &\to \mathbb{R}\\ x, y &\mapsto xy,\end{aligned}$$

then

$$\begin{aligned}\mathbf{d}F(u, v)[(x, y)] &= uy + vx\\ \mathbf{d}^2 F(u, v)[(x_1, y_1), (x_2, y_2)] &= x_1 y_2 + x_2 y_1\\ \mathbf{d}^n F &= 0 \;\text{ for }\; n \geq 3.\end{aligned}$$

Results

Theorem 17 (Equivalent definition of representatives)**.** *For $R \in \mathcal{E}[\Omega]$, we have $R \in \mathcal{E}_M[\Omega]$ iff the partial differentials $d^k R_\varepsilon$ have a moderate growth in the following sense: $\forall\, K \Subset \Omega$, $\alpha \in \mathbb{N}_0^d$, $k \in \mathbb{N}_0$ $\quad\exists\, N \in \mathbb{N}$ such that*

$$\partial^\alpha \mathrm{d}^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] = O(\varepsilon^{-N}) \quad (\,\varepsilon \searrow 0\,) \tag{2}$$

uniformly when $x \in K$, φ is in a bounded subset of $\mathcal{A}_0(\mathbb{R}^d)$ and $\psi_1, \dots, \psi_k$ are in a bounded subset of $\mathcal{A}(\mathbb{R}^d)$.

This means: if we include partial differentials in the definition of the moderate growth, we do not need to consider φ depending on x (unlike in Definition 8).

Proof: I. If the condition is fulfilled, we calculate $\left(\frac{\partial}{\partial x}\right)^\alpha R_\varepsilon(\varphi_x, x)$ using Theorem 12 on differentiation of a composition.

II. Suppose $R \in \mathcal{E}[\Omega]$ (Definition 8). We have to prove (2) for a suitable N (depending on α and k), uniformly for $x \in K$, $\varphi \in \mathcal{A}_0 \cap \mathcal{B}$ and $\psi_1, \dots, \psi_k \in \mathcal{A} \cap \mathcal{B}$ ($\mathcal{B}$ is a bounded subset of $\mathcal{D}(\mathbb{R}^d)$). For $a = (a_1, \dots, a_d) \in K$, $x = (x_1, \dots, x_d) \in \Omega$, $\varphi, \psi_1, \dots, \psi_k$ running over bounded sets as above and $t_1, \dots, t_k$ attaining values $0, 1, \dots, k$, let us define

$$\begin{aligned} \varphi_x = \varphi_{x,t_1,\dots,t_k} &:= \varphi + \sum_{j=1}^{k} \frac{t_j \psi_j}{(|\alpha| + k^2 + j)!}(x_d - a_d)^{|\alpha|+k^2+j}, \\ p &:= \sum_{j=1}^{k} \bigl(|\alpha| + k^2 + j\bigr) = k|\alpha| + k^3 + \binom{k+1}{2}. \end{aligned} \tag{3}$$

By Definition 8, there is a number $N \in \mathbb{N}_0$ (depending on K, α and p) such that

$$\left(\tfrac{\partial}{\partial x}\right)^\alpha \left(\tfrac{\partial}{\partial x_d}\right)^p R_\varepsilon(\varphi_x, x) = O(\varepsilon^{-N}) \qquad (\varepsilon \searrow 0) \tag{4}$$

uniformly if $x \in K$ and if $(\varphi_x)_{x \in \Omega}$ runs over a bounded subset of $\mathcal{E}\bigl(\Omega \rightarrow \mathcal{A}_0(\mathbb{R}^d)\bigr)$.

We will only use it for $x = a \in K$. The derivative at the left hand side of (4) is the value of the differential with respect to the vectors

$$h_1, h_2, \dots, h_{|\alpha|+p} \tag{5}$$

such that exactly α_j of them are equal to the coordinate unit vector $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ $(j = 1, \dots, d-1)$ and $\alpha_d + p$ of them are equal to e_d. We apply Theorem 12 on the differentiation of a composition to the composition of R with $x \mapsto (\varphi_x, x)$ at $x = a$. The inner mapping $x \mapsto (\varphi_x, x)$ has the following value and derivatives at $x = a$:

$$\begin{aligned} (\varphi_x, x) &= (\varphi, a) \\ \frac{\partial}{\partial x_j}(\varphi_x, x) &= (0, e_j) \\ \Bigl(\frac{\partial}{\partial x_d}\Bigr)^{|\alpha|+k^2+j}(\varphi_x, x) &= (t_j \psi_j, 0) \quad (\,j = 1, 2, \dots, k\,). \end{aligned} \tag{6}$$

The other derivatives with respect to coordinate unit vectors are $= 0$ at $x = a$. So, only those decompositions $\mathcal{I}$ of the multi-index $I = (1, 2, \dots, |\alpha| + p)$ can give non-zero terms in the sum in Theorem 12, that every element of $\mathcal{I}$ either is a singleton (i.e. has the cardinality 1) or has the cardinality $|\alpha| + k^2 + j$ for some $j = 1, \dots, k$. Moreover, every $h_i \neq e_d$ must belong to a singleton of $\mathcal{I}$. The number $\widetilde{k}$ of elements of $\mathcal{I}$ that are not singleton (even if they have the less possible cardinality $|\alpha| + k^2 + 1$) cannot be greater then k: if $\widetilde{k} = k + 1$ we would have $(k+1)(|\alpha| + k^2 + 1) \leq |\alpha| + p$, which contradicts (3). It follows from Theorem 12 that $\big(\frac{\partial}{\partial x}\big)^{\alpha}\big(\frac{\partial}{\partial x_d}\big)^{p} R_\varepsilon(\varphi_{x,t_1,\dots,t_k})$ at $x = a$ equals to a sum of terms of the form

$$\begin{aligned} \mathbf{d}^{\widetilde{k}+|\widetilde{\alpha}|} R_\varepsilon(\varphi, a)\big[(t_{j_1}\psi_{j_1}, 0), \dots, (t_{j_{\widetilde{k}}}\psi_{j_{\widetilde{k}}}, 0),\ (0, h_{n_1}), \dots, (0, h_{n_{|\widetilde{\alpha}|}})\big] \\ = t_{j_1} \dots t_{j_{\widetilde{k}}} d^{\widetilde{k}} \partial^{\widetilde{\alpha}} R_\varepsilon(\varphi, a)\big[\psi_{j_1}, \dots, \psi_{j_{\widetilde{k}}}\big] \qquad (\widetilde{k} \leq k) \end{aligned} \tag{7}$$

(the numbers $\widetilde{k}, \widetilde{\alpha}, j_1, \dots, j_{\widetilde{k}}$ depend on $\mathcal{I}$ and can be the same for different decompositions $\mathcal{I}$). By (3) we see that there is at least one decomposition $\mathcal{I}$ of I giving the term $t_1 \cdot \ldots \cdot t_k\, \partial^{\alpha} d^k R_\varepsilon(\varphi, a)\big[\psi_1, \dots, \psi_k\big]$. Choose coefficients $c_0, \dots, c_k$ fulfilling the equations

$$\begin{aligned} &\sum_{j=0}^{k} c_j \cdot j \ \ = 1 \\ &\sum_{j=0}^{k} c_j \cdot j^n = 0 \quad \text{for} \quad n = 0 \ \text{ or } \ 2, 3, \dots, k. \end{aligned} \tag{8}$$

It follows from (4) that

$$\sum_{t_1=0}^{k} \cdots \sum_{t_k=0}^{k} c_{t_1} \dots c_{t_k} \big(\tfrac{\partial}{\partial x}\big)^{\alpha}\big(\tfrac{\partial}{\partial x_d}\big)^{p} R_\varepsilon(\varphi_{x,t_1,\dots,t_k}, x) = O(\varepsilon^{-N}) \tag{9}$$

(uniformly under the requirements as above). By (7) and (8), the left hand side is the sum only of terms of the form $\partial^{\widetilde{\alpha}} d^k R_\varepsilon(\varphi, a)\big[\psi_1, \dots, \psi_k\big]$ for some multi-index $\widetilde{\alpha}$ and there is at least once the term $\partial^{\alpha} d^k R_\varepsilon(\varphi, a)\big[\psi_1, \dots, \psi_k\big]$. As every $h_i \neq e_d$ belongs to a singleton, we have $\widetilde{\alpha}_j = \alpha_j$ for $j < d$. Considering the cardinalities of the elements of $\mathcal{I}$, we see that $|\widetilde{\alpha}| = |\alpha|$, so $\widetilde{\alpha} = \alpha$. Thus the left hand side of (9) is a natural multiple of $\partial^{\alpha} d^k R_\varepsilon(\varphi, a)\big[\psi_1, \dots, \psi_k\big]$ and this is what we wanted to prove. □

18. While we had to modify the definition of $\mathcal{E}_M[\Omega]$ in [5], there is no need to do the same with the definition of the ideal $\mathcal{N}$, serving as representatives of the null generalized function, thanks to the following equivalences.

