A generalization of Čech-complete spaces and Lindelöf Σ-spaces

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. The class of s-spaces is studied in detail. It includes, in particular, all Čech-complete spaces, Lindelöf p-spaces, metrizable spaces with the weight $\leq 2^{\omega}$, but countable non-metrizable spaces and some metrizable spaces are not in it. It is shown that s-spaces are in a duality with Lindelöf Σ-spaces: X is an s-space if and only if some (every) remainder of X in a compactification is a Lindelöf Σ-space [Arhangel'skii A.V., *Remainders of metrizable and close to metrizable spaces*, Fund. Math. **220** (2013), 71–81]. A basic fact is established: the weight and the networkweight coincide for all s-spaces. This theorem generalizes the similar statement about Čech-complete spaces. We also study hereditarily s-spaces, provide various sufficient conditions for a space to be a hereditarily s-space, and establish that every metrizable space has a dense subspace which is a hereditarily s-space. It is also shown that every dense-in-itself compact hereditarily s-space is metrizable.

Keywords: metrizable, Lindelöf p-space, Lindelöf Σ-space, remainder, compactification, σ-space, countable network, countable type, perfect mapping

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1. Introduction

We provide a systematic study of s-spaces. These spaces are intimately related, via the remainder duality, to Lindelöf Σ-spaces [13]. Various sufficient conditions for a space to be an s-space are provided (see Sections 2 and 3). This leads to a deeper study of hereditarily s-spaces in Sections 4 and 6 culminating in the following theorem: if a dense-in-itself Lindelöf Σ-space is a hereditarily s-space, then X has a countable base. As a part of technique, we provide new results on separation of Lindelöf Σ-subspaces by continuous mappings to separable metrizable spaces (Section 6, Theorem 6.1).

"A space" in this article stands for "a Tychonoff space" unless the restrictions on separation axioms are explicitly stated. A compactification bX of a space X is a compact space which contains X as a dense subspace. By a *remainder* of a space X we mean the subspace $bX \setminus X$ of a compactification bX of X.

A space X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ which has a countable base of open neighbourhoods in X. All metrizable spaces and all locally compact spaces are of countable type [7], [4]. Here is a famous classical result of M. Henriksen and J. Isbell [21]:

Theorem 1.1. *A space X is of countable type if and only if the remainder in any* (*in some*) *compactification of X is Lindelöf* [21].

A Lindelöf p-space is a preimage of a separable metrizable space under a perfect mapping [7]. Continuous images of Lindelöf p-spaces are called *Lindelöf Σ-spaces* [23]. For the definition of a p-space, see [7]. We denote by $w(X)$ the weight of X, $d(X)$ stands for the density of X, and $nw(X)$ is the networkweight of X. For further terminology and notation, see [19].

2. s-spaces and their basic properties

In this section, we provide a systematic study of s-spaces. We use a characterization of s-spaces given in [13] to establish some basic properties of s-spaces.

Suppose that $\mathcal{S}$ is a family of subsets of a space X. Then $\mathcal{S}_\delta$ denotes the family of all sets that can be represented as the intersection of some nonempty subfamily of $\mathcal{S}$, and $\mathcal{S}_{\delta\sigma}$ denotes the family of all sets that can be represented as the union of some subfamily of $\mathcal{S}_\delta$. We say that $\mathcal{S}$ is a *source for a space Y in X* if Y is a subspace of X such that $Y \in \mathcal{S}_{\delta\sigma}$.

