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Productivity of the Zariski topology on groups

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. This paper investigates the productivity of the Zariski topology $\mathfrak{Z}_G$ of a group G. If $\mathcal{G} = \{G_i \mid i \in I\}$ is a family of groups, and $G = \prod_{i\in I} G_i$ is their direct product, we prove that $\mathfrak{Z}_G \subseteq \prod_{i\in I} \mathfrak{Z}_{G_i}$. This inclusion can be proper in general, and we describe the doubletons $\mathcal{G} = \{G_1, G_2\}$ of abelian groups, for which the converse inclusion holds as well, i.e., $\mathfrak{Z}_G = \mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2}$.

If $e_2 \in G_2$ is the identity element of a group G_2, we also describe the class Δ of groups G_2 such that $G_1 \times \{e_2\}$ is an elementary algebraic subset of $G_1 \times G_2$ for every group G_1. We show among others, that Δ is stable under taking finite products and arbitrary powers and we describe the direct products that belong to Δ. In particular, Δ contains arbitrary direct products of free non-abelian groups.

Keywords: Zariski topology, (elementary, additively) algebraic subset, δ-word, universal word, verbal function, (semi) $\mathfrak{Z}$-productive pair of groups, direct product

Classification: Primary 20F70, 20K45; Secondary 20K25, 57M07

1. Introduction

1.1 Algebraic subsets of a group and the Zariski topology. Let G be a group. A self-map $G \to G$ of the form $g \mapsto g_1 g^{\varepsilon_1} g_2 g^{\varepsilon_2} \cdots g_n g^{\varepsilon_n} g_0$, where $n \in \mathbb{N}$, $g_0, g_1, \ldots, g_n \in G$, $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and $g \in G$, will be called a *verbal function* of G. Since these functions play a pivotal role in the paper, we give also a more formal definition as follows.

Taking x as a symbol for a variable, we denote by $G[x] = G * \langle x\rangle$ the free product of G and the infinite cyclic group $\langle x\rangle$ generated by x. A non-trivial element $w \in G[x]$ is given by

$$w(x) = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} g_0, \tag{1}$$

where $n \in \mathbb{N}$ and $g_0, g_1, \ldots, g_n \in G$, $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$. For simplicity, we write only w, when this leads to no misunderstanding. We call $G[x]$ *the group of words with coefficients in* G and its elements w are called *words in* G. We denote by $e_{G[x]}$ the neutral element (the trivial word) of $G[x]$.

In these terms, every word $w \in G[x]$ determines a verbal function of G, namely the associated evaluation function $f_w \colon G \to G$, mapping $g \mapsto w(g)$, where $w(g) \in$

G is obtained replacing x with g in (1) and taking products (and eventually inversions) in G (see [15] for more details on verbal functions).

Definition 1.1. If $w \in G[x]$, we let

$$E_w^G = f_w^{-1}(\{e_G\}) = \{g \in G \mid f_w(g) = e_G\} \subseteq G,$$

we call E_w^G *elementary algebraic subset of* G, and we will denote it simply by E_w when no confusion is possible.

We denote by $\mathbb{E}_G = \{E_w \mid w \in G[x]\} \subseteq \mathcal{P}(G)$ the family of elementary algebraic subsets of G, and by $\mathbb{E}_G^{\cup}$ the family of finite unions of elements of $\mathbb{E}_G$.

If $X \subseteq G$, we call X:

(a) *additively algebraic* if X is a finite union of elementary algebraic subsets of G, i.e. if $X \in \mathbb{E}_G^{\cup}$;
(b) *algebraic* if X is an intersection of additively algebraic subsets of G.

Obviously, every singleton is an elementary algebraic subset, so every finite subset is additively algebraic. Then the family of algebraic subsets is closed under finite unions and arbitrary intersections, and contains G and all finite subsets of G. So it can be taken as the family of closed sets of a unique T_1 topology $\mathfrak{Z}_G$ on G, called the *Zariski topology* ([5], [6], [7], [8], [9], [2], [15]).

While the definition of elementary algebraic, additively algebraic and algebraic subset goes back to Markov [11], he did not explicitly introduce the Zariski topology, although it was implicitly present in [11], [12], [13] (through the *algebraic closure* of a subset X, i.e., the smallest algebraic subset of the group G that contains X). It was explicitly introduced only in 1977 by Bryant [3] under the name *verbal topology*. Here we keep the name Zariski topology and the notation $\mathfrak{Z}_G$ for this topology.

The Zariski topology of the abelian groups was described and thoroughly studied in the abelian case in [7] (we recall some of the most relevant facts in the abelian case in §1.4). Here we provide examples in the non-abelian case.

Example 1.2. (1) If $g \in G$, its *centralizer* in G is the subgroup

$$C_G(g) = \{h \in G \mid gh = hg\}$$

consisting of the elements of G that commute with g. Then $C_G(g) = E_w$, where $w = gxg^{-1}x^{-1} \in G[x]$. Hence $C_G(g) \in \mathbb{E}_G$.

If $S \subseteq G$, the centralizer of S is the intersection $C_G(S) = \bigcap_{s\in S} C_G(s)$, consisting of the elements of G that commute with *every* element of S. Therefore, $C_G(S)$ is an algebraic subset of G.

In particular, the *center* $Z(G) = C_G(G)$ of G is an algebraic subset. We call *center-free* a group G such that $Z(G) = \{e_G\}$.

(2) For every $n \in \mathbb{Z}$, let

$$G[n] = \{g \in G \mid g^n = e_G\} \subseteq G.$$

For example, $G[1] = \{e_G\}$ and $G[0] = G$.

The word $x^n \in G[x]$ determines the verbal function $f_{x^n} \colon g \mapsto g^n$, and obviously $G[n] = E_{x^n}$.

If G is abelian, every $G[n]$ is a subgroup of G, and these (together with their cosets, of course) are all the non-empty elementary algebraic subsets of G (see (3) and §1.4).

(3) Let $n \in \mathbb{N}$. Here we shall provide some easy examples of cases when the elementary algebraic subset $E_{x^n} = G[n]$ is not a coset of a subgroup, by imposing that the subgroup generated by $G[n] \neq G$ is the whole group G (as $e_G \in G[n] \neq G$). To this end, it suffices to consider a *simple* group G with $\{e_G\} \neq G[n]$. Indeed, the subset $G[n]$ is invariant under conjugations, so the subgroup N generated by $G[n]$ is normal in G, and we conclude $N = G$.

To get an easy example to this effect take a non-abelian finite simple group G. Then $|G|$ is even (e.g., by Feit-Thompson theorem), so that $\{e_G\} \neq G[2] \neq G$.

As another example, let G be a compact, connected, simple Lie group (for example, the group $G = \mathrm{SO}_3(\mathbb{R})$ will do). Then G is covered by copies of the torus $\mathbb{R}/\mathbb{Z}$ (see for example [1]), so that $\{e_G\} \neq G[n] \neq G$ for every $n > 1$.

(4) By item 2, we have that $G[2] = E_{x^2}$. Here we slightly generalize this example studying E_w for a word $w = g_1 x g_2 x$ (note that $w = x^2$ when $g_1 = g_2 = e_G$).

Then $w = a^{-1}(g_2 x)^2$, for $a = g_2 g_1^{-1}$, so that

$$E_w = \{g \in G \mid (g_2 g)^2 = a\} = \{g_2^{-1} h \in G \mid h^2 = a\} = g_2^{-1}\{g \in G \mid g^2 = a\}$$

is a translate of the 'square roots' of the element $a \in G$.

If $E_w \neq \emptyset$, i.e. if $a = b^2$ for some $b \in G$, then $g_2^{-1}(C_G(b)[2])b \subseteq E_w$.

1.2 Preliminaries. We denote by $\mathbb{Z}$ the group of integers, by $\mathbb{N}_+$ the set of positive integers, by $\mathbb{N}$ the set of naturals, and by $\mathbb{P}$ the set of prime numbers.