Equivalent definitions. The ideal $\mathcal{N}[\Omega] \subset \mathcal{E}_M[\Omega]$ is defined to be the set of all representatives fulfilling one of the following equivalent conditions ($\mathcal{A}_q$ means $\mathcal{A}_q(\mathbb{R}^d)$).

1° (the definition in [4], where only the uniformity with respect to φ is not required). $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}\quad \exists q \in \mathbb{N}$ such that $\forall$ bounded $\mathcal{B} \subset \mathcal{D}(\mathbb{R}^d)$, we have

$$\partial^\alpha R_\varepsilon(\varphi, x) = O(\varepsilon^n)$$

uniformly for $\varphi \in \mathcal{A}_q \cap \mathcal{B},\ x \in K$.

2° (the same for the differentials). $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ k \in \mathbb{N}_0,\ n \in \mathbb{N}\quad \exists q \in \mathbb{N}$ such that $\forall$ bounded $\mathcal{B} \subset \mathcal{D}(\mathbb{R}^d)$, we have

$$\partial^\alpha d^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] = O(\varepsilon^n) \tag{10}$$

uniformly for

$$\varphi \in \mathcal{A}_q \cap \mathcal{B},\ \psi_1, \dots, \psi_k \in (\mathcal{A}_q - \mathcal{A}_q) \cap \mathcal{B},\ x \in K. \tag{11}$$

3° $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}\quad \exists q \in \mathbb{N}$ such that, for every bounded path $\big\{(\varphi_x^\varepsilon)_{x\in\Omega};\ \varepsilon \in\ ]0,1]\big\} \subset \mathcal{E}\big(\Omega{\to}\mathcal{A}_q(\mathbb{R}^d)\big)$, which is $\mathcal{C}^\infty$ with respect to ε, we have

$$\big(\tfrac{\partial}{\partial x}\big)^\alpha R_\varepsilon(\varphi_x^\varepsilon, x) = O(\varepsilon^n) \tag{12}$$

uniformly for $x \in K$.

4° (the definition in [5]). $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}\quad \exists q \in \mathbb{N}$ such that, for every bounded path $\big\{(\varphi_x^\varepsilon)_{x\in\Omega};\ \varepsilon \in\ ]0,1]\big\} \subset \mathcal{E}\big(\Omega{\to}\mathcal{A}_0(\mathbb{R}^d)\big)$, which is $\mathcal{C}^\infty$ with respect to ε and fulfills

$$a_q(\varphi_x^\varepsilon) = O(\varepsilon^q) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K$ (for a_q see Notation 2), we have (12) uniformly for $x \in K$.

Proof of 1° ⇔ 2°: ⇐ being evident, we are going to deduce 2° from 1° by induction. Denote by $S(k)\quad (k \in \mathbb{N}_0)$ the statement

$S(k)\ :\ \forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}\quad \exists q \in \mathbb{N}$ such that $\forall$ bounded $\mathcal{B} \subset \mathcal{D}(\mathbb{R}^d)$
(10) holds uniformly under the requirements (11).

$S(0)$ is the definition 1°. Supposing $S(k-1)$, we will prove $S(k)$ by contradiction $(k \in \mathbb{N})$. If $S(k)$ does not hold, choose $K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}$ such that $\forall q \in \mathbb{N}\quad \exists \mathcal{B}$ for which (10) does not hold uniformly under the requirements (11). Choose N by Theorem 17 (an equivalent definition of representatives) and then choose q by the induction hypothesis $S(k-1)$ such that

$$d^{k+1}\partial^\alpha R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_{k-1}, \psi_k, \psi_k] = O(\varepsilon^{-N}), \tag{13}$$

$$d^{k-1}\partial^\alpha R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_{k-1}] = O(\varepsilon^{2n+N+2}), \tag{14}$$

uniformly under the requirements (11) for any bounded $\mathcal{B} \subset \mathcal{D}(\mathbb{R}^d)$. Since, for this q, (10) does not hold uniformly, there are bounded sequences of test functions

$$\varphi_j \in \mathcal{A}_q,\, \psi_{1,j}, \ldots, \psi_{k,j} \in \mathcal{A}_q - \mathcal{A}_q \quad (j \in \mathbb{N})$$

and $x_j \in K$, $\varepsilon_j \searrow 0$, $\varepsilon_j \in]0,1]$ such that

$$\left| d^k \partial^\alpha R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1,j}, \ldots, \psi_{k,j}] \right| \geq 2\varepsilon_j^n. \tag{15}$$

By (13) we get

$$\left| d^{k+1} \partial^\alpha R_{\varepsilon_j}(\varphi_j + t\psi_{k,j}, x_j)[\psi_{1,j}, \ldots, \psi_{k-1,j}, \psi_{k,j}, \psi_{k,j}] \right| \leq \varepsilon_j^{-N-1}$$

for all j sufficiently great independently on $t \in [0,1]$. Therefore, for

$$0 < t \leq \varepsilon_j^{n+N+1}$$

we obtain from the Mean Value Theorem (Theorem 11)

$$\begin{gathered}\left| d^k \partial^\alpha R_{\varepsilon_j}(\varphi_j + t\psi_{k,j}, x_j)[\psi_{1,j}, \ldots, \psi_{k,j}] - d^k \partial^\alpha R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1,j}, \ldots, \psi_{k,j}] \right| \leq \\ \left| \sup_{t' \in [0,t]} d^{k+1} \partial^\alpha R_{\varepsilon_j}(\varphi_j + t'\psi_{k,j}, x_j)[\psi_{1,j}, \ldots, \psi_{k,j}, t\psi_{k,j}] \right| \leq \varepsilon_j^{-N-1} \varepsilon_j^{n+N+1} = \varepsilon_j^n.\end{gathered}$$

This means

$$d^k \partial^\alpha R_{\varepsilon_j}(\varphi_j + t\psi_{k,j}, x_j)[\psi_{1,j}, \ldots, \psi_{k,j}] \in \overline{B}(d_j, \varepsilon_j^n) \tag{16}$$

(the closed ball in $\mathbb{R}$), where we have denoted by

$$d_j := d^k \partial^\alpha R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1,j}, \ldots, \psi_{k,j}]. \tag{17}$$

Again from the Mean Value Theorem and (16), we get

$$\begin{gathered} d^{k-1} \partial^\alpha R_{\varepsilon_j}(\varphi_j + \varepsilon_j^{n+N+1}\psi_{k,j},\, x_j)[\psi_{1,j}, \ldots, \psi_{k-1,j}] \\ - d^{k-1} \partial^\alpha R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1,j}, \ldots, \psi_{k-1,j}] \in \\ \overline{\operatorname{conv}} \left\{ d^k \partial^\alpha R_{\varepsilon_j}(\varphi_j + t\psi_{k,j}, x_j)[\psi_{1,j}, \ldots, \psi_{k-1,j}, \varepsilon_j^{n+N+1}\psi_{k,j}];\, t \in (0, \varepsilon_j^{n+N+1}) \right\} \\ \subset \overline{B}(\varepsilon_j^{n+N+1} d_j,\, \varepsilon_j^{2n+N+1}). \end{gathered}$$

Thanks to (15) and (17), it follows

$$\begin{aligned} &\Big| d^{k-1} \partial^\alpha R_{\varepsilon_j}(\varphi_j + \varepsilon_j^{n+N+1}\psi_{k,j},\, x_j)[\psi_{1,j}, \ldots, \psi_{k-1,j}] \\ &- d^{k-1} \partial^\alpha R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1,j}, \ldots, \psi_{k-1,j}] \Big| \geq \varepsilon_j^{2n+N+1}. \end{aligned} \tag{18}$$

The functions $\varphi_j + \varepsilon_j^{n+N+1}\psi_{k,j}$ form a bounded set, hence by (14) the left hand side of (18) should be $= O(\varepsilon_j^{2n+N+2})$. This contradicts (18). ☐

Proof of 1° or 2° ⇔ 3°: 2° ⇒ 3° can be calculated using Theorem 12 (on the differentiation of a composition) and 14.