Clearly, the same family $\mathcal{S}$ may serve as a source for many spaces. A source $\mathcal{S}$ for Y in X is *open* (*closed*) if every member of $\mathcal{S}$ is an open (respectively, closed) subset of the space X. A source $\mathcal{S}$ is *countable* if $\mathcal{S}$ is countable. For some early appearances of open sources in literature, see [1], [16], and [19, 3.9.E]. The concepts of an open source and of a countable open source have been considered lately in [15] in connection with embeddings in compacta. A space X is called an *s-space* if there exists a countable open source for X in some compactification bX of X [15], [13]. Obviously, we have:

Proposition 2.1. *Every Čech-complete space is an s-space.*

For a space Z, we denote by $Bor_1(Z)$ the smallest family of subsets of Z such that every open subset of Z is in $Bor_1(Z)$, and for any countable subfamily γ of $Bor_1(Z)$ we have $\bigcap\gamma \in Bor_1(Z)$ and $\bigcup\gamma \in Bor_1(Z)$. Members of $Bor_1(Z)$ are called *Borelian subsets of Z of the first type*. A space X is a *Borelian space of the first type* if for some compactification bX of X, $X \in Bor_1(bX)$.

Proposition 2.2. *Every Borelian space X of the first type is an s-space. Even more, every member of $Bor_1(bX)$ is an s-space.*

Proposition 2.3. *Every separable metrizable space is an s-space.*

In fact, the next more general statement holds [15]:

Proposition 2.4. *Every Lindelöf p-space is an s-space.*

Thus, the class of s-spaces is very wide and includes many classical objects. We also see that to be an s-space is not a completeness type property.

We show below that Lindelöf Σ-spaces need not be s-spaces. But they can be characterized in terms of closed sources [23]:

Proposition 2.5. *A space X is a Lindelöf Σ-space if and only if there exists a countable closed source for X in some (in every) compact space B such that X is a subspace of B.*

The next statement is obvious.

Proposition 2.6. *Let X be a subset of a set B, $Y = B \setminus X$, and $\mathcal{S}$ be a family of subsets of B. Then $\mathcal{S}$ is a source for X in B if and only if $\{B \setminus P : P \in \mathcal{S}\}$ is a source for Y in B.*

Theorem 2.7. *Suppose that X is a space with a remainder Y in some compactification bX of X. Then the following conditions are equivalent:*

(i) *X is an s-space,*
(ii) *Y is a Lindelöf Σ-space,*
(iii) *X has a countable open source in bX.*

Proof: (i) ⇒ (ii). Since X is an s-space, there exists a compactification b_1X of X such that $Y_1 = b_1X \setminus X$ is a Lindelöf Σ-space. It follows that Y is also a Lindelöf Σ-space, since the class of Lindelöf Σ-spaces is preserved by perfect mappings in both directions.

(ii) ⇒ (iii). By Proposition 2.5, there exists a countable closed source for Y in bX. Hence, by Proposition 2.6, X has a countable open source in bX.

(iii) ⇒ (i). This follows immediately from the definition of an s-space. □

By Theorem 2.7, the definition of s-space can be complemented as follows:

Proposition 2.8. *If X is an s-space, then X has a countable open source in every compactification bX of X.*

The next result follows directly from Theorem 2.7:

Corollary 2.9. *Suppose that X is a nowhere locally compact space with a remainder Y. Then X is a Lindelöf Σ-space if and only if Y is an s-space.*

Clearly, the assumption in 2.9 that X is nowhere locally compact cannot be dropped.

We can use the well known facts of the theory of Lindelöf Σ-spaces to establish properties of s-spaces. We also get an idea how to extend the concepts of s-space and Lindelöf Σ-space from the countable case to the case of an arbitrary infinite cardinal τ: we want Theorem 2.7 to remain true in the general situation.

A space X will be called an *s_τ-space* if there exists an open source $\mathcal{S}$ for X in some compactification bX of X such that $|\mathcal{S}| \leq \tau$. A space X will be called an *n_τ-space* if there exists a closed source $\mathcal{S}$ for X in some compactification bX of X such that $|\mathcal{S}| \leq \tau$. It can be shown that these definitions do not depend on the choice of the compactification bX. All statements about s-spaces and Lindelöf Σ-spaces presented so far remain valid when they are reformulated for s_τ-spaces and n_τ-spaces. We continue to formulate the results for the countable case. The next statement is obvious.