Given two elements g, h of a group G, their commutator element is $[g, h] = ghg^{-1}h^{-1} \in G$. Note that $[g, h] = e_G$ if and only if $gh = hg$, i.e. g and h commute.

A *torsion group* is a group in which each element has finite order. All finite groups are torsion.

The *exponent* $\exp(G)$ of a torsion group G is the least common multiple, if it exists, of the orders of the elements of G. In this case, the group is called *bounded*, and $\exp(G) > 0$. Otherwise, or if G is not even torsion, it will be called *unbounded*, and we conventionally define $\exp(G) = 0$. Any finite group has positive exponent: it is a divisor of $|G|$.

Definition 1.3. Let $w \in G[x]$ be as in (1).

If $g_i \neq e_G$ whenever $\varepsilon_{i-1} = -\varepsilon_i$ for $i = 2, \dots, n$, we say that w is a *reduced word* in the free product $G[x] = G * \langle x \rangle$ and we define the *lenght* of w by $\mathrm{l}(w) = n$, where $n \in \mathbb{N}$ is the least natural number such that w is as in (1).

We call *constant* a word w with $\mathrm{l}(w) = 0$, i.e. a word of the form $w = g_0 \in G$.

The proof of the following standard fact can be found in [15] and will be used in Lemma 3.1.

Proposition 1.4 ([15]). *Let $\phi\colon G_1 \to G_2$ be a group homomorphism. Then there exists a unique group homomorphism $F\colon G_1[x] \to G_2[x]$ such that $F \restriction_{G_1} = \phi$, $F(x) = x$.*

The map $F\colon G_1[x] \to G_2[x]$ can be explicitly described as the assignment

$$G_1[x] \ni g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} g_0 \mapsto \phi(g_1) x^{\varepsilon_1} \phi(g_2) x^{\varepsilon_2} \cdots \phi(g_n) x^{\varepsilon_n} \phi(g_0) \in G_2[x].$$

Remark 1.5. If $w \in G[x]$ and $g \in G$, then also $w' = gwg^{-1} \in G[x]$, and $E_w = E_{w'}$. As a consequence, if w is a non-constant word as in (1), *we will assume $g_0 = e_G$.*

We also introduce the following notions.

- The *constant term of* w is $\mathrm{ct}(w) = w(e_G) = g_1 g_2 \cdots g_n \in G$.
- The *content* of w is $\epsilon(w) = \sum_{j=1}^{n} \varepsilon_j \in \mathbb{Z}$, which will also be denoted simply by ϵ when no confusion is possible.

If $w = g \in G$, then $\mathrm{ct}(w) = w(e_G) = g$, and we define $\epsilon(w) = 0$. We call *singular* a word w such that $\epsilon(w) = 0$. By definition, all constant words are singular.

Definition 1.6. Let G be a group. A word $w \in G[x]$ is called *universal*, if $E_w = G$. We denote by $\mathcal{U}_G$ the normal subgroup of $G[x]$ consisting of the universal words of G.

Note that w is universal if and only if $f_w \equiv e_G$ is the constant function e_G on G.

1.3 The Zariski topology and subgroups. If H is a subgroup of a group G, then H carries its own Zariski topology $\mathfrak{Z}_H$, as well as the induced topology $\mathfrak{Z}_G \restriction_H$. If $w \in H[x]$, then one can consider w also in $G[x]$, so that both E_w^H and E_w^G make sense, and $E_w^H = E_w^G \cap H$. From this, one can deduce the inclusion $\mathfrak{Z}_H \subseteq \mathfrak{Z}_G \restriction_H$. To better describe the cases when the two topologies $\mathfrak{Z}_H$ and $\mathfrak{Z}_G \restriction_H$ on H coincide, the following definition was given in [6].

Definition 1.7 ([6, Definition 2.1]). A subgroup H of a group G is called *Zariski embedded in G* if $\mathfrak{Z}_G \restriction_H = \mathfrak{Z}_H$.

Note that H is Zariski embedded in G if and only if $\mathfrak{Z}_G \restriction_H \subseteq \mathfrak{Z}_H$. This condition is also equivalent to ask $E_w^G \cap H$ to be an algebraic subset of H for every word $w \in G[x]$.

As a consequence of [6, Theorem 3.4] and [9, Proposition 2.7(c)] one can immediately obtain the following result we will use in Corollary 4.13. For the reader's convenience, we give a direct proof here.

Proposition 1.8. *Every central subgroup is Zariski embedded.*

Proof: Let G be a group, and $H \leq Z(G)$ be a subgroup of G. We will prove that $E_w^G \cap H \in \mathbb{E}_H$ for every word $w \in G[x]$.

Let $w \in G[x]$. Then $w(h) = \mathrm{ct}(w)h^{\epsilon(w)}$ as $H \leq Z(G)$, so that

$$E_w^G \cap H = \{x \in H \mid w(h) = e_G\} = \{x \in H \mid \mathrm{ct}(w)h^{\epsilon(w)} = e_G\}.$$

If $\mathrm{ct}(w) \in G \setminus H$, then $E_w^G \cap H = \emptyset$ and there is nothing to prove.

Otherwise, let $\mathrm{ct}(w) = h_0 \in H$. Then $w_0 = h_0 x^{\epsilon(w)} \in H[x]$, and the above equation shows that $E_w^G \cap H = E_{w_0}^H$. □

1.4 The Zariski topology on abelian groups. Here we resume some results from [7] on the Zariski topology of an abelian group.

Let $(G, +, 0_G)$ be an abelian group. Then the elementary algebraic subset $G[n] = \{g \in G \mid ng = 0_G\}$ is a subgroup of G, called the *n-socle of G*.

It can be easily verified that the family of verbal functions of G is $\{f_{g+nx} \mid g \in G, n \in \mathbb{Z}\}$. The elementary algebraic subset of G determined by f_{g+nx} is

$$E_{g+nx} = \begin{cases} \emptyset & \text{if } g + nx = 0_G \text{ has no solution in } G, \\ G[n] + x_0 & \text{if } x_0 \text{ is a solution of } g + nx = 0_G. \end{cases} \tag{2}$$

On the other hand, if $n \in \mathbb{Z}$, and $g \in G$, then $G[n] + g = E_{-ng+nx}$. So the non-empty elementary algebraic subsets of G are exactly the cosets of the n-socles of G:

$$\mathbb{E}_G \setminus \{\emptyset\} = \{G[n] + g \mid n \in \mathbb{N}, g \in G\}. \tag{3}$$

One can verify that $\mathbb{E}_G$ is stable under taking finite intersections, and satisfies the descending chain condition. Using this fact, the authors of [7] proved that $\mathbb{E}_G^{\cup}$ is the family of all the $\mathfrak{Z}_G$-closed subsets of an abelian group G. In other words, *every algebraic subset of G is additively algebraic.*

Theorem 1.9 ([7]). *If G is an abelian group, then the family of $\mathfrak{Z}_G$-closed sets is $\mathbb{E}_G^{\cup}$.*

Remark 1.10. It follows from (2) that if G is abelian, and $w \in G[x]$ is singular, then either $E_w = G$ or $E_w = \emptyset$.

The following result from [14] classifies the class of abelian groups that have a cofinite Zariski topology. Recall that G is said to be *almost torsion-free*, if $G[n]$ is finite for every $n \neq 0$.

Proposition 1.11 ([14, Theorem 5.1]). *Let G be an abelian group. Then $\mathfrak{Z}_G$ is the cofinite topology if and only if either G is almost torsion-free, or* $\exp(G) \in \mathbb{P}$.

Finally, every subgroup of an abelian group is Zariski embedded by Proposition 1.8.

1.5 Productivity of the Zariski topology. Consider the group $\mathbb{Z}$ of integers, and the product $G = \mathbb{Z} \times \mathbb{Z}$. Then the Zariski topology of G is the cofinite topology by Proposition 1.11, so neither $\mathbb{Z} \times \{0\}$ nor $\{0\} \times \mathbb{Z}$ are Zariski closed in G, whereas they are certainly closed in the product topology $\mathfrak{Z}_{\mathbb{Z}} \times \mathfrak{Z}_{\mathbb{Z}}$.