If 1° does not hold, there are $K \Subset \Omega$, $\alpha \in \mathbb{N}_0^d$, $n \in \mathbb{N}$ such that for every $q \in \mathbb{N}$ we can find sequences $\varepsilon_j \searrow 0$ with $\varepsilon_1 > \varepsilon_2 > \dots$, $x_j \in K$ and $\{\varphi_j\}$ bounded in $\mathcal{A}_q$ such that

$$\partial^\alpha R_{\varepsilon_j}(\varphi_j, x_j) \neq O(\varepsilon^n). \tag{19}$$

Choose a decomposition of unity $\sum \lambda_j = 1$ on the interval $]0,1]$ with test functions $\lambda_j \in \mathcal{D}(]\varepsilon_{j+1}, \varepsilon_{j-1}[)$ $\quad (j = 2,3,\dots)$, $\lambda_1 \in \mathcal{D}(]\varepsilon_2, \infty[)$, $\lambda_j(\varepsilon_j) = 1$. Then the path of constant $\mathcal{A}_q$-valued functions

$$\left(\textstyle\sum_{j=1}^\infty \lambda_j(\varepsilon)\varphi_j\right)_{x\in\Omega} \in \mathcal{E}\big(\Omega \to \mathcal{A}_q(\mathbb{R}^d)\big) \qquad (\,\varepsilon \in \,]0,1]\,)$$

has, for $\varepsilon = \varepsilon_j$ and $x = x_j$, the values φ_j, therefore due to (19) it does not satisfy 3°. ☐

Proof of 3° ⇔ 4°: ⇐ being evident, we are proving ⇒. For the given K and α take first a number N by Theorem 17 (an equivalent definition of the representatives) such that

$$\partial^\alpha \mathrm{d}R_\varepsilon(\varphi, x)[\psi] = O(\varepsilon^{-N}) \tag{20}$$

uniformly if $x \in K$ and if φ, ψ run over bounded sets in $\mathcal{A}_0(\mathbb{R}^d)$, $\mathcal{A}(\mathbb{R}^d)$ respectively. Then, having chosen n, let q satisfy 3° and at the same time

$$q \geq n + N. \tag{21}$$

Let $B \subset \mathbb{R}^d$ be a bounded set containing the supports of all φ_x^ε with $x \in K$ and let $\varepsilon_0 > 0$ be such that $R_\varepsilon(\varphi, x)$ is always defined whenever $0 < \varepsilon \leq \varepsilon_0$, $\varphi \in \mathcal{A}_0(B)$ and $x \in K$.

Recall a known lemma of functional analysis ([14, II.3, Lemma 5): If linear forms $f_0, f_1, \dots, f_k$ on a linear space E are linearly independent, then there is a point $x \in E$ such that $f_0(x) = 1$, $f_1(x) = \dots = f_k(x) = 0$.

Since the functions $x \mapsto x^\beta$ considered as distributions $\in \mathcal{D}'(B)$ with $\beta \in \mathbb{N}_0^d$, $0 \leq |\beta| \leq q$, are linearly independent, there are test functions $\psi_\alpha \in \mathcal{D}(B)$, $0 \leq |\alpha| \leq q$, fulfilling

$$\int \psi_\alpha(\xi) \cdot \xi^\alpha \,\mathrm{d}\xi = 1 \tag{22}$$

$$\int \psi_\alpha(\xi) \cdot \xi^\beta \,\mathrm{d}\xi = 0 \quad \text{for} \quad \beta \neq \alpha,\ 0 \leq |\beta| \leq q. \tag{23}$$

Hence $\psi_\alpha \in \mathcal{A}(B)$, except for ψ_0, which we will not need. If we denote

$$c_{\alpha,x,\varepsilon} := \int \varphi_x^\varepsilon(\xi)\xi^\alpha \,\mathrm{d}\xi, \tag{24}$$

we have

$$\kappa_x^\varepsilon := \varphi_x^\varepsilon - \sum_{\substack{\alpha\in\mathbb{N}_0^d\\ 1\le|\alpha|\le q}} c_{\alpha,x,\varepsilon}\psi_\alpha \in \mathcal{A}_q. \tag{25}$$

By the hypothesis of 4°, the definition of a_q (in Notation 2) and (24), we have

$$c_{\alpha,x,\varepsilon} = O(\varepsilon^q). \tag{26}$$

Let us order the summation indexes α in (25) into a sequence $\alpha_1, \dots, \alpha_m$. Using the Mean Value Theorem 11, we have

$$\partial^\alpha R_\varepsilon\big(\varphi_x^\varepsilon, x\big) - \partial^\alpha R_\varepsilon\big(\kappa_x^\varepsilon, x\big) =$$
$$\sum_{j=1}^m \Big(\partial^\alpha R_\varepsilon\big(\kappa_x^\varepsilon + \textstyle\sum_{i=1}^j c_{\alpha_i,x,\varepsilon}\psi_{\alpha_i}, x\big) - \partial^\alpha R_\varepsilon\big(\kappa_x^\varepsilon + \textstyle\sum_{i=1}^{j-1} c_{\alpha_i,x,\varepsilon}\psi_{\alpha_i}, x\big)\Big) \in$$
$$\sum_{j=1}^m \overline{\mathrm{conv}}\Big\{\partial^\alpha \mathrm{d}R_\varepsilon\big(\kappa_x^\varepsilon + \textstyle\sum_{i=1}^{j-1} c_{\alpha_i,x,\varepsilon}\psi_{\alpha_i} + tc_{\alpha_j,x,\varepsilon}\psi_{\alpha_j}\big)\big[c_{\alpha_j,x,\varepsilon}\psi_{\alpha_j}\big];\ t\in\,]0,1]\Big\}.$$

Due to (26) and (20), it follows

$$\partial^\alpha R_\varepsilon\big(\varphi_x^\varepsilon, x\big) - \partial^\alpha R_\varepsilon\big(\kappa_x^\varepsilon, x\big) = O(\varepsilon^{q-N})$$

uniformly for $x \in K$. It is also $= O(\varepsilon^n)$ due to (21). Now, by (25) and 3°, $\partial^\alpha R_\varepsilon\big(\kappa_x^\varepsilon, x\big) = O(\varepsilon^n)$, hence also $\partial^\alpha R_\varepsilon\big(\varphi_x^\varepsilon, x\big) = O(\varepsilon^n)$. □

19. We can easily see like in [4] that $\mathcal{N}[\Omega]$ is an ideal in the algebra $\mathcal{E}_M[\Omega]$, so we can define $\mathcal{G}[\Omega]$ as follows.

Definition. The space of generalized functions on Ω is the quotient algebra $\mathcal{G} = \frac{\mathcal{E}_M[\Omega]}{\mathcal{N}[\Omega]}$.

Notation. The generalized function with the representative R, i.e. the class of the representatives defining the same generalized function as R, is denoted by $\langle R\rangle$.

Proposition 1° (*Moderate growth as a local property*). *A function $R \in \mathcal{E}[\Omega]$ belongs to $\mathcal{E}_M[\Omega]$ iff $\forall\, x \in \Omega$ there is an open neighborhood U of x in Ω such that $R \in \mathcal{E}_M[U]$ (after the restriction of R on $\mathcal{A}_0(U) \times U$).*

2°. *($\mathcal{N}$ as a local property). A representative $\mathcal{N}$ belongs to $\mathcal{N}[\Omega]$ iff $\forall\, x \in \Omega$ there is an open neighborhood U of x in Ω such that $R \in \mathcal{N}[U]$ (after the restriction of R on $\mathcal{A}_0(U) \times U$).*

3°. *$\mathcal{G}$ is a sheaf.*

Proof: The statements 1° and 2° are an easy consequence of the following observation (see Notation 6): If $\varepsilon \searrow 0$, then $\operatorname{supp} \varphi_{x,\varepsilon}$ tends to $\{x\}$ uniformly with respect to φ running over a bounded subset of $\mathcal{A}_0(\mathbb{R}^d)$.
3° is similar as in [4] (■1.3, Local properties ...). □

Notation 20 (the values of a representative which matter). **1°**. Let $x \mapsto q_x \in \mathbb{N}_0$ be an upper semi-continuous function on Ω and $(U_x)_{x\in\Omega}$ be a family of open neighborhoods of points x, contained in Ω, which are locally uniform in the following sense: for every $x \in \Omega$ there is a neighborhood V of x such that $\bigcap\limits_{y\in V} (U_y - y)$ is a neighborhood of 0. Under these hypotheses we denote

$$\mathfrak{U} = \mathfrak{U}\big((U_x, q_x)_{x\in\Omega}\big) := \big\{\, (\varphi, x);\ x \in \Omega,\ \operatorname{supp} \varphi \subset U_x,\ \varphi(\bullet{-}x) \in \mathcal{A}_{q_x} \big\}\,.$$

If $R \in \mathcal{E}[\Omega]$ is a representative, then we can check from Definition 18.1° of $\mathcal{N}$ that only the values $R(\varphi, x)$ matter for which $(\varphi, x) \in \mathfrak{U}$. This means that if two representatives are equal for these pairs (φ, x), they determine the same generalized function.

