Subgroups of $\mathbb{R}$-factorizable groups

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Abstract. The properties of $\mathbb{R}$-factorizable groups and their subgroups are studied. We show that a locally compact group G is $\mathbb{R}$-factorizable if and only if G is σ-compact. It is proved that a subgroup H of an $\mathbb{R}$-factorizable group G is $\mathbb{R}$-factorizable if and only if H is z-embedded in G. Therefore, a subgroup of an $\mathbb{R}$-factorizable group need not be $\mathbb{R}$-factorizable, and we present a method for constructing non-$\mathbb{R}$-factorizable dense subgroups of a special class of $\mathbb{R}$-factorizable groups. Finally, we construct a closed G_δ-subgroup of an $\mathbb{R}$-factorizable group which is not $\mathbb{R}$-factorizable.

Keywords: $\mathbb{R}$-factorizable group, z-embedded set, $\aleph_0$-bounded group, P-group, Lindelöf group

Classification: Primary 54H11, 22A05; Secondary 22D05, 54C50

1. Introduction

A topological group G is called $\mathbb{R}$-*factorizable* ([7], [8]) if for every continuous function $g: G \to \mathbb{R}$ there exist a continuous homomorphism $\pi: G \to H$ of G onto a second-countable topological group H and a continuous function $h: H \to \mathbb{R}$ such that $g = h \circ \pi$. The reals $\mathbb{R}$ in this definition can be substituted by any second countable regular space X, thus giving us a possibility to factorize continuous functions $f: G \to X$ via continuous homomorphism onto second countable topological groups ([8]). The class of $\mathbb{R}$-factorizable groups is sufficiently wide; it contains all totally bounded groups, σ-compact groups (or, more generally, Lindelöf groups) and arbitrary subgroups of Lindelöf Σ-groups ([7], [8]). It is known, however, that subgroups of $\mathbb{R}$-factorizable groups do not inherit this property ([7, Example 2]).

In fact, some results on topological groups proved before 1990 can now be reformulated in terms of $\mathbb{R}$-factorizability. For example, the theorem proved on pages 118–119 of [6] is equivalent to say that every compact topological group is $\mathbb{R}$-factorizable. Theorem 1.2 of [2] implies, in particular, that every pseudocompact topological group is $\mathbb{R}$-factorizable. Note that every pseudocompact group is totally bounded ([2, Theorem 11]).

Our aim is to study $\mathbb{R}$-factorizable groups and their subgroups. We show first that a locally compact group is $\mathbb{R}$-factorizable if and only if it is σ-compact (Theorem 2.3). Then we characterize the subgroups of $\mathbb{R}$-factorizable groups which

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inherit this property: a subgroup H of an $\mathbb{R}$-factorizable group G is $\mathbb{R}$-factorizable if and only if H is z-embedded in G (Theorem 2.4). A slight modification of a construction in [7] gives us a lot of dense subgroups of $\mathbb{R}$-factorizable groups which are not $\mathbb{R}$-factorizable (see Theorem 3.1). We also construct a closed G_δ-subgroup of an Abelian $\mathbb{R}$-factorizable group which is not $\mathbb{R}$-factorizable (Example 3.2).

Finally, we consider a formally weaker notion of a semi-$\mathbb{R}$-factorizable group and show that every semi-$\mathbb{R}$-factorizable group is $\mathbb{R}$-factorizable.

2. z-embedded subgroups of topological groups

The notion of an $\aleph_0$-*bounded* topological group introduced by Guran ([3]) plays an important rôle in our considerations.

Definition 2.1. A topological group G is said to be $\aleph_0$-*bounded* if for each neighborhood U of the identity, there exists a countable subset $M \subseteq G$ such that $G = M \cdot U$.

It is known ([3]) that a topological group G is $\aleph_0$-bounded if and only if it embeds into a cartesian product of second countable topological groups as a topological subgroup. Although the following result was mentioned in [8], its proof was only sketched there.