Moreover, as the topology $\mathfrak{Z}_{\mathbb{Z}} \times \mathfrak{Z}_{\mathbb{Z}}$ is T_1, it contains the cofinite topology $\mathfrak{Z}_G$, so that $\mathfrak{Z}_{\mathbb{Z}\times\mathbb{Z}} \subseteq \mathfrak{Z}_{\mathbb{Z}} \times \mathfrak{Z}_{\mathbb{Z}}$. We prove that this inequality holds in the general case (see the comments below).

If $\{G_i \mid i \in I\}$ is a non-empty family of groups, we denote by $e_i \in G_i$ the identity element of G_i. We consider the direct product $G = \prod_{i\in I} G_i$, and we denote G by H^I when all the groups G_i coincide with a group H.

We denote by $\prod_{i\in I} \mathfrak{Z}_{G_i}$ the product topology on G of the Zariski topologies $\mathfrak{Z}_{G_i}$ on each factor G_i. Then the Zariski topology $\mathfrak{Z}_G$ of the direct product is coarser than the product topology $\prod_{i\in I} \mathfrak{Z}_{G_i}$. For more details, see Theorem 3.4, where we give also a description of the elementary algebraic subsets of the product G.

As we noted above, these two topologies $\mathfrak{Z}_G$ and $\prod_{i\in I} \mathfrak{Z}_{G_i}$ on a product group $G = \prod_{i\in I} G_i$ need not coincide even in very simple cases. These observations motivated the following definitions.

Definition 1.12. Let G_1, G_2 be groups, and $G = G_1 \times G_2$. Then the pair G_1, G_2 will be called:

- $\mathfrak{Z}$*-productive*, if $\mathfrak{Z}_G = \mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2}$;
- *semi* $\mathfrak{Z}$*-productive*, if both $G_1 \times \{e_2\}$ and $\{e_1\} \times G_2$ are $\mathfrak{Z}_G$-closed subsets of G.

The pair G_1, G_2 is $\mathfrak{Z}$-productive exactly when $\mathfrak{Z}_{G_1\times G_2} \supseteq \mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2}$, as the other inclusion always holds by Theorem 3.4.

From the definitions, it immediately follows that a $\mathfrak{Z}$-productive pair is semi $\mathfrak{Z}$-productive. We are interested in studying when the converse implication holds true, so we explicitly state the following question.

Question 1. Let G_1, G_2 be a semi $\mathfrak{Z}$-productive pair. Is G_1, G_2 then $\mathfrak{Z}$-productive?

Theorem A below answers the above question when G_1, G_2 are abelian, thus classifying the abelian $\mathfrak{Z}$-productive pairs.

Theorem A. Let G_1, G_2 be abelian groups, and $G = G_1 \times G_2$. Then the following conditions are equivalent:

(a) the pair G_1, G_2 is $\mathfrak{Z}$-productive;
(b) the pair G_1, G_2 is semi $\mathfrak{Z}$-productive;
(c) G_1 and G_2 are bounded, $G_1 \cong F_1 \times G_1^*$, and $G_2 \cong F_2 \times G_2^*$, for finite subgroups $F_i \leq G_i$ for $i = 1, 2$, and subgroups $G_i^* \leq G_i$ for $i = 1, 2$ such that $(\exp(G_1^* \times G_2^*), |F_1|) = 1$, $(\exp(G_1^* \times G_2^*), |F_2|) = 1$, $(\exp(G_1^*), \exp(G_2^*)) = 1$.

This theorem will be proved in §4.3.

To study when a pair of groups G_1, G_2 is (semi) $\mathfrak{Z}$-productive, we have also considered the cases when $G_1 \times \{e_2\} \in \mathbb{E}_{G_1\times G_2}$. To this end, we give the following definition.

Definition 1.13. Let G be a group. A word $w \in G[x]$ is called a *δ-word for G* if w is singular, and $E_w^G = \{e_G\}$.

Let us immediately see that a non-trivial abelian group G has no δ-words. Indeed, if $w \in G[x]$ is singular, then $E_w \neq \{e_G\}$ by Remark 1.10.

The class Δ of the groups that admit a δ-word can be characterized as follows.

Theorem B. Let G_2 be a non-trivial group. Then, the following conditions are equivalent:

(a) G_2 belongs to Δ;
(b) $G_1 \times \{e_2\} \in \mathbb{E}_{G_1 \times G_2}$ for every group G_1.

In what follows, we will deduce Theorem B from some more general results proved in Theorem 4.6 and Corollary 4.7.

We prove that the class Δ is stable under taking finite products (Corollary 3.9) and under taking arbitrary powers (Theorem 3.10 Theorem 3.10). Moreover, we characterize the infinite direct products that belong to Δ (Theorem 3.12). This implies that every direct product of free non-abelian groups belongs to Δ (see Proposition 3.11 and its proof).

2. δ-Words

We begin this section giving the definition and a few properties of the *Taĭmanov topology* of a group.

Definition 2.1. The *Taĭmanov topology* $\mathcal{T}_G$ on a group G is the topology having the family of the centralizers of the elements of G as a subbase of the filter of the neighborhoods of e_G.

It is easy to check that $\mathcal{T}_G$ is a group topology, and for every element $g \in G$ the subgroup $C_G(g)$ is a $\mathcal{T}_G$-open (hence, closed) subset of G. In particular, $\overline{\{e_G\}}^{\mathcal{T}_G} = Z(G)$, so $\mathcal{T}_G$ need not be Hausdorff.

Lemma 2.2 ([4, Lemma 4.1])**.** *If G is a group, then the following hold for $\mathcal{T}_G$.*

(1) *$\mathcal{T}_G$ is Hausdorff if and only if G is center-free.*
(2) *$\mathcal{T}_G$ is indiscrete if and only if G is abelian.*

We have already noted that a non-trivial abelian group does not admit any δ-word. In the following lemma, we give a much more precise result.

Lemma 2.3. *If a group $G \in \Delta$, then its Taĭmanov topology $\mathcal{T}_G$ is discrete. In particular, G has trivial center.*

Proof: Assume $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} \in G[x]$ to be a δ-word for G. Then in particular $\epsilon(w) = 0$, and $e_G \in E_w$, i.e. $\mathrm{ct}(w) = e_G$.

Let $C = C_G(g_1, g_2, \dots, g_n)$ be the centralizer of $g_1, g_2, \dots, g_n$, and assume $g \in C$. Then

$$w(g) = \mathrm{ct}(w)\, g^{\epsilon(w)} = e_G g^0 = e_G,$$

so that $g \in E_w$, which yields $g = e_G$. This proves $C = \{e_G\}$. As C is a $\mathcal{T}_G$-neighborhood of e_G, we conclude that $\mathcal{T}_G$ coincides with the discrete topology of G. □

Remark 2.4. Note that a δ-word has even length, being singular. It is immediate to verify that the only group having a δ-word w with $\mathrm{l}(w) = 0$ is the trivial group, and w is the trivial word.

Now we show that no group has a δ-word with $\mathrm{l}(w) = 2$. Assume by contradiction $w \in G[x]$ to be a δ-word with $\mathrm{l}(w) = 2$. As $\mathrm{ct}(w) = e_G$, we can assume $w = gxg^{-1}x^{-1}$, so that $w = [g, x]$ and Example 1.2, item 1, gives

$$\{e_G\} = E_w = C_G(g).$$

This forces $g = e_G$, hence w to be trivial, which contradicts $\mathrm{l}(w) = 2$.

In the following proposition we show a δ-word with length 4 for every free non-abelian group.

Proposition 2.5. *Let F be a free non-abelian group, generated by the elements $\{a_i \mid i \in I\}$, and let $a \neq b$ be two of them. Then*

$$w = [a, x][b, x] = axa^{-1}x^{-1}bxb^{-1}x^{-1} \in F[x]$$

is a δ-word for F.