Proposition 2.10. *Suppose that X is an s-space. Then:*

(a) *every closed subspace of X is an s-space;*
(b) *every open subspace of X is an s-space;*
(c) *the intersection of any countable family of s-subspaces of X is an s-space.*

Proposition 2.11. *Every s-space is a space of countable type.*

PROOF: This is so by Theorem 1.1, since every remainder of X is Lindelöf. □

Hence, any countable non-first-countable space is a non-s-space. We also see that the union of a countable family of s-spaces need not be an s-space. Here is another corollary to Proposition 2.11:

Corollary 2.12. *Every s-space X is a k-space* (*even, a k_2-space*).

PROOF: Indeed, every space of point-countable type is a k-space (even, a k_2-space) [4], [3]. So it is enough to apply Proposition 2.11. □

Theorem 2.13. *Suppose that f is a perfect mapping of a space X onto a space Y. Then Y is an s-space if and only if X is an s-space.*

PROOF: We can take the continuous extension $f^* : \beta X \to \beta Y$ of f to the Stone-Čech compactifications of X and Y. The restriction of f^* to the remainders rX and rY of X and Y in βX and βY is a perfect mapping of rX onto rY. Therefore, rX is a Lindelöf Σ-space if and only if rY is a Lindelöf Σ-space. It remains to apply Theorem 2.7. □

The union of two subspaces which are s-spaces need not be an s-space. To see this, take the countable Fréchet-Urysohn fan. However, the next addition theorem holds:

Theorem 2.14. *If a space X is the union of a countable family η of dense subspaces of X such that each $Z \in \eta$ is an s-space, then X is also an s-space.*

PROOF: Fix a compactification bX of X. Take any $Z \in \eta$. Clearly, Z is dense in bX, that is, bX is a compactification of Z. Since Z is an s-space, we can fix a countable open source $\mathcal{S}_Z$ for Z in bX. Then, obviously, the family $\mathcal{S} = \bigcup\{\mathcal{S}_Z : Z \in \eta\}$ is a countable open source for X in bX. □

Theorem 2.15. *The topological product of any countable family η of s-spaces is an s-space.*

PROOF: Let $\eta = \{X_i : i \in \omega\}$. For each $i \in \omega$, fix a compactification b_iX_i of X_i, and let X be the topological product of the family η, and B be the topological product of the family $\{b_iX_i : i \in \omega\}$. Clearly, B is compact and X is dense in B. Thus, B is a compactification bX of X. Put $Y = B \setminus X$, $P_i(i) = b_iX_i \setminus X_i$, for

$i \in \omega$, and $P_j(i) = b_j X_j$ for $i, l \in \omega$ such that $j \neq i$. Then, clearly, the topological product of the family $\{P_j(i) : j \in \omega\}$ is a Lindelöf Σ-space. We denote this space by H_i. Obviously, $Y = \bigcup\{H_i : i \in \omega\}$. Hence, Y is a Lindelöf Σ-space as well [23]. Since B is compact, it follows from Theorem 2.7 that X is an s-space. □

3. Some subclasses of the class of s-spaces

We have seen that the class of s-spaces lies between the class of Čech-complete spaces and the class of spaces of countable type. Below we establish further results on which spaces are s-spaces.

A space X is *perfect* if every closed subset of X is a G_δ-set in X. We have [15]:

Theorem 3.1. *Every perfect s-space is a p-space.*

Observe that an s-space need not be a p-space. This follows from the next result [13]:

Theorem 3.2. *If X is a space with a σ-disjoint base such that $|X| \leq 2^\omega$, then X is an s-space.*

The famous Michael line [19] satisfies the restrictions in the above theorem and hence, is an s-space, but is not a p-space (otherwise it would have been metrizable).