2°. Let $(V_i)_{i\in I}$ be an open covering of Ω with $V_i \subset \Omega$ for all $i \in I$, where I is an arbitrary set of indexes, and let $\{q_i\}_{i\in I}$ be a family of numbers $q_i \in \mathbb{N}_0$. Denote by

$$\begin{gathered}\mathfrak{V} = \mathfrak{V}\big((V_i, q_i)_{i\in I}\big) := \\ \big\{\, (\varphi, x);\ \exists\, i \in I \text{ such that } x \in V_i,\ \operatorname{supp} \varphi \subset V_i,\ \varphi(\bullet{-}x) \in \mathcal{A}_{q_i} \big\}\,.\end{gathered}$$

If $R \in \mathcal{E}[\Omega]$ is a representative, then only the values $R(\varphi, x)$ matter for determining $\langle R\rangle$, for which $(\varphi, x) \in \mathfrak{V}$ (see the following proposition).

Proposition. *For each set $\mathfrak{U}$ according to 1° there is a set $\mathfrak{V} \subset \mathfrak{U}$ according to 2°. For each set $\mathfrak{V}$ according to 2° there is a set $\mathfrak{U} \subset \mathfrak{V}$ according to 1°.*

Proof: I. Using the uniformity condition in 1°, for $x \in \Omega$ choose its neighborhood V_x such that every U_y for $y \in V_x$ contains an open ball $B(y, r)$ ($r > 0$ is independent on y). Then change V_x for a smaller one so that its diameter

$$\operatorname{diam} V_x \le r \tag{27}$$

and that the function $y \mapsto q_y$ is bounded on V_x by a number $\widetilde{q}_x \in \mathbb{N}_0$. Thus we obtain $\mathfrak{V} = \mathfrak{V}\big((V_x, \widetilde{q}_x)_{x\in\Omega}\big) \subset \mathfrak{U}$. Indeed, if $(\varphi, y) \in \mathfrak{V}$, we have for some x: $y \in V_x$, $\operatorname{supp} \varphi \subset V_x \subset B(y, r)$ by (27). Hence $V_x \subset U_y$ and $\varphi(\bullet{-}y) \in \mathcal{A}_{\widetilde{q}_x} \subset \mathcal{A}_{q_y}$. This means $(\varphi, y) \in \mathfrak{U}$.

II. Taking a refining, we can suppose without a loss of generality that $(V_i)_{i\in I}$ is locally finite and V_i are relatively compact in Ω. Let $(W_i)_{i\in I}$ be an open covering of Ω with $\overline{W}_i \subset V_i$ for every $i \in I$ (this is possible for instance according to [7, Chapter 5, p. 207, Lemma 1] in a normal space (even with a point finite open covering)). We define $\mathfrak{U}$ as follows:

$$U_x := \bigcap_{x\in\overline{W_i}} V_i, \qquad q_x := \max\left\{q_i;\, x \in \overline{W_i}\right\},$$

where the intersection and the maximum are extended over those i for which $x \in \overline{W_i}$. Fix a point $x \in \Omega$. As $(W_i)_{i\in I}$ is locally finite, there is an open neighborhood V of x such that $\overline{V}$ is compact in Ω and does not meet any $\overline{W_i}$ with $x \notin \overline{W_i}$. Thus, for $y \in \overline{V}$, it is $U_y \supset U_x$ and $q_y \le q_x$. Hence the function $x \mapsto q_x$ is upper semi-continuous. Since U_x is an open neighborhood of the compact set $\overline{V}$, the neighborhoods $U_x - y$ of points $y \in \overline{V}$ are uniform. Therefore, $U_y - y$ are uniform as well. □

21. The following useful theorem shows that a representative need not be defined on the whole set $\mathcal{A}(\Omega) \times \Omega$. It is sufficient for determining $\langle R\rangle$ only to define R on a set $\mathfrak{U}$ or $\mathfrak{V}$ defined in Notation 20.

Theorem. 1°. *Let R° be a $\mathcal{C}^\infty$ function defined on a set $\mathfrak{V}\big((V_i, q_i)_{i\in I}\big)$. Then there is a $\mathcal{C}^\infty$ function R on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$ coinciding with R° on some set $\mathfrak{U}\big((U_x, q_x)_{x\in\Omega}\big)$.*

2°. *Suppose in addition that R° satisfies the moderate growth condition in Definition 8, whenever $(\varphi_x)_{x\in\Omega}$ runs over such a bounded set of $\mathcal{E}\big(\Omega\to\mathcal{A}_0(\mathbb{R}^d)\big)$ that $\big(\frac{\partial}{\partial x}\big)^\alpha (R^\circ)_\varepsilon(\varphi_x, x)$ is defined for $x \in K$ and ε sufficiently small independently on $x \in K$. Then R from the part 1° can be chosen in addition $\in \mathcal{E}_M[\Omega]$.*

Proof of 1°: Taking a refining, we can suppose that $(V_i)_{i\in I}$ is in addition locally finite and that $\overline{V_i} \Subset \Omega$. Choose a locally finite open covering $(W_i)_{i\in I}$ of Ω, with $\overline{W_i} \subset V_i$, and a smooth partition of unity $(\tau_i)_{i\in I}$ subordinated to $(W_i)_{i\in I}$: $\tau_i \in \mathcal{D}(W_i)$, $\sum \tau_i = 1$ on Ω. The idea of the proof is to define

$$\begin{aligned} (28) \qquad & R(\varphi, x) := \sum \tau_i(x)\, R_i(\varphi, x) \\ \text{with} \quad & R_i(\varphi, x) := R^\circ\big(\pi_i(\varphi), x\big) \qquad (\,x \text{ in a neighborhood of } \operatorname{supp}\tau_i\,), \end{aligned}$$

where π_i is an appropriate mapping (depending on x) of $\mathcal{A}_0(\mathbb{R}^d)$ into $\mathcal{A}_0(V_i)$. If $x \notin \operatorname{supp}\tau_i$, then the term $\tau_i(x)\, R_i(\varphi, x)$ is considered to be $= 0$ even if $R_i(\varphi, x)$ is not defined. For the sake of simplicity of the notation, we do not indicate the dependence of π_i on x. Here are all required properties of π_i:

(i) the map $\varphi, x \mapsto \pi_i(\varphi)$ is defined and $\mathcal{C}^\infty$ for x in a neighborhood of $\operatorname{supp}\tau_i$ and for all $\varphi \in \mathcal{A}_0(\mathbb{R}^d)$;

(ii) $\operatorname{supp}\pi_i(\varphi) \subset V_i$;
(iii) $\pi_i(\varphi)(x{+}\bullet) \in \mathcal{A}_{q_i}$;
(iv) if $\varphi(x{+}\bullet) \in \mathcal{A}_{q_i}$ and $\operatorname{supp}\varphi \subset W_i$, then $\pi_i(\varphi) = \varphi$.

Under these requirements (which we will prove), we have $R(\varphi, x) = R^\circ(\varphi, x)$ whenever

$$\varphi(x{+}\bullet) \in \mathcal{A}_{\max q_i} \qquad \text{and} \qquad \operatorname{supp}\varphi \subset \bigcap_{x \in \operatorname{supp}\tau_i} W_i,$$

where max and $\bigcap$ are taken over those i for which $x \in \operatorname{supp}\tau_i$. Thus, we have got $U_x := \bigcap\limits_{x \in \operatorname{supp}\tau_i} W_i$ and $q_x := \max\limits_{x \in \operatorname{supp}\tau_i} q_i$. For proving the first part of the theorem, we only have to construct the map π_i with the required properties.