Proof: Obviously w is singular, $w(e_F) = e_F$, and we have to prove that $f_w(g) \neq e_G$ for every $g \in F$, $g \neq e_F$. To this end, let $f_1 = f_{w_1}$ and $f_2 = f_{w_2}$, where

$$w_1 = [a, x]^{-1} = [x, a] = xax^{-1}a^{-1} \in F[x],$$
$$w_2 = [b, x] = bxb^{-1}x^{-1} \in F[x].$$

As $w = w_1^{-1}w_2$, we have that $f_w = (f_1)^{-1}f_2$, and so $f_w(g) = e_G$ if and only if $f_1(g) = f_2(g)$, for every $g \in F$. So it suffices to prove that $f_1(g) \neq f_2(g)$ for every $g \in F$, $g \neq e_F$.

So let $e_F \neq g \in F$, and we are going to show that $f_1(g) \neq f_2(g)$. We can assume $g \notin \bigcup_{i \in I}\langle a_i \rangle$, so let $g = a_i^n h a_j^m$ be the reduced form of g, for $h \in F$, $0 \neq n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. (In particular, if $h = e_F$, then $g = a_i^n a_j^m$, with $i \neq j$.) Then

$$f_1(g) = a_i^n h a_j^m \cdot a \cdot (a_i^n h a_j^m)^{-1} \cdot a^{-1} = a_i^n h \, \underline{a_j^m \cdot a \cdot a_j^{-m}} h^{-1} \underline{a_i^{-n} \cdot a^{-1}},$$
$$f_2(g) = b \cdot a_i^n h a_j^m \cdot b^{-1} \cdot (a_i^n h a_j^m)^{-1} = \underline{b \cdot a_i^n} h \, \underline{a_j^m \cdot b^{-1} \cdot a_j^{-m}} h^{-1} a_i^{-n}.$$

As the only possible cancellations are between underlined elements, we can immediately say that $f_1(g)$ begins with $a_i^n h \ldots$; on the other hand, $f_2(g)$ either begins with $a_i^{n+1} h \ldots$ (if $a_i = b$), or it begins with $b \cdot a_i^n h \ldots$ (if $a_i \neq b$). In either case, $f_1(g) \neq f_2(g)$. □

Although Theorem B characterizes the class Δ, it is desirable to have another description of Δ.

Problem 1. Find an alternative description of the class Δ.

A necessary condition for $G \in \Delta$ is given by Lemma 2.3 in terms of Taĭmanov topology of G.

As a free non-abelian group contains cyclic (hence, abelian) subgroups, Δ is not stable under taking subgroups. According to Proposition 2.5, Δ is not stable under taking quotients either (as every group is a quotient of a free non-abelian group).

The class Δ is stable under taking finite products (Corollary 3.9), and under taking arbitrary powers (Theorem 3.10), while Theorem 3.12 characterizes which infinite products belong to Δ. In particular, every product of free non-abelian groups belongs to Δ by Proposition 3.11.

3. The Zariski topology on products

If $I \neq \emptyset$ is a set, and $\{G_i \mid i \in I\}$ is a family of groups, throughout this section we will consider the direct product $G = \prod_{i\in I} G_i$.

Lemma 3.1. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. Then there exists a canonical map $\vartheta\colon G[x] \to \prod_{i\in I}(G_i[x])$.*

Proof: For every $i \in I$, let $p_i\colon G \to G_i$ be the i-th canonical projection. Apply Proposition 1.4 to obtain the homomorphism $\pi_i\colon G[x] \to G_i[x]$, such that $\pi_i \restriction_G = p_i$, and $\pi_i(x) = x$. Finally, consider the diagonal map ϑ of the family $\{\pi_i \mid i \in I\}$, so that $\vartheta\colon G[x] \to \prod_{i\in I}(G_i[x])$.

The map $\vartheta\colon G[x] \to \prod_{i\in I}(G_i[x])$ has the following explicit form. Let

$$w = g^{(1)}x^{\varepsilon_1}g^{(2)}x^{\varepsilon_2}\cdots g^{(n)}x^{\varepsilon_n} \in G[x],$$

where $g^{(j)} = (g_i^{(j)})_{i\in I} \in G$ for elements $g_i^{(j)} = p_i(g^{(j)}) \in G_i$, for $i \in I$ and $j = 1, \dots, n$. Let

$$w_i = g_i^{(1)}x^{\varepsilon_1}g_i^{(2)}x^{\varepsilon_2}\cdots g_i^{(n)}x^{\varepsilon_n} \in G_i[x]$$

be the word in G_i obtained by taking the i-th coordinate of the coefficients of w. Then $w_i = \pi_i(w)$, and $\vartheta(w) = (w_i)_{i\in I} \in \prod_{i\in I}(G_i[x])$. □

Definition 3.2. In the notation of Lemma 3.1, we call $\vartheta(w) = (w_i)_{i\in I}$ the *coordinates of* w in $\prod_{i\in I}(G_i[x])$. Note that $\epsilon(w_i) = \epsilon(w)$ for every $i \in I$.

The map ϑ in Lemma 3.1 is not injective if $|I| > 1$ and the groups under consideration are not trivial (we discuss $\ker(\vartheta)$ in Example 3.5 below). Nonetheless, Lemma 3.1 suffices to obtain the following corollary which describes the verbal functions of a direct product as products of verbal functions of each component.

Corollary 3.3. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. If $w \in G[x]$ has coordinates $\vartheta(w) = (w_i)_{i\in I} \in \prod_{i\in I}(G_i[x])$, then the verbal function*

$f_w\colon G \to G$ is the mapping $(g_i)_{i\in I} \mapsto (f_{w_i}(g_i))_{i\in I}$, i.e., the product of the verbal functions f_{w_i}.

In the following theorem we show that the elementary algebraic subset E_w of a direct product is the cartesian product of the elementary algebraic subsets E_{w_i}, where $(w_i)_{i\in I}$ are the coordinates of w in $\prod_{i\in I}(G_i[x])$.

Theorem 3.4. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. If $w \in G[x]$, and $(w_i)_{i\in I}$ are the coordinates of w in $\prod_{i\in I}(G_i[x])$, then E_w^G has the form*

$$E_w^G = \prod_{i\in I} E_{w_i}^{G_i}. \tag{4}$$

In particular, $w \in \mathcal{U}_G$ (resp., w is a δ-word) if and only if $w_i \in \mathcal{U}_{G_i}$ (resp., w_i is a δ-word) for every $i \in I$.

As a consequence, the Zariski topology $\mathfrak{Z}_G$ of the direct product is coarser than the product topology $\prod_{i\in I} \mathfrak{Z}_{G_i}$.

PROOF: By Corollary 3.3, $g = (g_i)_{i\in I} \in G$ satisfies $w(g) = e_G$ if and only if $g_i \in G_i$ satisfies $w_i(g_i) = e_i$ for every $i \in I$. Thus E_w^G is as in (4), and $E_w^G = G$ if and only if $E_{w_i}^{G_i} = G_i$ for every $i \in I$, while $E_w^G = \{e_G\}$ if and only if $E_{w_i}^{G_i} = \{e_i\}$ for every $i \in I$, and $\epsilon(w_i) = \epsilon(w)$ for every $i \in I$.

By (4), it follows that E_w^G is closed in the product topology $\prod_{i\in I} \mathfrak{Z}_{G_i}$. Being $\mathbb{E}_G$ a subbase for $\mathfrak{Z}_G$-closed sets, we conclude that $\mathfrak{Z}_G \subseteq \prod_{i\in I} \mathfrak{Z}_{G_i}$. □

Example 3.5. Let G_1, G_2 be non-trivial groups, $g_i \in G_i \setminus \{e_i\}$, and $G = G_1 \times G_2$. Consider the word

$$w = (g_1^{-1}, e_2)x(e_1, g_2)x^{-1}(g_1, e_2)x(e_1, g_2^{-1})x^{-1} \in G[x],$$

and note that $w \neq e_{G[x]}$ is non-trivial, in fact $\mathrm{l}(w) = 4$. As

$$w_1 = \pi_1(w) = g_1^{-1}xe_1x^{-1}g_1xe_1x^{-1} = e_{G_1[x]},$$
$$w_2 = \pi_2(w) = e_2xg_2x^{-1}e_2xg_2^{-1}x^{-1} = e_{G_2[x]},$$

we have $w \in \ker(\vartheta)$, in the notation of Lemma 3.1.