Lemma 2.2. *Every $\mathbb{R}$-factorizable group is $\aleph_0$-bounded.*

PROOF: Let G be an $\mathbb{R}$-factorizable group. It suffices to show that G can be embedded as a topological subgroup into a product of second countable groups. Let $\mathcal{N}(e)$ be a neighborhood base at the identity e of G. For every neighborhood $U \in \mathcal{N}(e)$, let $f_U\colon G \to \mathbb{R}$ be a continuous function such that $f(e) = 1$ and $f(G \setminus U) = \{0\}$. Since G is $\mathbb{R}$-factorizable, there exist a second countable group H_U, a continuous homomorphism $\pi_U\colon G \to H_U$ and a continuous function $h\colon H_U \to \mathbb{R}$ such that $f = h \circ \pi_U$. Observe that the diagonal product $\varphi = \Delta\{\pi_U : U \in \mathcal{N}(e)\}$ is a topological monomorphism of G to the group $\Pi = \prod\{H_U : U \in \mathcal{N}(e)\}$.

Since second countable groups H_U are $\aleph_0$-bounded, the group Π is $\aleph_0$-bounded as well. Now, subgroups of $\aleph_0$-bounded groups are $\aleph_0$-bounded, so G inherits this property. □

Theorem 2.3. *A locally compact $\mathbb{R}$-factorizable group is σ-compact.*

PROOF: Suppose that G is a locally compact $\mathbb{R}$-factorizable group. Then there exists a neighborhood U of the identity of G such that $\overline{U}$ is compact. Since every $\mathbb{R}$-factorizable group is $\aleph_0$-bounded (Lemma 2.2), there is a countable subset $C \subseteq G$ such that $C \cdot U = G$. Therefore, $\{g \cdot \overline{U} : g \in C\}$ is a countable family of compact sets whose union is G. □

Tkačenko [7] showed that subgroups of $\mathbb{R}$-factorizable groups are not necessarily $\mathbb{R}$-factorizable. On the other hand, an $\mathbb{R}$-factorizable subgroup of an arbitrary topological group G is z-embedded in G ([4]). In the following theorem we give a complete characterization of subgroups of ℝ-factorizable groups which preserve the property of ℝ-factorizability. Let X be a topological space and let be $A \subseteq X$. We say that A is z-embedded in X if every cozero set B in A is of the form $B = A \cap C$, where C is a cozero set in X.

Theorem 2.4. *A subgroup H of an ℝ-factorizable group G is ℝ-factorizable if and only if H is z-embedded in G.*

Proof: We shall only give the proof of the fact that z-embedding is a sufficient condition for the subgroup H to be ℝ-factorizable because the proof of necessity appears as Theorem 3.1 of [4]. Let $f\colon H \to \mathbb{R}$ be a continuous function. Consider the family γ of all open intervals in $\mathbb{R}$ with rational end points. For every $U \in \gamma$, let V_U be a cozero set in G such that $V_U \cap H = f^{-1}(U)$. There exists a continuous function $g_U\colon G \to \mathbb{R}$ such that $g_U^{-1}(U) = V_U$. The diagonal product $g = \Delta_{U\in\gamma} g_U$ is a continuous mapping of G to the second countable space $\mathbb{R}^\gamma$ and, by ℝ-factorizability of G, there exist a continuous homomorphism π of G onto a second countable topological group G^* and a continuous function $g^*\colon G^* \to \mathbb{R}^\gamma$ such that $g = g^* \circ \pi$.