Denote by $B_1 = B(0,1)$ the open unit ball in $\mathbb{R}^d$ and by $B = B(0,\rho)$ the ball in $\mathbb{R}^d$ with Lebesgue measure $\Lambda(B) = 1$. Thus

$$\Lambda(B_1) = \rho^{-d}. \tag{29}$$

Fix $i \in I$ and choose a number $r_i > 0$ such that

$$\operatorname{supp}\tau_i + \tfrac{r_1}{2}\overline{B_1} \subset W_i \qquad \text{and} \qquad \overline{W_i} + \tfrac{r_1}{2}\overline{B_1} \subset V_i. \tag{30}$$

Choose $0 \le \vartheta_i \in \mathcal{D}(V_i)$ with $\vartheta_i = 1$ on W_i, and $0 \le \vartheta \in \mathcal{D}([-1,1])$ with $\vartheta = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. We will define π_i and R_i for

$$x \in \left\{x;\, x + \tfrac{r_i}{2}\overline{B_1} \subset W_i\right\}. \tag{31}$$

By (30), this is a neighborhood of $\operatorname{supp}\tau_i$, contained in W_i. For $\varphi \in \mathcal{A}_0(\mathbb{R}^d)$ put

$$\varphi^\circ := \vartheta_i \cdot \varphi \qquad (\,\text{so} \quad \varphi^\circ \in \mathcal{D}(V_i)\,), \tag{32}$$

$$k := \left(\frac{\|\varphi^\circ\|^2}{\rho^d} + \frac{\vartheta\big((\frac{r_i}{\rho})^d \cdot \frac{\|\varphi^\circ\|^2}{2}\big)}{r_i^d} \right)^{\frac{1}{d}} \qquad (\,\|\ \|\text{ is the } \mathcal{L}^2\text{-norm}). \tag{33}$$

We have $k \ge \frac{1}{r_i}$ since either $\frac{\|\varphi^\circ\|^2}{\rho^d} \ge \frac{1}{r_i^d}$ or $\vartheta(\dots) = 1$. Let $\psi_\alpha \in \mathcal{D}(B_1)$ be functions fulfilling (22), (23) for $0 \le |\alpha| \le q_i$. Put

$$\pi_i(\varphi) := \varphi^\circ - \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ 0 \le |\alpha| \le q_i}} c_\alpha \psi_\alpha\big((\bullet{-}x)k\big) \tag{34}$$

with such coefficients c_α that $\pi_i(\varphi)(x{+}\bullet) \in \mathcal{A}_{q_i}$. By (22) and (23), c_α are well defined for $0 \le |\alpha| \le q_i$ and we have

$$\begin{aligned} k^d(\textstyle\int \varphi^\circ - 1) \qquad &= c_0 \\ k^{|\alpha|+d} \int \varphi^\circ(x+\xi)\xi^\alpha \,\mathrm{d}\xi &= c_\alpha \qquad (1 \le |\alpha| \le q_i). \end{aligned} \tag{35}$$

The properties (i) and (iii) are evident. As $k \geq \frac{1}{r_i}$ and $\operatorname{supp}\psi_\alpha \subset B_1$, (ii) easily follows from (30) and (32), if x fulfills (31). Now, let $\varphi(x+\bullet) \in \mathcal{A}_{q_i}$ and $\operatorname{supp}\varphi \subset W_i$. Then by (32) and the definition of ϑ_i, we have $\varphi = \varphi^\circ$. Evidently $\pi_i(\varphi) = \varphi^\circ = \varphi$ and so the requirements (i)–(iv) are proved. ☐

Proof of 2°: Since the sum in (28) is locally finite, it suffices to prove the moderate growth of R_i. If the vector valued function $(\varphi_x)_{x\in\Omega} \in \mathcal{E}\big(\Omega\to\mathcal{A}_0(\mathbb{R}^d)\big)$ runs over a bounded set, its values φ_x for $x \in \overline{V_i} \Subset \Omega$ remain in a bounded set of $\mathcal{A}_0$, so their supports are contained in a common ball $B(0,A) \subset \mathbb{R}^d$ $(A>0)$. Hence, if $\varepsilon_0 := \frac{r_i}{2A}$, then $\forall\,\varepsilon \in\,]0,\varepsilon_0]$, $x \in \overline{V_i}$ the support of the function

$$\varphi_{x,\varepsilon} = \varepsilon^{-d}\varphi_x\Big(\frac{\bullet - x}{\varepsilon}\Big) \tag{36}$$

is contained in $x + \frac{r_i}{2}B_1 \subset W_i$ by (31) (see Notation 6 defining R_ε). By (32) we have $\varphi^\circ_{x,\varepsilon} = \varphi_{x,\varepsilon}$. The Hölder inequality gives

$$\|\varphi_{x,\varepsilon}\| \cdot \|\chi_{x+\frac{r_i}{2}B_1}\| \geq \int \varphi_{x,\varepsilon} = 1 \qquad (\chi \text{ is the characteristic function}).$$

Due to (29), this means

$$\|\varphi_{x,\varepsilon}\|^2 \cdot \Big(\frac{r_i}{2\rho}\Big)^d \geq 1. \tag{37}$$

By (28) and Notation 6, we have

$$\begin{aligned}(R_i)_\varepsilon(\varphi_x, x) &= R_i(\varphi_{x,\varepsilon},\, x) = R^\circ\big(\pi_i(\varphi_{x,\varepsilon}),\, x\big)\\ &= (R^\circ)_\varepsilon\big(\varepsilon^d\pi_i(\varphi_{x,\varepsilon})(x+\bullet\varepsilon),\, x\big)\end{aligned} \tag{38}$$

where π_i is defined by (34) and (35). Denote the number k in (33) for the function $\varphi_{x,\varepsilon} = \varphi^\circ_{x,\varepsilon}$ by k_ε, taking into account that it depends on x, too. It follows from (37), due to the definition of ϑ, that k_ε is given by a simpler formula then (33):

$$k_\varepsilon = \frac{\|\varphi_{x,\varepsilon}\|^{2/d}}{\rho}. \tag{39}$$

Considering that $\varphi^\circ_{x,\varepsilon} = \varphi_{x,\varepsilon}$, we get from (38), (36) and (34)

$$\big(R_i\big)_\varepsilon(\varphi_x, x) = \big(R^\circ\big)_\varepsilon\big(\varphi_x - \varepsilon^d \textstyle\sum c_\alpha\psi_\alpha(\bullet\varepsilon k_\varepsilon),\, x\big). \tag{40}$$

From (39) and (36) we calculate $\varepsilon k_\varepsilon = \varepsilon_0 k_{\varepsilon_0}$. From (35) we calculate $c_0 = 0$ and, for $|\alpha| \geq 1$,

$$\varepsilon^d c_\alpha = \varepsilon^d k_\varepsilon^{|\alpha|+d}\int \varphi_{x,\varepsilon}(x+\xi)\xi^\alpha\,\mathrm{d}\xi = \varepsilon^{|\alpha|+d}k_\varepsilon^{|\alpha|+d}\int \varphi_x\big(\tfrac{\xi}{\varepsilon}\big)\cdot\big(\tfrac{\xi}{\varepsilon}\big)^\alpha\varepsilon^{-d}\,\mathrm{d}\xi,$$

which does not depend on ε due to the preceding result. So the test function on the right hand side of (40) does not depend on ε and remains bounded in $\mathcal{E}\big(\Omega\to\mathcal{A}_0(\mathbb{R}^d)\big)$ if $(\varphi_x)_x$ runs over a bounded set. Moreover, the right hand side of (40) is defined, being equal to $R^\circ\big(\pi_i(\varphi_{x,\varepsilon}),\, x\big)$ with $\big(\pi_i(\varphi_{x,\varepsilon}),\, x\big) \in \mathfrak{V}$, thanks to the points (ii) and (iii) of the first part of the proof. Hence, by hypothesis, it has a moderate growth. ☐

Remark 22 (Definition of the derivative)**.** We define the derivative $\partial_{e_j}\langle R\rangle$ of a generalized function $\langle R\rangle$ (with respect to the j-th coordinate unit vector e_j) in the same way as it is defined in [4]: If R_1 is a representative of $\langle R\rangle$ according to the definitions in [4], a representative of $\partial_{e_j}\langle R\rangle$ is defined there to be $\varphi, x \mapsto \frac{\partial}{\partial x_j} R_1(\varphi, x)$. As a consequence of Change 4 in notation, we have $R_1(\varphi, x) = R(\varphi(\bullet - x), x)$. It follows

$$\frac{\partial}{\partial x_j} R_1(\varphi, x) = \mathrm{d}R(\varphi(\bullet - x), x)[-\partial_{e_j}\varphi\,(\bullet - x)] + \partial_{e_j} R(\varphi(\bullet - x), x).$$

Hence (in our notation) $\partial_{e_j}\langle R\rangle = \langle R'\rangle$ with

$$R'(\varphi, x) = -\mathrm{d}R(\varphi, x)[\partial_{e_j}\varphi] + \partial_{e_j} R(\varphi, x).$$

Recall our definition of the canonical embedding of $\mathcal{D}'$ into $\mathcal{G}$: the canonical image of a distribution f in $\mathcal{G}$ has the function $\varphi, x \mapsto \langle f, \varphi\rangle$ (independent on x) as a representative. Thus, with the usual definition of the differentiation of the distributions (by [16]) and with the definition above, the canonical embedding evidently commutes with the differentiation.