If $w \in \ker(\vartheta)$, then $w_i = e_{G_i[x]}$ is the trivial word for every $i \in I$, so that in particular $w_i \in \mathcal{U}_{G_i}$. Then also $w \in \mathcal{U}_G$ by Theorem 3.4.

Corollary 3.6. *Let G_1, G_2 be non-trivial groups, and $G = G_1 \times G_2$. Then G has a singular, non-trivial universal word.*

PROOF: Consider the singular, non-trivial word $w \in G[x]$ defined in Example 3.5. Its coordinates in $G_1[x] \times G_2[x]$ are $(w_1, w_2) = (e_{G_1[x]}, e_{G_2[x]})$, so that equation (4) gives $E_w^G = E_{e_{G_1[x]}}^{G_1} \times E_{e_{G_2[x]}}^{G_2} = G_1 \times G_2$. □

The next definition will be used in the following Lemma 3.8 to give a sufficient condition on an element $(w_i)_{i\in I} \in \prod_{i\in I}(G_i[x])$ to belong to $\vartheta(G[x])$, where $\vartheta\colon G[x] \to \prod_{i\in I}(G_i[x])$ is the map defined in Lemma 3.1.

Definition 3.7. Let G be an arbitrary group and $w \in G[x]$. If $\mathrm{l}(w) = n \in \mathbb{N}_+$ and $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} g_0 \in G[x]$, we define $\vec{\epsilon}(w) = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{1, -1\}^n$.

Lemma 3.8. *Let $n \in \mathbb{N}_+$, $\vec{\epsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{1, -1\}^n$, and $\{G_i \mid i \in I\}$ be a family of groups. For every $i \in I$, let $w_i \in G_i[x]$ be such that $\mathrm{l}(w_i) = n$ and $\vec{\epsilon}(w_i) = \vec{\epsilon}$. Then, with $G = \prod_{i \in I} G_i$,*

(a) *$(w_i)_{i \in I} = \vartheta(w)$ for a word $w \in G[x]$;*
(b) *if every $w_i \in G_i[x]$ is a δ-word (resp., a universal word) for G_i, then also $w \in G[x]$ is a δ-word (resp., a universal word) for G.*

Proof: (a). We have to prove that there exists $w \in G[x]$ such that $(w_i)_{i \in I}$ are the coordinates of w in $\prod_{i \in I}(G_i[x])$. By assumption, for every $i \in I$, the word w_i has the form

$$w_i = g_i^{(1)} x^{\varepsilon_1} g_i^{(2)} x^{\varepsilon_2} \cdots g_i^{(n)} x^{\varepsilon_n} \in G_i[x].$$

Defining $g^{(j)} = (g_i^{(j)})_{i \in I} \in G$ for $j = 1, \ldots, n$, the word $w = g^{(1)} x^{\varepsilon_1} g^{(2)} x^{\varepsilon_2} \cdots g^{(n)} x^{\varepsilon_n} \in G[x]$ satisfies $\vartheta(w) = (w_i)_{i \in I}$, i.e. $(w_i)_{i \in I}$ are the coordinates of w in $\prod_{i \in I}(G_i[x])$.

(b). By (4), w is a δ-word (resp., a universal word) for G, if every $w_i \in G_i[x]$ is a δ-word (resp., a universal word). □

Now we prove that the class Δ is stable under taking finite products, using the idea of the proof of Lemma 3.8.

Corollary 3.9. *If $G \in \Delta$, and $H \in \Delta$, then also $P = G \times H \in \Delta$.*

Proof: Let

$$w_1 = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} \in G[x],$$
$$w_2 = h_1 x^{\delta_1} h_2 x^{\delta_2} \cdots h_m x^{\delta_m} \in H[x]$$

be δ-words respectively for G and H.

Let

$$v_1 = e_G x^{\delta_1} e_G x^{\delta_2} \cdots e_G x^{\delta_m} g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} \in G[x],$$
$$v_2 = h_1 x^{\delta_1} h_2 x^{\delta_2} \cdots h_m x^{\delta_m} e_H x^{\varepsilon_1} e_H x^{\varepsilon_2} \cdots e_H x^{\varepsilon_n} \in H[x].$$

Note that $\epsilon(v_i) = \epsilon(w_i) = \epsilon(w_1) + \epsilon(w_2) = 0$ and $E^G_{v_1} = E^G_{w_1} = \{e_G\}$, $E^H_{v_2} = E^H_{w_2} = \{e_H\}$, so that also v_1, v_2 are δ-words respectively for G and H.

Let $p_j = (e_G, h_j) \in P$ for $j = 1, \ldots, m$, and $p_{m+j} = (g_j, e_H) \in P$ for $j = i, \ldots, n$, and consider

$$w = p_1 x^{\delta_1} p_2 x^{\delta_2} \cdots p_m x^{\delta_m} p_{m+1} x^{\varepsilon_1} p_{m+2} x^{\varepsilon_2} \cdots p_{m+n} x^{\varepsilon_n} \in P[x].$$

Obviously, $\epsilon(w) = 0$, and (v_1, v_2) are the coordinates of w in $G[x] \times H[x]$, so that $E^P_w = E^G_{v_1} \times E^H_{v_2} = \{e_P\}$ by Theorem 3.4. So w is a δ-word for P. □

In the following theorem we show that a group G has a δ-word (in other words, $G \in \Delta$) if and only if G^I does.

Theorem 3.10. *Let G be a group, and I be a set. Then $G \in \Delta$ if and only if $G^I \in \Delta$.*

Proof: Let $w \in G[x]$ be a δ-word. Then Lemma 3.8 gives a word $v \in G^I[x]$ such that $(w)_{i\in I} \in G[x]^I$ are the coordinates of v, and v is a δ-word for G^I.

By (4), $w \in G^I[x]$ with coordinates $(w_i)_{i\in I}$ is a δ-word if and only if $w_i \in G[x]$ is a δ-word for every $i \in I$. □

As a consequence of Proposition 2.5 and Theorem 3.10, we get that every power of a free non-abelian group has a δ-word, i.e., belongs to Δ. In the following result, we show that Δ contains all *products* of free non-abelian groups.

Proposition 3.11. *Let $\{G_i \mid i \in I\}$ be a family of free non-abelian groups. Then $G = \prod_{i\in I} G_i$ belongs to Δ.*

Proof: For every $i \in I$, let $a_i, b_i \in G_i$ be two of the generators of G_i, and $w_i = [a_i, x][b_i, x] = a_i x a_i^{-1} x^{-1} b_i x b_i^{-1} x^{-1} \in G_i[x]$ be the δ-word for G_i constructed in Proposition 2.5. As $\mathrm{l}(w_i) = 4$, and $\vec{\epsilon}(w_i) = (1, -1, 1, -1)$ for every $i \in I$, Lemma 3.8 applies, so there exists a δ-word $w \in G[x]$ such that $(w_i)_{i\in I}$ are the coordinates of w in $\prod_{i\in I}(G_i[x])$. □

Let $\Delta_m \subseteq \Delta$ be the class of groups G having a δ-word $w \in G[x]$ with $\mathrm{l}(w) \le m$. Then $\Delta_{2k} = \Delta_{2k+1}$ for every $k \in \mathbb{N}$, and $\Delta_0 = \Delta_2$ only contains the trivial group $\{e\}$ by Remark 2.4. Moreover, Δ_4 contains every product of free non-abelian groups by Proposition 3.11. Then

$$\Delta_0 = \Delta_2 = \big\{\{e\}\big\} \subsetneq \Delta_4 \subseteq \Delta_6 \subseteq \ldots \subseteq \bigcup_{m\in\mathbb{N}} \Delta_m = \Delta. \tag{5}$$

In the following theorem, we characterize which products belong to the class Δ.