Corollary 3.3 ([13]). *If the cardinality of a metrizable space X does not exceed 2^ω, then X is an s-space.*

The cardinality restriction in the last statement is essential. Indeed, it has been established in [12] that a remainder of a metrizable space can fail to be a Lindelöf Σ-space. Theorem 2.7 permits to reformulate this result in the following way:

Theorem 3.4. *There exists a metrizable space which is not an s-space.*

Example 3.5. Theorem 3.2 cannot be extended to spaces X with a point-countable base such that $|X| \leq 2^\omega$. To see this, we take a non-metrizable hereditarily Lindelöf space X with a point-countable base constructed under $[CH]$ in [18]. This space X is not a p-space, since otherwise it would have been metrizable as a Lindelöf p-space with a point-countable base. Notice that X is a perfect space, since it is hereditarily Lindelöf. Therefore, by Theorem 3.1, X is not an s-space. It has been shown in [13] that no remainder of this space X is a Lindelöf Σ-space.

Theorem 3.6. *Every countably compact s-space X is Čech-complete.*

Proof: Fix a compactification bX of X. Then $Y = bX \setminus X$ has a countable closed source $\mathcal{S}$ in bX. We fix it, and put $\mathcal{S}_y = \{P \in \mathcal{S} : y \in P\}$ for each $y \in Y$. The family $\mathcal{S}_y$ has the following properties:

(i) every member of $\mathcal{S}_y$ is a closed compact subset of bX,

(ii) $y \in \bigcap\{P : P \in \mathcal{S}_y\} \subset Y$,

(iii) $\mathcal{S}_y$ is countable.

It follows from (i), (ii), and (iii) that $\eta_y = \{P \cap X : P \in \mathcal{S}_y\}$ is a countable family of closed subsets of X with the empty intersection. Therefore, since X is countably compact, the intersection of some finite subfamily of η_y is empty. Hence, $X \cap (\bigcap \mu_y) = \emptyset$, for some finite subfamily μ_y of $\mathcal{S}$. Since y is an arbitrary point of Y, we have: $Y = \bigcup\{K : K \in \mathcal{K}\}$, where $\mathcal{K}$ is the family of all compact subspaces of Y which are intersections of some finite subfamily of $\mathcal{S}$. Clearly, $\mathcal{K}$ is countable, since $\mathcal{S}$ is countable. Hence, Y is σ-compact, and X is Čech-complete. □

Corollary 3.7. *If a remainder of a nowhere locally compact Lindelöf Σ-space X is countably compact, then X is σ-compact.*

Proof: It is enough to apply Theorem 3.6 and Corollary 2.9. □

Here is another application of Theorem 3.6:

Theorem 3.8. *If a topological group G is a Lindelöf Σ-space, and some remainder Y of G is normal, then either G is σ-compact, or G is a Lindelöf p-space.*

Proof: By the Dichotomy Theorem for remainders of topological groups [11], Y is either Lindelöf or pseudocompact. In the first case G must be a p-space, since G is a topological group [11]. In the second case Y is countably compact, since it is normal and pseudocompact. We may assume that G is nowhere locally compact, since otherwise it is locally compact and hence, is a p-space. Now it follows from Corollary 3.7 that G is σ-compact. □

4. T_0-separators and hereditarily s-spaces

In this section, we provide a few sufficient conditions for a space to be a hereditarily s-space.

Remark. Now we need some terminology more general than that of sources. Let X and Y be subspaces of a space Z, and γ be a family of subsets of Z such that for any distinct x, y, where $x \in X$ and $y \in Y$, there exists $P \in \gamma$ such that $x \in P$ and $y \notin P$. Then we say that γ is a T_0-*separator* in Z for the pair (X, Y). If, in the same situation, one can find disjoint $P, Q \in \gamma$ such that $x \in P$ and $y \in Q$, we say that γ is a *Hausdorff separator* in Z for (X, Y). We call a separator *open* (*closed*) if all its members are open (respectively, closed) (in Z). Clearly, if $Y = Z \setminus X$, then γ is a T_0-separator for the pair (X, Y) if and only if γ is a source for X in Z.