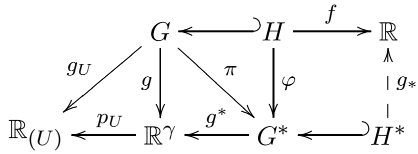

Diagram 1

We claim that for any x_0, $x_1 \in H$, $f(x_0) = f(x_1)$ whenever $\pi(x_0) = \pi(x_1)$. Assume the contrary, let $f(x_0) \neq f(x_1)$ for some x_0, $x_1 \in H$ with $\pi(x_0) = \pi(x_1)$. We can also assume that $f(x_0) < f(x_1)$. If r_0, r_1 and r_2 are rationals and $r_0 < f(x_0) < r_1 < f(x_1) < r_2$, consider the intervals $U_0 = (r_0, r_1) \in \gamma$ and $U_1 = (r_1, r_2) \in \gamma$. Let $p_{U_i}\colon \mathbb{R}^\gamma \to \mathbb{R} = \mathbb{R}_{U_i}$ be the natural projections, $g \circ p_{U_i} = g_{U_i}$ $(i = 0, 1)$. On the one hand, the sets $g_{U_0}^{-1}(U_0) \cap H = f^{-1}(U_0)$ and $g_{U_1}^{-1}(U_1) \cap H = f^{-1}(U_1)$ are disjoint. This is equivalent to say that $g^{-1}(O_0) \cap H$ and $g^{-1}(O_1) \cap H$ are disjoint, where $O_i = p_{U_i}^{-1}(U_i) \ni g(x_i)$ $(i = 0, 1)$. In particular, $g(x_0) \neq g(x_1)$. On the other hand, $g = g^* \circ \pi$, whence $g(x_0) = g(x_1)$, a contradiction.

Put $H^* = \pi(H)$. The assertion just proved implies that there exists a function $g_*\colon H^* \to \mathbb{R}$ such that $f = g_* \circ \pi \restriction_H$. It remains to verify that g_* is continuous. Let $U \in \gamma$ be arbitrary. Then

$$g_*^{-1}(U) = \pi\big(f^{-1}(U)\big) = \pi\big(g_U^{-1}(U) \cap H\big) = (g^*)^{-1}\big(p_U^{-1}(U)\big) \cap \pi(H)$$

is open in $\pi(H) = H^*$. Since γ is a base for $\mathbb{R}$, this proves the continuity of g_*. Thus, we have $f = g_* \circ \varphi$, where $\varphi = \pi \restriction_H$ is a continuous homomorphism of H onto the second countable group $H^* \subseteq G^*$, and hence H is ℝ-factorizable. □

It is clear that every retract of a space X is z-embedded in X. Indeed, if $r: X \to X$ is a retraction and $Y = r(X)$, then for each continuous function $f: Y \to \mathbb{R}$, the function $\hat{f} = f \circ r$ is a continuous extension of f to X. Note also that if G is a topological group and H is an open subgroup of G, then H is a retract of G. Indeed, in every left coset U of H in G, pick a point $x_U \in U$. Define $r: G \to H$ in the following way: if $g \in H$, then $f(g) = g$; if $g \in U$ and $U \neq H$, put $r(g) = x_U^{-1} g$. Since the left cosets are open and disjoint, the continuity of r is immediate. From these two observations we deduce the following results.

Corollary 2.5. *Let G be an $\mathbb{R}$-factorizable group and H a subgroup of G. If H is a retract of G, then H is $\mathbb{R}$-factorizable.*

Corollary 2.6. *An open subgroup of an $\mathbb{R}$-factorizable group is $\mathbb{R}$-factorizable.*

3. Some examples

By Corollary 1.13 of [8], every Lindelöf topological group is $\mathbb{R}$-factorizable. Let us call a topological group G a *P-group* if any intersection of countably many open sets in G is open. Making use of the existence of a special Lindelöf P-group $\widehat{G}$ of weight $\aleph_1$ (see [1]), Tkačenko [7] constructed an example of a proper dense subgroup of $\widehat{G}$ which was not $\mathbb{R}$-factorizable. Our aim is to show that *any* proper dense subgroup of an arbitrary Lindelöf P-group of weight $\aleph_1$ is not $\mathbb{R}$-factorizable.

Theorem 3.1. *If H is a proper dense subgroup of a Lindelöf P-group G of weight $\aleph_1$, then H is not $\mathbb{R}$-factorizable.*

Proof: Since G is a P-group, it is zero-dimensional. Therefore, we choose a base $\mathcal{B} = \{O_\alpha : \alpha < \omega_1\}$ at the identity e of G satisfying the following conditions for each $\alpha < \omega_1$:

(1) O_α is a clopen set;
(2) $O_\alpha = \bigcap_{\beta<\alpha} O_\beta$ for any limit ordinal $\alpha < \omega_1$;
(4) $O_{\alpha+1}^2 \subset O_\alpha$;
(3) $O_\alpha \setminus O_{\alpha+1} = A_\alpha \cup B_\alpha$ where A_α and B_α are nonempty disjoint clopen sets.