Theorem 3.12. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. Then the following are equivalent:*

1. *$G \in \Delta$;*
2. *there exists $m \in \mathbb{N}$ such that $G \in \Delta_m$;*
3. *there exists $m \in \mathbb{N}$ such that $G_i \in \Delta_m$ for every $i \in I$.*

Proof: The equivalence between conditions 1 and 2 follows from the definitions, while 2 implies 3 (with the same m) by Theorem 3.4.

So we only have to prove that 3 implies 1. Let $w_i \in G_i[x]$ be a δ-word, with $\mathrm{l}(w_i) = l_i \le m$.

For $1 \le k \le m$, let $I_k = \{i \in I \mid \mathrm{l}(w_i) = k\}$, and note that $\vec{\epsilon}(w_i) \in \{-1, 1\}^k$ for every $i \in I_k$. So for every $\vec{\epsilon} \in \{-1, 1\}^k$, let also $I_{k,\vec{\epsilon}} = \{i \in I_k \mid \vec{\epsilon}(w_i) = \vec{\epsilon}\}$.

Note that $I = \bigcup_{k=1}^m \bigcup_{\vec{\epsilon}\in\{-1,1\}^k} I_{k,\vec{\epsilon}}$ is a partition of I into finitely many subsets $I_{k,\vec{\epsilon}}$. If $I_{k,\vec{\epsilon}}$ is empty, let $G_{k,\vec{\epsilon}} = \{e\}$ be the trivial group, otherwise let $G_{k,\vec{\epsilon}} =$

$\prod_{i\in I_{k,\vec{\epsilon}}} G_i$. Then

$$G = \prod_{\substack{k=1,\dots,m\\ \vec{\epsilon}\in\{-1,1\}^k}} G_{k,\vec{\epsilon}}$$

is a finite product of the groups $G_{k,\vec{\epsilon}}$.

Then we can apply Lemma 3.8 to the family $\{G_i \mid i \in I_{k,\vec{\epsilon}}\}$, obtaining that $G_{k,\vec{\epsilon}} \in \Delta$.

Finally, $G \in \Delta$ by Corollary 3.9. □

Note that both Theorem 3.10 and Proposition 3.11 can be obtained as corollaries of Theorem 3.12.

Remark 3.13.
(a) The class Δ is stable under taking arbitrary products if and only if $\Delta = \Delta_m$ for some $m \in \mathbb{N}$, i.e. the chain (5) stabilizes after finitely many steps.
(b) We do not know if the equivalent conditions in item (a) do hold, for example we do not even know if $\Delta_4 \subsetneq \Delta_6$.

Motivated by Remark 3.13, one can ask the following question.

Question 2. Does the equality $\Delta = \Delta_m$ hold for some $m \in \mathbb{N}$? Or, equivalently, is it true that for every integer $m \geq 2$ there exists a group $G_m \in \Delta_{2m+2} \setminus \Delta_{2m}$?

We conclude this part with an easy result on the Zariski topology of a direct product.

Lemma 3.14. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $X_i \subseteq G_i$ be a subset for every $i \in I$. If $G = \prod_{i\in I} G_i$, then $\prod_{i\in I} C_{G_i}(X_i)$ is a $\mathfrak{Z}_G$-closed subgroup of G.*

In particular, if G_{i_0} is center-free for some $i_0 \in I$, then $\{e_{i_0}\} \times \prod_{i_0\neq i\in I} G_i$ is $\mathfrak{Z}_G$-closed.

Proof: It follows from the fact that $\prod_{i\in I} C_{G_i}(X_i) = C_G(\prod_{i\in I} X_i)$. Then Example 1.2, item 1, applies.

In the special case when G_{i_0} is center-free,

$$\{e_{i_0}\} \times \prod_{i_0\neq i\in I} G_i = C_G\Big(G_{i_0} \times \prod_{i_0\neq i\in I} \{e_i\}\Big).$$ □

4. $\mathfrak{Z}$-productivity

4.1 The class Δ and $\mathfrak{Z}$-productivity.

Lemma 4.1. *Let G_1, G_2 be groups, with $G_2 \in \Delta$ and $G = G_1 \times G_2$. Then $G_1 \times \{e_2\} = E_w^G$, for a singular word $w \in G[x]$.*

Proof: Let $w_0 = g_1x^{\varepsilon_1}g_2x^{\varepsilon_2}\cdots g_nx^{\varepsilon_n} \in G_2[x]$ be a δ-word for G_2. For $i = 1, 2, \dots, n$ define the elements $\widetilde{g_i} = (e_1, g_i) \in G$ and let $w = \widetilde{g_1}x^{\varepsilon_1}\widetilde{g_2}x^{\varepsilon_2}\cdots\widetilde{g_n}x^{\varepsilon_n} \in$

$G[x]$. The coordinates of w in $G_1[x]\times G_2[x]$ are (w_1, w_0), where $w_1 = e_1x^{\varepsilon_1}e_1x^{\varepsilon_2}\cdots e_1x^{\varepsilon_n} = x^{\epsilon(w_0)} = x^0$ is the neutral element of $G_1[x]$.

Then $E_w^G = E_{w_1}^{G_1} \times E_{w_0}^{G_2} = G_1 \times \{e_2\}$ and $\epsilon(w) = \epsilon(w_0) = 0$. □

Example 4.2. Let G_2 be a product of free non-abelian groups, G_1 be an arbitrary group, and $G = G_1 \times G_2$. By Proposition 3.11, $G_2 \in \Delta$, so that $G_1 \times \{e_2\} \in \mathbb{E}_G$ by Lemma 4.1.

In particular, $G_1 \times \{e_2\}$ is a $\mathfrak{Z}_G$-closed subset of G for every group G_1. In Theorem 4.11 we prove that the groups G_2 with this property are exactly the center-free groups.

Lemma 4.3. *Let G be an abelian group, and assume that G is a finite union of elementary algebraic subsets determined by non-singular words. Then G is bounded.*

Proof: Let $G = \bigcup_{i=1}^k G[n_i] + g_i$ for elements $g_i \in G$ and integers $n_i \in \mathbb{N}_+$, as $1 \le i \le k$. If $m = n_1n_2\cdots n_k$, then $G[n_i] \subseteq G[m]$, so that $G = \bigcup_{i=1}^k G[m] + g_i$. Then $[G : G[m]]$ is finite, and so $mG \cong G/G[m]$ is finite. As $m \neq 0$, we deduce that G is bounded. □

As a consequence of Lemma 4.3, if G is an abelian unbounded group, and G is a finite union of elementary algebraic subsets, then at least one of them is determined by a singular word. This motivates the following definition introducing the class $\mathcal{W}_0^*$ of groups in the general case.

Definition 4.4. We say that a group $G \in \mathcal{W}_0^*$ if G satisfies the following property: for every $k \in \mathbb{N}_+$, if $w_1, w_2, \ldots, w_k \in G[x]$ are such that $G = \bigcup_{i=1}^k E_{w_i}$, then w_i is singular for some $i = 1, 2, \ldots, k$.

Here we give some necessary and sufficient conditions on a group G to belong to $\mathcal{W}_0^*$.

Remark 4.5.
- If $G \in \mathcal{W}_0^*$, then every universal word of G is singular. In particular, if G is a bounded group, and $n = \exp(G)$, then $n > 0$ and $x^n \in \mathcal{U}_G$ is non-singular, so that $G \notin \mathcal{W}_0^*$.
- On the other hand, if G is an abelian unbounded group, then $G \in \mathcal{W}_0^*$ by Lemma 4.3. So if G is abelian then $G \in \mathcal{W}_0^*$ if and only if G is unbounded.

In the following theorem we prove that the converse of Lemma 4.1 holds for groups $G_1 \in \mathcal{W}_0^*$.

Theorem 4.6. *Let $G_1 \in \mathcal{W}_0^*$ and G_2 be groups. If $G = G_1 \times G_2$, then the following conditions are equivalent:*

(a) $G_2 \in \Delta$;
(b) $G_1 \times \{e_2\} = E_w^G$, *for a singular word* $w \in G[x]$;
(c) $G_1 \times \{e_2\} \in \mathbb{E}_G$;
(d) $G_1 \times \{e_2\} \in \mathbb{E}_G^{\cup}$.