Lemma 4.1. *Suppose that X is a subspace of a space Z. Then any two of the following conditions are equivalent:*

(i) *X has a (countable) open T_0-separator in X for (X, X),*
(ii) *X has a (countable) closed T_0-separator in X for (X, X),*
(iii) *X has a (countable) open T_0-separator in Z for (X, X),*
(iv) *X has a (countable) closed T_0-separator in Z for (X, X).*

Proof: To see that (iv) ⇒ (ii) and (iii) ⇒ (i), we take the traces on X of members of a T_0-separator in Z. To show that (i) ⇒ (iii), we expand each member V of a T_0-separator $\mathcal{S}$ in X to an open subset U_V of Z such that $U_V \cap X = V$, and take $\{U_V : V \in \mathcal{S}\}$. The equivalence of (i) and (ii) is also obvious. □

The role of countable T_0-separators in X for (X, X) is emphasized by the next fact:

Proposition 4.2. *If X is an s-space with a countable T_0-separator γ in X for (X, X) which is either open or closed, and bX is an arbitrary compactification of X, then there exists a countable family $\mathcal{S}$ of open subsets of bX such that $\mathcal{S}$ is a source for every subspace M of X. Hence, X is a hereditarily s-space.*

Proof: Fix a countable open source $\mathcal{S}_1$ for X in bX. By Lemma 4.1, we can also find a countable open T_0-separator γ in bX for (X, X). It is easy to verify that the family $\mathcal{S} = \mathcal{S}_1 \cup \gamma$ is a countable open source for M in bX. □

A natural sufficient condition for a space X to have a countable open T_0-separator in X for (X, X) is the existence of a countable pseudobase in X. In particular, we have:

Proposition 4.3. *If there exists a one-to-one continuous mapping f of a space X onto a T_1-space H with a countable base $\mathcal{B}$, then X has a countable open T_0-separator in X for (X, X).*

Proof: Clearly, $\mathcal{S} = \{f^{-1}(V) : V \in \mathcal{B}\}$ is a countable open T_0-separator in X for (X, X). □

Corollary 4.4. *Let X be an s-space satisfying at least one of the following conditions:*

(a) *X admits a one-to-one continuous mapping f onto a T_1-space H with a countable base;*
(b) *X has a countable pseudobase.*

Then every subspace of X is an s-space.

Corollary 4.5. *Suppose that B is a compact space, and that a Lindelöf Σ-space L is a subspace of B. Furthermore, suppose that the subspace $M = B \setminus L$ admits a one-to-one continuous mapping f onto a T_1-space H with a countable base. Then any subspace X of B such that $L \subset X$ is a Lindelöf Σ-space.*

Proof: It is enough to refer to Propositions 4.2, 2.5, and 2.6. □

Corollary 4.6. *Let B be a compact space, and M be a metrizable subspace of B dense in B with $w(M) \leq 2^\omega$.*

Then any $X \subset B$ such that $B \setminus M \subset X$ is a Lindelöf Σ-space.

Proof: This is so by Corollary 4.5, since every metrizable space X of the cardinality not greater than 2^ω is an s-space [13] and admits a one-to-one continuous mapping onto a separable metrizable space. □

5. Point-finite and boundedly point-finite families of sets, T_0-separators, and hereditarily s-spaces

We provide below further sufficient conditions for a space X to have a countable closed T_0-separator in X for (X, X) and, based on them, sufficient conditions for a space to be a hereditarily s-space. These conditions are related to the following fact established in [13]: if X is a space with a σ-disjoint base such that $|X| \leq 2^\omega$, then every subspace of X is an s-space (see Theorem 3.2). A typical result in this section is the next statement which follows from Corollary 5.4 below.

Proposition 5.1. *If X is a space with a σ-discrete network, and $|X| \leq 2^\omega$, then X has a countable closed T_0-separator in X for (X, X).*

The next fact is essentially known, and its standard proof is omitted.