Action of a $\mathcal{C}^\infty$ diffeomorphism

Change 23 (which we will not always keep)**.** In the expression $R(\varphi, x)$, we consider the test function φ as a test density ([8]). While we are not dealing with coordinate diffeomorphisms, this change has no influence, as there is a one-to-one correspondence between a test function φ and the corresponding test density $\underline{\varphi}$ given by the formula

$$\underline{\varphi}(x) = \varphi(x)\underline{\mathrm{d}}x, \tag{41}$$

where $\underline{\mathrm{d}}x$ stands for the Lebesgue measure on $\mathbb{R}^d$. According to [10], we denote all odd differential forms (including densities) by underline letters. In the same way we denote also the spaces of odd differential forms. For instance, $\overset{d}{\underline{\mathcal{D}}}(\mathbb{R}^d)$ is the space of all test densities on $\mathbb{R}^d$. When the first variable of a representative is a test density, we will denote the representative and the spaces of representatives by underline letters as well, for instance $\underline{R}(\underline{\varphi}, x)$. We have to use this type of representatives, when we deal with generalized functions on a $\mathcal{C}^\infty$ manifold (different from $\Omega \subset \mathbb{R}^d$), but this is not necessary for generalized functions on $\Omega \subset \mathbb{R}^d$. Recall that similarly, for defining the distributions on a $\mathcal{C}^\infty$ manifold of the dimension d, the space of the test functions $\mathcal{D}$ is replaced with $\overset{d}{\underline{\mathcal{D}}}$. Thanks to the notion of density, we can define the image by a coordinate diffeomorphism in an easy and natural way.

Definitions and notations 24. Let $\Omega, \widetilde{\Omega}$ be open subsets of $\mathbb{R}^d$ and $\mu : \widetilde{\Omega} \to \Omega$ be a $\mathcal{C}^\infty$ diffeomorphism. If f is a function on Ω, then the function $\widetilde{f} = f{\circ}\mu$ on $\widetilde{\Omega}$ is denoted also by $\mu^* f$ (inverse image of f). If $\underline{\varphi}$ is a test density (or more generally an integrable density) on Ω, then its inverse image $\underline{\widetilde{\varphi}}$, denoted by $\underline{\varphi}{\circ}\mu$ or $\mu^*\underline{\varphi}$ or $\underline{\varphi}(\mu(\widetilde{x}))$ $\ (\widetilde{x} \in \widetilde{\Omega}\,)$ is defined to be the density on $\widetilde{\Omega}$ given by the formula

$$\int (\tau{\circ}\mu)(\underline{\varphi}{\circ}\mu) = \int \tau\underline{\varphi} \qquad \forall\, \tau \in \mathcal{D}(\Omega).$$

Or directly, if $\underline{\varphi}$ corresponds to a function φ by (41), then $\underline{\varphi}{\circ}\mu$ corresponds to the function $(\varphi{\circ}\mu)|J_\mu|$ where J_μ is the Jacobian of μ. Using another notation, $\underline{\widetilde{\varphi}}(\widetilde{x}) = \varphi(\mu(\widetilde{x}))\underline{\mathrm{d}}\mu(\widetilde{x})$, where $\underline{\mathrm{d}}\mu(\widetilde{x}) = |J_\mu(\widetilde{x})|\,\underline{\mathrm{d}}\widetilde{x}$ stands for the inverse image of the Lebesgue measure $\underline{\mathrm{d}}x$. If the space of distributions $\mathcal{D}'(\Omega)$ is defined to be the dual space to the space of the test densities $\overset{d}{\underline{\mathcal{D}}}(\Omega)$, then the inverse image of a distribution $f \in \mathcal{D}'(\Omega)$, denoted by $f{\circ}\mu$ or $\mu^* f$ or $f(\mu(\widetilde{x}))$, is defined by the formula

$$\langle f{\circ}\mu, \underline{\varphi}{\circ}\mu\rangle = \langle f, \underline{\varphi}\rangle \quad (\,\underline{\varphi} \in \overset{d}{\underline{\mathcal{D}}}(\Omega)\,).$$

Denote by $\overset{d}{\underline{\mathcal{A}}}_0$ the set of all $\underline{\varphi} \in \overset{d}{\underline{\mathcal{D}}}$ for which $\int \underline{\varphi} = 1$. If $\langle \underline{R}\rangle \in \mathcal{G}$ is a generalized function with $\underline{R}$ as a representative, its inverse image $\langle \underline{\widetilde{R}}\rangle \in \mathcal{G}(\widetilde{\Omega})$ is defined to have as a representative the function $\underline{\widetilde{R}}$ defined by the formula

$$\underline{\widetilde{R}}(\underline{\varphi}{\circ}\mu,\, \widetilde{x}) = \underline{R}(\underline{\varphi},\, \mu(\widetilde{x})) \qquad (\,\underline{\varphi} \in \overset{d}{\underline{\mathcal{A}}}_0(\Omega)),$$

i.e.

$$\underline{\widetilde{R}}(\underline{\widetilde{\varphi}},\, \widetilde{x}) = \underline{R}(\,\underline{\widetilde{\varphi}}{\circ}\mu^{-1},\, \mu(\widetilde{x})) \qquad (\,\underline{\widetilde{\varphi}} \in \overset{d}{\underline{\mathcal{A}}}_0(\widetilde{\Omega})).$$

Using the notation with the test functions, we obtain

$$\widetilde{R}(\widetilde{\varphi},\, \widetilde{x}) = R\big(\,|J_{\mu^{-1}}|\,\widetilde{\varphi}{\circ}\mu^{-1},\, \mu(\widetilde{x})\big) \qquad (\,\widetilde{\varphi} \in \mathcal{A}_0(\widetilde{\Omega})).$$

We denote $\widetilde{R}$ by $\mu^* R$, $\underline{\widetilde{R}}$ by $\mu^*\underline{R}$.

Remark 25. In [5] it is proved that $\mu^* R \in \mathcal{E}_M[\widetilde{\Omega}]$ for $R \in \mathcal{E}_M[\Omega]$ and that the inverse image of an element of $\mathcal{N}$ is again an element of $\mathcal{N}$, but the basic definitions in [5] are not exactly the same as we have here. From the last formula, using the definition of R_ε in Notation 6, we deduce the relation between $\widetilde{R}_\varepsilon$ and R_ε:

$$\widetilde{R}_\varepsilon(\widetilde{\varphi},\, \widetilde{x}) = R_\varepsilon(\varphi, x)$$

with

$$x = \mu(\widetilde{x}), \quad \varphi(\xi) = \Big|J_{\mu^{-1}}(x + \xi\varepsilon)\Big|\; \widetilde{\varphi}\Big(\frac{\mu^{-1}(x + \xi\varepsilon) - \widetilde{x}}{\varepsilon}\Big), \tag{42}$$

provided

$$\widetilde{\varphi} \in \mathcal{A}_0\Big(\frac{\widetilde{\Omega} - \widetilde{x}}{\varepsilon}\Big).$$

More precisely: The diffeomorphism

$$\xi \mapsto \frac{\mu^{-1}(x + \xi\varepsilon) - \widetilde{x}}{\varepsilon} \qquad (\,\widetilde{x} \in \widetilde{\Omega},\ x = \mu(\widetilde{x}) \in \Omega\,) \tag{43}$$

has the open set $\frac{\Omega - x}{\varepsilon}$ as its domain and $\frac{\widetilde{\Omega} - \widetilde{x}}{\varepsilon}$ as its rang. Thus, φ is defined by (42) on $\frac{\Omega - x}{\varepsilon}$, only. We extend φ, putting $\varphi(\xi) = 0$ if φ is not defined by (42). Thus, φ is a smooth function on $\mathbb{R}^d$, $\varphi \in \mathcal{A}_0\big(\frac{\Omega - x}{\varepsilon}\big)$.

(42) is the formula (2) in [5], by which Colombeau defined the inverse image of a generalized function and proved that the inverse image of an element of $\mathcal{N}$ is again an element of $\mathcal{N}$. This proof will be valid for us, too, when only we have proved the following proposition, because our Definition 18.4° does not differ essentially from the definition in [5].

Proposition. *We have* $\widetilde{R} \in \mathcal{E}_M[\widetilde{\Omega}]$ *for* $R \in \mathcal{E}_M[\Omega]$.