PROOF: (a) $\Rightarrow$ (b) follows by Lemma 4.1.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are trivial.

(d) $\Rightarrow$ (a). Assume $G_1 \times \{e_2\} = \bigcup_{i=1}^k E_{w_i}^G$ for a positive integer k, and words $w_i \in G[x]$ for $i = 1, \dots, k$ with $E_{w_i}^G \neq \emptyset$.

By (4), every elementary algebraic subset E_w^G of G has the form $E_w^G = E_{w'}^{G_1} \times E_{w''}^{G_2}$ for words $w' \in G_1[x]$ and $w'' \in G_2[x]$. So $G_1 \times \{e_2\} = \bigcup_{i=1}^k E_{w_i'}^{G_1} \times E_{w_i''}^{G_2}$, from which we deduce

$$G_1 = \bigcup_{i=1}^k E_{w_i'}^{G_1}, \tag{6}$$

$$\text{and } \{e_2\} = \bigcup_{i=1}^k E_{w_i''}^{G_2}, \text{ i.e. } E_{w_i''}^{G_2} = \{e_2\} \text{ for every } i = 1, \dots, k. \tag{7}$$

As $G_1 \in \mathcal{W}_0^*$, (6) implies that w_i' is singular for some $i = 1, \dots, k$. This implies that also w_i'' is singular. By (7), w_i'' is a δ-word for G_2. □

Lemma 4.1 and Theorem 4.6 immediately imply Corollary 4.7 below. In particular, the equivalence between its items (b) and (c) provides a converse to Lemma 4.1. Moreover, the equivalence between items (a) and (b) is Theorem B.

Corollary 4.7. *Let G_2 be a group. Then, the following conditions are equivalent:*

(a) $G_2 \in \Delta$;
(b) $G_1 \times \{e_2\} \in \mathbb{E}_{G_1 \times G_2}$ *for every group* G_1;
(c) $G_1 \times \{e_2\} \in \mathbb{E}_{G_1 \times G_2}$ *for every* $G_1 \in \mathcal{W}_0^*$;
(d) $G_1 \times \{e_2\} \in \mathbb{E}_{G_1 \times G_2}$ *for some* $G_1 \in \mathcal{W}_0^*$.

By Theorem 3.10, every power G_2^I has the same properties as those of G_2 stated in the above corollary.

Corollary 4.8. *Let G_1, G_2 be abelian groups, with G_1 unbounded and G_2 non-trivial. Then $G_1 \times \{0_2\}$ is not a Zariski closed subset of $G = G_1 \times G_2$.*

PROOF: We have $G_1 \in \mathcal{W}_0^*$ by Lemma 4.3, while the abelian group G_2 has no δ-words by Lemma 2.3. Then $G_1 \times \{0_{G_2}\} \notin \mathbb{E}_G^{\cup}$ by Theorem 4.6, so that Theorem 1.9 applies. □

Remark 4.9. (a) The implication in Corollary 4.8 need not hold if one of the groups G_1, G_2 is not abelian. Indeed, consider an arbitrary group G_1, a product G_2 of free non-abelian groups, and let $G = G_1 \times G_2$. By Example 4.2, we have that $G_1 \times \{e_2\}$ is $\mathfrak{Z}_G$-closed, independently on G_1.
(b) One can relax the hypothesis "non-trivial abelian" for G_2 to $Z(G_2) \neq \{e_2\}$, but then only the weaker conclusion "$G_1 \times \{e_2\}$ is not additively algebraic" can be obtained.

We anticipate the following result from [10] about the Zariski closure of $G_1 \times \{e_2\}$ in the product $G_1 \times G_2$, when $G_1 \in \mathcal{W}_0^*$.

Proposition 4.10 ([10]). *Let $G_1 \in \mathcal{W}_0^*$. Then $\overline{G_1 \times \{e_2\}}^{\mathfrak{Z}_{G_1 \times G_2}} = G_1 \times Z(G_2)$ for every group G_2.*

By Corollary 4.7, a group $G_2 \in \Delta$ if and only if $G_1 \times \{e_2\} \in \mathbb{E}_{G_1 \times G_2}$ for every group G_1. In particular, $G_1 \times \{e_2\}$ is a Zariski closed subset of $G_1 \times G_2$ for every group G_1. The next theorem characterizes the groups G_2 with the latter (weaker) property.

Theorem 4.11. *For a group G_2 the following are equivalent:*

(a) *G_2 is center-free;*
(b) *$G_1 \times \{e_2\}$ is a Zariski closed subset of $G_1 \times G_2$ for every group G_1.*

Proof: (b) $\Rightarrow$ (a). Proposition 4.10, applied with $H = G_1 = \mathbb{Z}$, implies $Z(G_2) = \{e_2\}$.

(a) $\Rightarrow$ (b). Since G_2 is a center-free group, Lemma 3.14 applies to conclude that G_2 satisfies (b). □

4.2 Semi $\mathfrak{Z}$-productive pairs.

Lemma 4.12. *Let G_1, G_2 be groups, $H_i \leq G_i$, for $i = 1, 2$ be subgroups, $G = G_1 \times G_2$ and $H = H_1 \times H_2$. If H is Zariski embedded in G, then the following hold.*

(1) *If the pair G_1, G_2 is semi $\mathfrak{Z}$-productive, then also the pair H_1, H_2 is semi $\mathfrak{Z}$-productive.*
(2) *If the pair G_1, G_2 is $\mathfrak{Z}$-productive, then also the pair H_1, H_2 is $\mathfrak{Z}$-productive.*

Proof: (1) By assumption, $G_1 \times \{e_2\}$ is a $\mathfrak{Z}_G$-closed subset of G, so $H_1 \times \{e_2\}$ is a $\mathfrak{Z}_G \restriction_H$-closed subsets of H. As $\mathfrak{Z}_G \restriction_H = \mathfrak{Z}_H$, this proves that $H_1 \times \{e_2\}$ is a $\mathfrak{Z}_H$-closed subset of H. The same argument holds for $\{e_1\} \times H_2$.

(2) Note that $\mathfrak{Z}_{G_1 \restriction_{H_1}} \times \mathfrak{Z}_{G_2 \restriction_{H_2}} \supseteq \mathfrak{Z}_{H_1} \times \mathfrak{Z}_{H_2}$. Then

$$\mathfrak{Z}_H = \mathfrak{Z}_G \restriction_H = (\mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2}) \restriction_H = \mathfrak{Z}_{G_1 \restriction_{H_1}} \times \mathfrak{Z}_{G_2 \restriction_{H_2}} \supseteq \mathfrak{Z}_{H_1} \times \mathfrak{Z}_{H_2},$$

where the first equality holds as H is Zariski embedded in G, while the second equality holds as G_1, G_2 is $\mathfrak{Z}$-productive.

From Theorem 3.4 and the above equation, it follows that $\mathfrak{Z}_H = \mathfrak{Z}_{H_1} \times \mathfrak{Z}_{H_2}$. □

Corollary 4.13. *If G_1, G_2 is a (semi) $\mathfrak{Z}$-productive pair, and $H_i \leq Z(G_i)$, for $i = 1, 2$ are subgroups, then also H_1, H_2 is (semi) $\mathfrak{Z}$-productive.*

In particular, if G_1, G_2 is an abelian (semi) $\mathfrak{Z}$-productive pair, and $H_i \leq G_i$, for $i = 1, 2$ are subgroups, then also H_1, H_2 is (semi) $\mathfrak{Z}$-productive.

Proof: As central subgroups are Zariski embedded by Proposition 1.8, we have that $H = H_1 \times H_2 \leq Z(G_1) \times Z(G_2) = Z(G_1 \times G_2)$ is Zariski embedded in $G_1 \times G_2$.