Lemma 5.2. *Suppose that γ is a family of subsets of a set X such that $|\gamma| \leq 2^\omega$. Then there exists a countable family $\mathcal{W}$ of subfamilies of γ satisfying the following condition:*

- (u) *For any finite subfamilies μ_1, μ_2 of γ such that $\mu_1 \cap \mu_2 = \emptyset$, there exists $\eta \in \mathcal{W}$ such that $\mu_1 \subset \eta$ and $\mu_2 \cap \eta = \emptyset$.*

Proposition 5.3. *Suppose that X is a set, M is a subset of X, and $\mathcal{S} = \{\gamma_n : n \in \omega\}$ is a sequence of families of subsets of X satisfying the following conditions for every $n \in \omega$:*

- (i) $|\gamma_n| \leq 2^\omega$,
- (ii) *γ_n is point-finite,*
- (iii) *for any pair (x, y) of distinct points such that $x \in M$ and $y \in X$, there exists $k \in \omega$ such that $x \in \bigcup \gamma_k$, and no member of γ_k contains both x and y.*

Then X has a countable T_0-separator $\mathcal{E}$ in X for (M, X) such that $\mathcal{E} \subset \bigcup\{\mathcal{U}_n : n \in \omega\}$, where $\mathcal{U}_n = \{\bigcup \eta : \eta \subset \gamma_n\}$.

Proof: Fix $n \in \omega$. By Lemma 5.2, we can find a countable family $\mathcal{W}_n$ of subfamilies of γ_n satisfying the following condition.

(u) For any pair of finite subfamilies μ_1, μ_2 of γ_n such that $\mu_1 \cap \mu_2 = \emptyset$, there exists $\eta \in \mathcal{W}_n$ such that $\mu_1 \subset \eta$ and $\mu_2 \cap \eta = \emptyset$. Clearly, we can also assume that $\gamma_n \in \mathcal{W}_n$.

Put $\mathcal{E} = \{\bigcup \eta : \eta \in \mathcal{W}_n, n \in \omega\}$. Clearly, $\mathcal{E}$ is a countable family of subsets of X, and $\mathcal{E} \subset \bigcup\{\mathcal{U}_n : n \in \omega\}$.

It remains to show that $\mathcal{E}$ is a T_0-separator in X for (M, X).

Take any distinct points x, y such that $x \in M$ and $y \in X$. By (iii), there exists $k \in \omega$ such that $x \in \bigcup \gamma_k$, and no member of γ_k contains both x and y. If $y \notin \bigcup \gamma_k$, then we have nothing to prove, since $\bigcup \gamma_k \in \mathcal{E}$ and $x \in \bigcup \gamma_k$.

Let $y \in \bigcup \gamma_k$. Put $\mu_1 = \{V \in \gamma_k : x \in V\}$ and $\mu_2 = \{V \in \gamma_k : y \in V\}$. Both families μ_1, μ_2 are finite by (ii). It follows from (iii) that $\mu_1 \cap \mu_2 = \emptyset$. Hence, by condition (u), there exists $\eta \in \mathcal{W}_k$ such that $\mu_1 \subset \eta$ and $\mu_2 \cap \eta = \emptyset$. Put

$W = \bigcup \eta$. Then $W \in \mathcal{E}$, and $x \in \bigcup \mu_1 \subset W$. On the other hand, $y \notin W$, since no member of μ_2 is in η. Hence, $\mathcal{E}$ is a T_0-separator in X for (M, X). □

Usually, applying the last result we will assume that $M = X$.