Now define U' and V' by $U' = (G \setminus O_0) \cup (\bigcup_{\alpha<\omega_1} A_\alpha)$ and $V' = \bigcup_{\alpha<\omega_1} B_\alpha$. From conditions (1) and (4) it follows that U' and V' are open sets. Conditions (2) and (4) imply that $U' \cup V' = G \setminus \{e\}$. Finally, (3) guarantees that U' and V' are nonempty.

Pick a point $g \in G \setminus H$ and define $U = gU' \cap H$ and $V = gV' \cap H$. Then U and V are non-empty open subsets of H and $H = U \cup V$. Let f be the function on H defined by the rule $f(x) = 0$ if $x \in U \cap H$ and $f(x) = 1$ if $x \in V \cap H$. It is easy to see that f is continuous. Let $\pi: H \to K$ be a continuous homomorphism of H to a metrizable group K. Then the kernel of π is a G_δ-set in H, and hence is an open neighborhood of e. So, we can find $\alpha < \omega_1$ such that $O_\alpha \cap H \subseteq \ker \pi$. Pick points $a \in H \cap gA_{\alpha+1}$ and $b \in H \cap gB_{\alpha+1}$. Then $ab^{-1} \in O_\alpha$ by (3) and

(4), which in turn implies that $\pi(a) = \pi(b)$, whereas $f(a) = 0$ and $f(b) = 1$. This means that the group H is not $\mathbb{R}$-factorizable. □

The above theorem shows that there are many subgroups of $\mathbb{R}$-factorizable groups which are not $\mathbb{R}$-factorizable. In special classes of $\mathbb{R}$-factorizable groups the situation changes: by Corollary 1.13 of [8], every subgroup of a σ-compact topological group is $\mathbb{R}$-factorizable. Intuitively, G_δ-subgroups of a topological group seem close to be z-embedded in it. Thus, Theorem 2.4 might suggest the conjecture that a closed G_δ-subgroup of an $\mathbb{R}$-factorizable group is $\mathbb{R}$-factorizable as well. We show below that this is not the case.

Example 3.2. Let H be an $\aleph_0$-bounded Abelian group of weight $\aleph_1$ which is not $\mathbb{R}$-factorizable ([7, Example 2.1]). By a theorem of Guran [3], H can be considered as a subgroup of a product $\Pi = \prod_{\alpha<\omega_1} G_\alpha$, where each G_α is a second countable Abelian group. Let $G = \Pi^\omega$. The subgroup H' of G that consists of all elements of the form $(h, h, \dots)$ with $h \in H$ is isomorphic to H.

By the Hewitt–Marczewski–Pondiczery theorem there exists a countable dense subset S of Π. Consider the subset D of G of all elements $x \in G$ such that for a finite set of $n_1, \dots, n_k \in \omega$, $x(n_i) \in S$ and $x(n) = 0$ for other indices n. It is easy to see that the set D is countable and dense in G. Let $K = \langle D\rangle$ be the subgroup of G generated by D. Then K is a countable dense subgroup of G and $K \cap H' = \{e_G\}$. Since any dense subgroup of a product of second countable groups is $\mathbb{R}$-factorizable ([8, Corollary 1.10]), we conclude that the subgroup $L = K + H'$ of G is $\mathbb{R}$-factorizable. On the other hand, since the diagonal $\Delta = \{(x, x, \dots) : x \in G\}$ of the group $G = \Pi^\omega$ is closed in G and $H' \subseteq \Delta$, we have $\overline{H'} \subseteq \Delta$ and $\Delta \cap K = \{e_G\}$, whence $\overline{H'} \cap L = H'$. This means that H' is a closed subgroup of L. For each $x \in K$, $x + H'$ is a closed subset of L and it is easy to see that

$$H' = \bigcap_{x\in K\setminus\{e_G\}} L \setminus (x + H').$$

Hence, $H' \simeq H$ is a closed G_δ-subgroup of the $\mathbb{R}$-factorizable group $L = K + H'$, which is not $\mathbb{R}$-factorizable.