Finally, Lemma 4.12 applies. □

4.3 Abelian $\mathfrak{Z}$-productive pairs.

Lemma 4.14. *Let G_1, G_2 be bounded abelian groups having coprime exponents. Then G_1, G_2 is $\mathfrak{Z}$-productive.*

Proof: Let $G = G_1 \times G_2$, and $\exp(G_i) = m_i$ for $i = 1, 2$. By (3), the $\mathfrak{Z}_{G_1}$- (resp., $\mathfrak{Z}_{G_2}$)-closed subsets are generated by the cosets of the n-socles $G_1[n]$ (resp., $G_2[n]$), for $n \in \mathbb{N}$. So it will suffice to show that, for every $n \in \mathbb{N}$, the subgroups $G_1[n] \times G_2$ and $G_1 \times G_2[n]$ are $\mathfrak{Z}_G$-closed subsets. Indeed $G_1[n] \times G_2$ is an elementary algebraic subset of G, as

$$G_1[n] \times G_2 = G_1[n] \times G_2[nm_2] = G_1[nm_2] \times G_2[nm_2] = G[nm_2],$$

where the first equality holds as $m_2 = \exp(G_2)$, and the second one as $(\exp(G_1), m_2) = 1$. Similarly, $G_1 \times G_2[n] = G_1[nm_1] \times G_2[nm_1] = G[nm_1]$. □

If $\{G_i \mid i \in I\}$ is a family of groups, for an element $g = (g_i)_{i \in I} \in G = \prod_{i \in I} G_i$, we denote by $\mathrm{supp}(g) = \{i \in I \mid g_i \neq e_i\} \subseteq I$ the set of indexes such that the correspondent coordinates of g are non-trivial.

The subgroup S of G consisting of the elements g such that $\mathrm{supp}(g)$ is finite will be called *direct sum* of $\{G_i \mid i \in I\}$, and denoted by $S = \bigoplus_{i \in I} G_i$. Obviously, $S = G$ when I is finite.

For an abelian group G, recall that $\pi(G) = \{p \in \mathbb{P} \mid G[p] \neq \{0\}\}$, and for $p \in \mathbb{P}$ it is defined the subgroup

$$G_p = \{g \in G \mid \exists n \in \mathbb{N} \ \ p^n g = 0\} = \bigcup_{n \in \mathbb{N}} G[p^n].$$

It is well-known that if G is a torsion, then $G \cong \bigoplus_{p \in \pi(G)} G_p$.

Now we are ready to prove Theorem A. It will give a positive answer to Question 1 for abelian $\mathfrak{Z}$-productive pairs, and provides a description of the structure of abelian groups G_1, G_2 such that the pair G_1, G_2 is $\mathfrak{Z}$-productive. Moreover, the implication (b) ⇒ (c) is a 'symmetric' form of Corollary 4.8, giving a much more precise conclusion.

Proof of Theorem A: We have to prove that if G_1, G_2 are abelian groups, and $G = G_1 \times G_2$, then the following conditions are equivalent:

(a) the pair G_1, G_2 is $\mathfrak{Z}$-productive;
(b) the pair G_1, G_2 is semi $\mathfrak{Z}$-productive;
(c) G_1 and G_2 are bounded, $G_1 = F_1 \oplus G_1^*$, and $G_2 = F_2 \oplus G_2^*$, for finite subgroups $F_i \leq G_i$ for $i = 1, 2$, and subgroups $G_i^* \leq G_i$ for $i = 1, 2$ such that $(\exp(G_1^* \oplus G_2^*), |F_1|) = 1$, $(\exp(G_1^* \oplus G_2^*), |F_2|) = 1$, $(\exp(G_1^*), \exp(G_2^*)) = 1$.

(a) ⇒ (b) follows by the definitions.

(b) $\Rightarrow$ (c). As both $G_1 \times \{0_2\}$ and $\{0_1\} \times G_2$ are $\mathfrak{Z}_G$-closed subsets of G, then both G_1 and G_2 are bounded by Corollary 4.8. Let $G_i = \bigoplus_{p\in\pi(G_i)} G_{i,p}$, where $\pi(G_i)$ is finite, for $i = 1, 2$.

Let $\pi = \pi(G_1) \cap \pi(G_2)$. If $\pi = \emptyset$, let F_1 and F_2 be the trivial subgroups of G_1 and G_2 respectively. Otherwise, let

$$F_1 = \bigoplus_{p\in\pi} G_{1,p} \quad \text{and} \quad F_2 = \bigoplus_{p\in\pi} G_{2,p}.$$

Set

$$G_1^* = \bigoplus_{p\in\pi(G_1)\setminus\pi(G_2)} G_{1,p} \quad \text{and} \quad G_2^* = \bigoplus_{p\in\pi(G_2)\setminus\pi(G_1)} G_{2,p},$$

so that

$$G_1 = F_1 \oplus G_1^* \quad \text{and} \quad G_2 = F_2 \oplus G_2^*.$$

It only remains to prove that both F_1, F_2 are finite groups, that is: if $p \in \pi$, then both $G_{1,p}$ and $G_{2,p}$ are finite.

So let $p \in \pi$ and by contradiction assume $G_{1,p}$ to be infinite. If $H_1 = G_1[p] \leq G_{1,p}$, then also H_1 is infinite. Fix an element $x \in G_2$ of order p, and let $H_2 = \langle x\rangle \leq G_2$. Finally, let $H = H_1 \times H_2$, and note that $\exp(H) = p$, so that $\mathfrak{Z}_H$ is the cofinite topology by Proposition 1.11. Being $H_0 = H_1 \times \{0_2\}$ an infinite proper subgroup of H, it is not $\mathfrak{Z}_H$-closed. This contradicts Corollary 4.13.

(c) $\Rightarrow$ (a). Assume $G_1 = F_1 \oplus G_1^*$ and $G_2 = F_2 \oplus G_2^*$, with F_1, F_2 finite, G_1^*, G_2^* bounded, with coprime exponents as in the statement of (c). Then $\mathfrak{Z}_{G_i} = \mathfrak{Z}_{F_i} \times \mathfrak{Z}_{G_i^*}$ for $i = 1, 2$ by Lemma 4.14, so that

$$\mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2} = \mathfrak{Z}_{F_1} \times \mathfrak{Z}_{G_1^*} \times \mathfrak{Z}_{F_2} \times \mathfrak{Z}_{G_2^*}.$$

Finally, let $F = F_1 \times F_2$ and note that $\mathfrak{Z}_F = \mathfrak{Z}_{F_1} \times \mathfrak{Z}_{F_2}$ is the discrete topology on the finite group F. So

$$\mathfrak{Z}_{G_1\times G_2} = \mathfrak{Z}_{F_1\oplus G_1^*\times F_2\oplus G_2^*} = \mathfrak{Z}_{F\times G_1^*\times G_2^*} \overset{(*)}{=} \mathfrak{Z}_F \times \mathfrak{Z}_{G_1^*} \times \mathfrak{Z}_{G_2^*} = \mathfrak{Z}_{F_1} \times \mathfrak{Z}_{F_2} \times \mathfrak{Z}_{G_1^*} \times \mathfrak{Z}_{G_2^*},$$

where the equality $(*)$ follows again from Lemma 4.14, as the three groups F, G_1^* and G_2^* are all bounded with mutually coprime exponents. This concludes the proof. □

Corollary 4.15. *Let G_1, G_2 be an abelian semi $\mathfrak{Z}$-productive pair. Then neither G_1, nor G_2, can contain as a subgroup any of the following groups: the group of integers $\mathbb{Z}$; the p-Prüfer group $\mathbb{Z}_{p^\infty}$; $\bigoplus_{n=1}^{\infty} \mathbb{Z}_{p^n}$ for a prime number $p \in \mathbb{P}$; $\bigoplus_{n=1}^{\infty} \mathbb{Z}_{p_n}$ for infinitely many different prime numbers $p_n \in \mathbb{P}$, as $n \in \mathbb{N}$.*

It follows from Theorem A that for every non-trivial abelian group G there exists a bounded abelian group H such that G, H is not a $\mathfrak{Z}$-productive pair.

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