Recall that a family γ of closed subsets of a space X is called *conservative* if the union of every subfamily of γ is closed. Proposition 5.1 obviously follows from the next more general statement which is a corollary to the last result:

Corollary 5.4. *Suppose that X is a space with a family $\mathcal{S} = \{\gamma_n : n \in \omega\}$ of families of closed subsets of X satisfying the following conditions for every $n \in \omega$:*

(i) $|\gamma_n| \leq 2^\omega$,
(ii) *γ_n is point-finite and conservative,*
(iii) *for any pair (x, y) of distinct points of X, there exists $k \in \omega$ such that $x \in \bigcup \gamma_k$, and no member of γ_k contains both x and y.*

Then:

(a) *X has a countable closed T_0-separator $\mathcal{E}$ in X for (X, X);*
(b) *if, in addition, X is an s-space, then every subspace of X is an s-space.*

Proof: Clearly, Proposition 5.3 is applicable. Under the assumptions made in 5.4, the countable T_0-separator $\mathcal{E}$ in X for (X, X) constructed in the proof of Proposition 5.3 is closed. □

Corollary 5.5. *Suppose that X is an s-space such that $|X| \leq 2^\omega$, and $\mathcal{S} = \{\gamma_n : n \in \omega\}$ is a family of families of closed subsets of X satisfying the following conditions for every $n \in \omega$:*

(i) *γ_n is point-finite and conservative,*
(ii) *for any $x \in X$ and any open neighbourhood Ox of x in X, there exists $k \in \omega$ such that $x \in \bigcup\{V \in \gamma_k : x \in V\} \subset Ox$.*

Then every subspace of X is an s-space and a p-space.

Proof: Proposition 4.2, and Corollary 5.4 imply that every subspace of X is an s-space. It follows from conditions (i) and (ii) that the space X is perfect. Hence, every subspace of X is perfect. By Theorem 3.1, every subspace of X is a p-space. □

Corollary 5.6. *Suppose that X is a space, M is a subset of X, and $\mathcal{S} = \{\gamma_n : n \in \omega\}$ is a family of point-finite families of open subsets of X satisfying the following conditions for every $n \in \omega$:*

(i) $|\gamma_n| \leq 2^\omega$,
(ii) *for any pair (x, y) of distinct points where $x \in M$ and $y \in X$, there exists $k \in \omega$ such that $x \in \bigcup \gamma_k$, and no member of γ_k contains both x and y.*

Then X has a countable open T_0-separator $\mathcal{E}$ in X for (M, X).

Proof: Clearly, Proposition 5.3 is applicable. Under the additional assumptions made in Corollary 5.6, the countable T_0-separator $\mathcal{E}$ in X for (M, X) constructed in the proof of Proposition 5.3 is open. □

We will now apply the above techniques to a class of spaces which is more wide than the class of spaces with a σ-disjoint base, and to a natural subclass of the class of spaces with a uniform base.

Let X be a space, and γ be a family of subsets of X. We will say that the *order of γ at $x \in X$ does not exceed $n \in \omega$* if x belongs to at most n members of γ. We will write in this case $ord_x(\gamma) \leq n$. If there exists $n \in \omega$ such that $ord_x(\gamma) \leq n$ for every $x \in X$, then we say that the *pointwise order of γ on X is uniformly bounded (by n)*.

Proposition 5.7. *Suppose that X is a space with a sequence $\mathcal{S} = \{\gamma_n : n \in \omega\}$ of families of open subsets of X satisfying the following conditions for every $n \in \omega$:*

(i) $|\gamma_n| \leq 2^\omega$,
(ii) *γ_n is of uniformly bounded pointwise order on X,*
(iii) *for any $x \in X$ and any open neighbourhood Ox of x, there exists $k \in \omega$ such that $x \in \bigcup \gamma_k$ and $St_{\gamma_k}(x) \subset Ox$.*

Furthermore, suppose that bX is a compactification of X.

Then X has a countable open T_0-separator $\mathcal{E}$ in bX for (X, bX).

Proof: Take any $n \in \omega$. For any $V \in \gamma_n$, fix an open subset U_V of bX such that $U_V \cap X = V$, and put $\xi_n = \{U_V : V \in \gamma_n\}$.

Claim 1: ξ_n is of uniformly bounded pointwise order on bX.

The pointwise order of γ_n is bounded by some $k \in \omega$. We claim that the pointwise order of ξ_n on bX is bounded by the same k.