4. Semi-$\mathbb{R}$-factorizable groups

The fact that a topological group G is $\mathbb{R}$-factorizable can be expressed in the following form equivalent to the original one: given a continuous function $f\colon G \to \mathbb{R}$, there exist a closed normal subgroup H of G, a Hausdorff second countable group topology τ for the quotient group G/H coarser than the quotient topology τ_q and a continuous function $h\colon (G/H, \tau) \to \mathbb{R}$ such that $f = h \circ \pi$, where $\pi\colon G \to G/H$ is the quotient homomorphism.

The motivation of the definition below arises if one omits the condition of normality of the subgroup $H \subseteq G$. Thus, we define a class of topological groups containing $\mathbb{R}$-factorizable groups. We will see, however, that the two classes coincide (Theorem 4.3).

Let H be a closed subgroup of a topological group G and $G/H = \{xH : x \in G\}$ a left coset space with the quotient topology τ_q. A topology $\tau \subseteq \tau_q$ for G/H is called *left-invariant* if the functions $\phi_a: G/H \to G/H$ defined by $\phi_a(xH) = axH$, $x \in G$, are continuous for al $a \in G$. This notation will be used in the proofs of Lemma 4.2 and Theorem 4.3.

Definition 4.1. A topological group G is said to be *semi-$\mathbb{R}$-factorizable* provided that for every continuous function $f: G \to \mathbb{R}$ there exist a closed subgroup H of G, a second countable left-invariant T_1 topology τ on the left coset space G/H coarser than the quotient topology and a continuous function $h: (G/H, \tau) \to \mathbb{R}$ such that $f = h \circ \pi$, where $\pi: G \to G/H$ is the natural projection.

Lemma 4.2. *Every semi-$\mathbb{R}$-factorizable group is $\aleph_0$-bounded.*

Proof: Let G be a semi-$\mathbb{R}$-factorizable group and V an open neighborhood of the identity e in G. Since a topological group is completely regular, there exists a continuous function $f: G \to [0,1]$ such that $f(e) = 1$ and $f(G \setminus V) = \{0\}$. Since G is semi-$\mathbb{R}$-factorizable, there exist a closed subgroup H of G, a left-invariant second countable T_1 topology τ on G/H and a continuous function $h: (G/H, \tau) \to \mathbb{R}$ such that $f = h \circ \pi$, where $\pi: G \to G/H$ is the natural projection. The set $U = h^{-1}(\frac{1}{2}, 1]$ is open in $(G/H, \tau)$ and $e \in \pi^{-1}(h^{-1}(\frac{1}{2}, 1]) = f^{-1}(\frac{1}{2}, 1] \subseteq V$. For each $g \in G$, the function $\sigma_g: G \to G$ defined by $\sigma_g(x) = gx$ is a homeomorphism of G onto G. Note that $\pi \circ \sigma_g = \phi_g \circ \pi$ and, therefore, $f \circ \sigma_g = h \circ \pi \circ \phi_g = h \circ \phi_g \circ \pi$. Since

$$(f \circ \sigma_{x^{-1}})^{-1}(\tfrac{1}{2}, 1] = \sigma_{x^{-1}}^{-1}(f^{-1}(\tfrac{1}{2}, 1]) = \sigma_x(f^{-1}(\tfrac{1}{2}, 1]) \subseteq \sigma_x(V) = xV,$$

we conclude that $U_x = \phi_{x^{-1}}^{-1}(h^{-1}(\frac{1}{2}, 1])$ is open in $(G/H, \tau)$ and $\pi^{-1}(U_x) \subseteq xV$. The collection $\{U_x : x \in G\}$ covers G/H. Since G/H has countable weight, there exists a sequence $x_0, x_1, \ldots$ of elements of G such that $G/H \subseteq \bigcup_{i=0}^{\infty} U_{x_i}$. Consequently, the family $\{\pi^{-1}(U_{x_i}) : i \in \omega\}$ covers G and, therefore, the corresponding family $\{x_i V : i \in \omega\}$ also covers G. This proves that G is $\aleph_0$-bounded. □