Proof: According to Definition 8, choose a compact $\widetilde{K} \Subset \widetilde{\Omega}$ and denote $K = \mu(\widetilde{K})$. For $(\widetilde{\varphi}_{\widetilde{x}})_{\widetilde{x} \in \widetilde{\Omega}} \in \mathcal{E}\big(\widetilde{\Omega} \to \mathcal{A}_0(\mathbb{R}^d)\big)$, define φ_x by (42) (if it is possible), depending on ε, so that $\widetilde{R}_\varepsilon(\widetilde{\varphi}_{\widetilde{x}}, \widetilde{x}) = R_\varepsilon(\varphi_x, x)$. We want to prove: If $(\widetilde{\varphi}_{\widetilde{x}})_{\widetilde{x} \in \widetilde{\Omega}}$ runs over a bounded subset of $\mathcal{E}\big(\widetilde{\Omega} \to \mathcal{A}_0(\mathbb{R}^d)\big)$ and if $\varepsilon \in \,]0, \varepsilon_0]$, with ε_0 sufficiently small, then $(\varphi_x)_x$ remains bounded in $\mathcal{E}\big(\Omega' \to \mathcal{A}_0(\mathbb{R}^d)\big)$ for some open neighborhood Ω' of K in Ω. Then we deduce the moderate growth of $\widetilde{R}_\varepsilon$ from the moderate growth of R_ε.

Choose $h > 0$ such that

$$\widetilde{K} + \overline{B}(0, 2h) \subset \widetilde{\Omega}, \quad K + \overline{B}(0, 2h) \subset \Omega.$$

As $(\widetilde{\varphi}_{\widetilde{x}})_{\widetilde{x} \in \widetilde{\Omega}}$ runs over a bounded subset of $\mathcal{E}\big(\widetilde{\Omega} \to \mathcal{A}_0(\mathbb{R}^d)\big)$, there is an open ball $B(0, r) \subset \mathbb{R}^d$ containing the supports of all $(\widetilde{\varphi}_{\widetilde{x}})$ with $\widetilde{x} \in \widetilde{K} + \overline{B}(0, h)$. Let $\ell \geq 1$ be a (Lipschitz) constant satisfying

$$\begin{aligned} &\widetilde{x}_1 \in \widetilde{K} + \overline{B}(0, h),\ \|\widetilde{x}_1 - \widetilde{x}_2\| \leq h \Rightarrow \|\mu(\widetilde{x}_1) - \mu(\widetilde{x}_2)\| \leq \ell\|\widetilde{x}_1 - \widetilde{x}_2\|, \\ &x_1 \in K + \overline{B}(0, h),\ \|x_1 - x_2\| \leq h \Rightarrow \|\mu^{-1}(x_1) - \mu^{-1}(x_2)\| \leq \ell\|x_1 - x_2\|. \end{aligned} \tag{44}$$

Let

$$\varepsilon \in \Big]0, \frac{h}{\ell^2 r}\Big]. \tag{45}$$

If $\|\xi\| < \ell r$, then $\|\xi\varepsilon\| \le \frac{h}{\ell} \le h$ and we see by (44) that the diffeomorphisms (43) are well defined for $x \in K + \overline{B}(0,h)$, and we have

$$\left\| \frac{\mu^{-1}(x+\xi\varepsilon) - \widetilde{x}}{\varepsilon} \right\| \le \ell \, \|\xi\| \qquad (\, \widetilde{x} = \mu^{-1}(x)\,). \tag{46}$$

On the other hand, if $\|\widetilde{\xi}\| < r$, we see similarly that

$$\frac{\mu^{-1}(x+\xi\varepsilon) - \widetilde{x}}{\varepsilon} = \widetilde{\xi}$$

for some well defined ξ and we have $\|\xi\| \le \ell\|\widetilde{\xi}\| < \ell r$. Hence, the supports of φ_x are contained in $B(0,\ell r)$. For proving the proposition, it is sufficient to prove that each derivative

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial \xi^\beta} \frac{\mu^{-1}(x+\xi\varepsilon) - \mu^{-1}(x)}{\varepsilon}$$

remains bounded, when $x \in K + B(0,h)$, $\|\xi\| \le \ell r$ and ε fulfills (45). For $|\beta| \ge 1$ this is evident and for $\beta = 0$ this can be deduced in exactly the same way as (46) for $\alpha = 0$. ☐

Now we see that the definition above of the inverse image of a generalized function is correct and $\mu^* : \mathcal{G}[\Omega] \to \mathcal{G}[\widetilde{\Omega}]$ is an isomorphism of linear spaces commuting with restriction (sheaf morphism). It is immediate that the canonical embedding

$$\begin{aligned} \mathcal{D}'(\Omega) &\to \mathcal{G} \\ f \ &\mapsto \langle R \rangle \quad \text{ with } \ R(\varphi,x) = \langle f, \varphi \rangle \quad (\,\text{independent on } x\,) \end{aligned}$$

is a sheaf isomorphism commuting with μ^*. This allows us to define, on a $\mathcal{C}^\infty$ manifold M, a space of generalized functions containing distributions, as it is done in [5]: Let $(\mu_i, \Omega_i)_{i\in I}$ be an atlas on M, where $\mu_i : \Omega_i \to \Omega'_i$ is a diffeomorphism of an open set $\Omega_i \subset M$ onto an open set $\Omega'_i \subset \mathbb{R}^d$. Then a generalized function F on M is defined by a family $(F_i)_i$ of generalized functions $F_i \in \mathcal{G}(\Omega'_i)$ fulfilling the compatibility conditions

$$F_i = F_j \circ (\mu_j \circ \mu_i^{-1}) \quad \text{ on } \quad \Omega'_i \cap \mu_i \circ \mu_j^{-1} \Omega'_j \tag{47}$$

(provided the latter intersection is non-empty). Similarly, a distribution f on M can be defined by a family $(f_i)_i$ of distributions $f_i \in \mathcal{D}'(\Omega'_i)$ fulfilling the same compatibility conditions. Since the compositions with the diffeomorphisms commute with the canonical embedding, we get the canonical embedding of the space of the distributions into the space of the generalized functions on M, as we have it on $\Omega'_i \subset \mathbb{R}^d$.

26. The following theorem provides us a global definition of a generalized function on a $\mathcal{C}^\infty$ manifold (besides the local definition above). A manifold is always supposed paracompact.

Theorem. *Let M be a $\mathcal{C}^\infty$ manifold with an atlas $(\mu_i, \Omega_i)_{i\in I}$ and let a generalized function F on M be defined by a family $(F_i)_i$ satisfying* (47). *Then there is a complex valued function $\underline{R}$, defined on $\overset{d}{\underline{\mathcal{A}}}_0(M)\times M$, such that the functions*

$$\begin{aligned} \underline{R}'_i\ &:\ \overset{d}{\underline{\mathcal{A}}}_0(\Omega'_i)\times\Omega'_i \to \mathbb{C} \\ \underline{\varphi}',\quad x'\ &\mapsto R(\underline{\varphi}'\circ\mu_i,\ \mu_i^{-1}(x')) \end{aligned} \tag{48}$$

are representatives of the generalized functions F_i.

Before proving the theorem, we introduce some notions and prove some preliminary results.

Notation 27. We have still an atlas $(\mu_i, \Omega_i)_{i\in I}$, where $\mu_i : \Omega_i \to \Omega'_i$ is a diffeomorphism of an open set $\Omega_i \subset M$ onto an open set $\Omega'_i \subset \mathbb{R}^d$. If a complex valued function $\underline{R}$ is defined at least on $\overset{d}{\underline{\mathcal{A}}}_0(\Omega_i)\times\Omega_i$ (for some $i\in I$), then, similarly to the notation $\mu^*\underline{R}$ introduced in Notations 24, the function $\underline{R}'_i$, defined by (48) in the theorem, will be denoted by $\mu_i^{-1*}\underline{R}$. Similarly, if $\underline{R}'_i : \overset{d}{\underline{\mathcal{A}}}_0(\Omega'_i)\times\Omega'_i \to \mathbb{C}$ is given, then the function $\underline{R}_i : \overset{d}{\underline{\mathcal{A}}}_0(\Omega_i)\times\Omega_i \to \mathbb{C}$, defined by

$$\underline{R}_i(\underline{\varphi}, x) := \underline{R}'_i(\,\underline{\varphi}\circ\mu_i^{-1},\ \mu_i(x)) \qquad (\,\underline{\varphi}\in\overset{d}{\underline{\mathcal{A}}}_0(\Omega_i),\ x\in\Omega_i\,),$$

will be denoted by $\mu_i^*\underline{R}'_i$.

Each function $\underline{R}$ in Theorem 26 is called a representative of F. We write $F = \langle\underline{R}\rangle$. Evidently, this notion (as well as the other notions introduced below) does not depend on the chosen atlas on M. We denote by $\underline{\mathcal{E}}[M]$ the linear space of all complex valued $\mathcal{C}^\infty$ functions on $\overset{d}{\underline{\mathcal{A}}}_0(M)\times M$. We denote by $\underline{\mathcal{E}_M}[M]$ the linear space of all $\underline{R}\in\underline{\mathcal{E}}[M]$ such that, for all $i\in I$, $\mu_i^{-1*}\underline{R}\in\underline{\mathcal{E}_M}[\Omega'_i]$. If $\underline{R}\in\underline{\mathcal{E}_M}[M]$, then the family $(F_i)_i$, with $F_i = \langle\mu_i^{-1*}\underline{R}\rangle$, fulfills the compatibility condition (47). Thus, it defines a generalized function F on M such that $\underline{R}$ is a representative of F.