Theorem 4.3. *Every semi-$\mathbb{R}$-factorizable group is $\mathbb{R}$-factorizable.*

Proof: Let G be a semi-$\mathbb{R}$-factorizable group and $f: G \to \mathbb{R}$ a continuous function. Then G has a closed subgroup H such that there exist a left-invariant second countable T_1 topology τ on G/H and a continuous function $h: (G/H, \tau) \to \mathbb{R}$ such that $f = h \circ \pi$, where $\pi: G \to G/H$ is the natural projection. If $\{W_i : i \in \omega\}$ is a local base of G/H at $\{H\}$, then $H = \bigcap_{i \in \omega} \pi^{-1}(W_i)$. Since G is $\aleph_0$-bounded (Lemma 4.2), for every $U_i = \pi^{-1}(W_i)$ there exist a continuous homomorphism $\pi_i: G \to H_i$ of G onto a second countable group H_i and a neighborhood V_i of the identity in H_i such that $\pi_i^{-1}(V_i) \subseteq U_i$ (see [3]). Then $N = \bigcap_{i \in \omega} \ker \pi_i$ is a closed normal subgroup of G and $N \subseteq H$. First, we define a second countable group topology t for G/N. Let $\varphi_i: G/N \to H_i$ be the homomorphism defined by $\varphi_i(aN) = \pi_i(a)$, $a \in G$. Note that φ_i is well-defined because if $b \in aN$ then $a^{-1}b \in N \subseteq \ker \pi_i$, and hence $\pi_i(a) = \pi_i(b)$. Let t be the weakest group topology on G/N that makes each of the homomorphisms φ_i continuous. It is clear that $(G/N, t)$ is a topological group because the topology t is generated by a family of homomorphisms, and t is second countable because each group H_i is second countable. We define the function $\tilde{h}: G/N \to \mathbb{R}$ by $\tilde{h}(aN) = h(aH)$, i.e., $\tilde{h} = h \circ \psi$, where $\psi: G/N \to G/H$ is given by $\psi(aN) = aH$. It is easy to see that ψ is well-defined because the left cosets of N in G are contained in the left cosets of H in G. Let π_N be the natural projection of G onto G/N. Then $\tilde{h} \circ \pi_N = h \circ \psi \circ \pi_N = h \circ \pi = f$ (see Diagram 2 below).

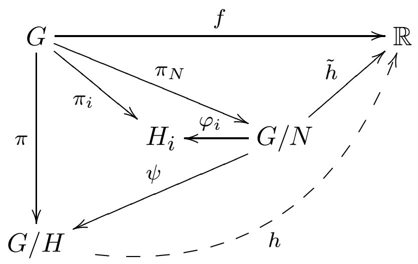

Diagram 2

Finally, we have to prove that the function $\tilde{h}$ is continuous. To this end, it suffices to show that ψ is continuous, that is, for each $A \in G/N$ and each open set $V \in \tau$ containing $\psi(A)$, there exists $U \in t$ with $A \in U$ such that $\psi(U) \subseteq V$. Since $A = gN$ for some $g \in G$, it follows from the definition of ψ that $\psi(A) = gH$. Since the topology τ on G/H is left-invariant, the set V has the form $\phi_g(V')$, where $H \in V' \in \tau$. There exists $i \in \omega$ such that $W_i \subseteq V'$. Recall that $\pi_i^{-1}(V_i) \subseteq U_i = \pi^{-1}(W_i)$ by the choice of the neighborhood V_i of the identity in H_i. Define $O = \varphi_i^{-1}(V_i)$ and $U = a \cdot O$, where $a = \pi_N(g)$. Then $A \in U \in t$ and

$$\begin{aligned}\psi(U) = \psi(a \cdot O) = \pi(g \cdot \pi_i^{-1}(V_i)) &= \phi_g(\pi(\pi_i(V_i))) \\ \subseteq \phi_g(\pi(U_i)) \subseteq \phi_g(\pi\pi^{-1}(W_i)) &= \phi_g(W_i) \subseteq \phi_g(V') = V.\end{aligned}$$

This implies the continuity of ψ, and hence the function $\tilde{h} = h \circ \psi$ is continuous as well. □

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