We denote by $\underline{\mathcal{N}}[M]$ the linear space of all $\underline{R}\in\underline{\mathcal{E}}[M]$ such that, for all $i\in I$, $\mu_i^{-1*}\underline{R}\in\underline{\mathcal{N}}[\Omega'_i]$. Evidently, $\underline{\mathcal{N}}[M]$ is an ideal in $\underline{\mathcal{E}_M}[M]$ and, when we have proved Theorem 26, we will see that the space of all generalized functions on M is equal to $\frac{\underline{\mathcal{E}_M}[M]}{\underline{\mathcal{N}}[M]}$.

28. We want to generalize Notation 20 replacing $\mathbb{R}^d$ with M. Note, however, that the spaces $\mathcal{A}_q$ cannot be defined independently of the coordinate system, except of $\overset{d}{\underline{\mathcal{A}}}_0$. So, we will accept only $q = 0$. The following is similar to Notation 20, Proposition 20 and Theorem 21.

Notation. Let Ω be an open set in M.

1°. Let $(U_x)_{x\in\Omega}$ be a family of open neighborhoods of points x in Ω, which are locally uniform in the following sense: for every coordinate chart (μ_i, Ω_i) and for every i, the family $\big(\mu_i(U_{\mu_i^{-1}(x')} \cap \Omega_i)\big)_{x'\in\Omega_i'}$ of open neighborhoods of points $x' \in \Omega_i' = \mu_i\Omega_i \subset \mathbb{R}^d$ is locally uniform. Under these hypotheses we denote by

$$\underline{\mathfrak{U}} = \underline{\mathfrak{U}}\big((U_x)_{x\in\Omega}\big) = \Big\{ (\underline{\varphi}, x);\ x \in \Omega,\ \underline{\varphi} \in \overset{d}{\underline{\mathcal{A}}}_0(U_x) \Big\}.$$

2°. Let $(V_j)_{j\in J}$ be an open covering of Ω, $V_j \subset \Omega$. Denote by

$$\underline{\mathfrak{V}} = \underline{\mathfrak{V}}\big((V_j)_{j\in J}\big) = \Big\{ (\underline{\varphi}, x);\ \exists\, j \in J \text{ such that } x \in V_j,\ \underline{\varphi} \in \overset{d}{\underline{\mathcal{A}}}_0(V_j) \Big\}.$$

Proposition. *For each set $\underline{\mathfrak{U}}$ according to 1° there is a set $\underline{\mathfrak{V}} \subset \underline{\mathfrak{U}}$ according to 2°. For each set $\underline{\mathfrak{V}}$ according to 2° there is a set $\underline{\mathfrak{U}} \subset \underline{\mathfrak{V}}$ according to 1°.*

This can be proved in the same way as Proposition 20: Taking a refining, we can suppose that every neighborhood U_x or V_j is a coordinate chart and we work, if needed, with the coordinate images.

Theorem. 1°. *Let $\underline{R}^\circ$ be a $\mathcal{C}^\infty$ function defined on a set $\underline{\mathfrak{V}}\big((V_j)_{j\in J}\big)$. Then there is a $\mathcal{C}^\infty$ function $\underline{R}$ on $\overset{d}{\underline{\mathcal{A}}}_0(M) \times \Omega$ coinciding with $\underline{R}^\circ$ on some set $\underline{\mathfrak{U}}\big((U_x)_{x\in\Omega}\big)$.*

2°. *Suppose in addition that, for every V_j, the function $\underline{R}^\circ$, restricted on $\overset{d}{\underline{\mathcal{A}}}_0(V_j) \times V_j$, belongs to $\underline{\mathcal{E}_M}[V_j]$. Then the function R, from the part 1° of the theorem, belongs to $\underline{\mathcal{E}_M}[\Omega]$.*

Proof: 1° can be proved in the same way as Theorem 21.1°, however the mapping π_i will be defined in a simpler way: we choose $\underline{\psi}_i \in \overset{d}{\underline{\mathcal{A}}}_0(V_i)$ and replace the formula (34) with

$$\pi_i(\underline{\varphi}) = \vartheta_i\underline{\varphi} - c_0\underline{\psi}_i, \qquad \pi_i(\underline{\varphi}) \in \overset{d}{\underline{\mathcal{A}}}_0(V_i). \tag{34'}$$

2° is evident. □

29.

Proof of Theorem 26: Without a loss of generality, we can suppose that $(\Omega_i)_{i\in I}$ is a locally finite open covering of M with relatively compact sets. Choose an open covering $(W_i)_{i\in I}$ with $\overline{W_i} \subset \Omega_i$, a decomposition of the unity $\sum \tau_i = 1$ on M with $\tau_i \in \mathcal{D}(W_i)$, and, for each $i \in I$, a representative $\underline{S}_i'$ of F_i on Ω_i'. Denote by $\underline{S}_i = \mu_i^*\underline{S}_i' \in \underline{\mathcal{E}_M}[\Omega_i]$, i.e.

$$\underline{S}_i(\underline{\varphi}, x) = \underline{S}_i'(\underline{\varphi}\circ\mu_i^{-1},\ \mu_i(x)) \qquad (\, x \in \Omega_i,\ \underline{\varphi} \in \overset{d}{\underline{\mathcal{A}}}_0(\Omega_i)\,).$$

Define

$$\underline{S}(\underline{\varphi}, x) := \sum_{i \in I} \tau_i(x) \underline{S}_i(\underline{\varphi}, x)$$

for

$$(\underline{\varphi}, x) \in \underline{\mathfrak{U}}\big((U_x)_{x \in M}\big) \qquad \text{with} \qquad U_x := \bigcap_{x \in \overline{W_i}} \Omega_i.$$

By Proposition 28 and Theorem 28, there is a function $\underline{R} \in \underline{\mathcal{E}_M}[M]$ which coincides with S on some set $\underline{\mathfrak{V}}\big((V_j)_{j \in J}\big) \subset \underline{\mathfrak{U}}\big((U_x)_{x \in M}\big)$. The latter inclusion means:

$$\text{(49)} \qquad \text{If} \qquad V_j \cap \overline{W_i} \neq \emptyset, \qquad \text{then} \qquad V_j \subset \Omega_i.$$

Having chosen an index $i_0 \in I$, we want to prove that $\underline{R}^\circ := \mu_{i_0}^{-1*} \underline{R} - \underline{S}'_{i_0} \in \mathcal{N}[\Omega'_{i_0}]$. We will use Proposition 19.2° ($\mathcal{N}$ as a local property) for this aim, proving that $\underline{R}^\circ \in \mathcal{N}[V']$ for each open set V' relatively compact in some set $\mu_{i_0}(V_j \cap \Omega_{i_0})$ $(j \in J)$. For $x \in V_j$ and $\underline{\varphi} \in \overset{d}{\underline{\mathcal{A}}}_0(V_j)$, we have $\underline{R}(\underline{\varphi}, x) = \underline{S}(\underline{\varphi}, x)$, and so for $x' \in V'$ and $\underline{\varphi}' \in \overset{d}{\underline{\mathcal{A}}}_0(V')$, we have

$$\underline{R}^\circ(\underline{\varphi}', x') = \sum_{i \in I} \tau_i(\mu_{i_0}^{-1}(x'))\big(\mu_{i_0}^{-1*} \underline{S}_i(\underline{\varphi}', x') - \underline{S}'_{i_0}(\underline{\varphi}', x')\big),$$

where the sum is locally finite (finite on $\overline{V'}$). If $\tau_i(\mu_{i_0}^{-1}(x')) \neq 0$ at a point $x' \in \overline{V'}$, then by (49), we have $V_j \subset \Omega_i$. Thus $\underline{S}_i = \mu_i^* \underline{S}'_i$ on $\overset{d}{\underline{\mathcal{A}}}_0(V_j) \times V_j$ and

$$\underline{R}^\circ(\underline{\varphi}', x') = \sum_{i \in I} \tau_i(\mu_{i_0}^{-1}(x'))\big((\mu_i \circ \mu_{i_0}^{-1})^* \underline{S}'_i(\underline{\varphi}', x') - \underline{S}'_{i_0}(\underline{\varphi}', x')\big).$$

This belongs to $\mathcal{N}[V']$ by the compatibility conditions (47). ☐

Acknowledgments. The author is very much indebted to Prof. J.F. Colombeau for a collaboration, which gave the main ideas for this paper.

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(*Received* April 22, 1998)