Assume the contrary. Then there exist distinct $V(i) \in \gamma_n$, $i = 1, \dots, k+1$, such that $W = \bigcap\{U_{V(i)} : i = 1, \dots, k+1\}$ is nonempty. Since W is open in bX, and X is dense in bX, we have $X \cap W \neq \emptyset$. For any $x \in W \cap X$ we have: $x \in \bigcap\{V(i) : i = 1, \dots, k+1\}$, a contradiction with $ord_x(\gamma_n) \leq k$. Claim 1 is proved.

Claim 2: X in the role of M, bX in the role of X, and $\mathcal{S}_1 = \{\xi_n : n \in \omega\}$ in the role of $\mathcal{S}$ satisfy all assumptions in Corollary 5.6. Indeed, we have seen that each ξ_n is a point-finite family of open subsets of bX (Claim 1). Clearly, $|\xi_n| \leq |\gamma_n| \leq 2^\omega$, that is, (i) holds. Let us verify (iii). Take any $x \in X$, $y \in bX$ such that $x \neq y$, and fix an open neighbourhood Ox of x in bX such that y is not in the closure of Ox in bX. By (iii) in 5.7, there exists $k \in \omega$ such that $x \in \bigcup \gamma_k$ and $St_{\gamma_k}(x) \subset Ox$. Since V is dense in U_V, we have:

$$St_{\xi_k}(x) \subset \overline{St_{\gamma_k}(x)} \subset \overline{Ox},$$

where the closure is taken in bX. Since $y \notin \overline{Ox}$, we conclude that $y \notin St_{\xi_k}(x)$. On the other hand, $x \in \bigcup \gamma_k \subset \bigcup \xi_k$. Thus, (iii) in 5.6 holds.

Now, by Corollary 5.6, X has a countable open T_0-separator $\mathcal{E}$ in bX for (X, bX). □

The next statement easily follows from the results already obtained, so its proof is omitted.

Theorem 5.8. *Suppose that X is a space with a sequence $\mathcal{S} = \{\gamma_n : n \in \omega\}$ of families of open subsets of X satisfying the following conditions for every $n \in \omega$:*

(i) $|\gamma_n| \leq 2^\omega$,
(ii) *γ_n is of uniformly bounded pointwise order on X,*
(iii) *for any $x \in X$ and any open neighbourhood Ox of x, there exists $k \in \omega$ such that $x \in \bigcup \gamma_k$ and $St_{\gamma_k}(x) \subset Ox$.*

Furthermore, suppose that bX is a compactification of X.
Then:

(a) *X is a hereditarily s-space,*
(b) *the remainder $bX \setminus X$, as well as any remainder of any subspace of X, is a Lindelöf Σ-space,*
(c) *there exists a countable family $\mathcal{S}$ of open subsets of bX such that $\mathcal{S}$ is a source in bX for any subspace of X.*

We call below a space X *boundedly submetacompact* if every open covering of X can be refined by an open covering $\gamma = \bigcup\{\gamma_n : n \in \omega\}$, where each γ_n is a family of uniformly bounded pointwise order. In particular, every screenable space is boundedly metacompact. Recall that a space X is called *screenable*, if every open covering of X can be refined by a σ-disjoint open covering [17]. The next result immediately follows from the last theorem.

Corollary 5.9. *Every boundedly submetacompact Moore space X with $w(X) \leq 2^\omega$ is a hereditarily s-space. Hence, all remainders of X are Lindelöf Σ-spaces.*

Recall that a *σ-space* is a space with a σ-discrete network.

Theorem 5.10. *Suppose that E is a first-countable σ-space which is also an s-space. Furthermore, suppose that M is a subspace of E with $|M| \leq 2^\omega$. Then every subspace of M is an s-space.*

Proof: Let X be the closure of M in E. Clearly, X is a σ-space and s-space, and $|X| \leq 2^\omega$. By Corollary 5.5, every subspace of X is an s-space. Hence, M is a hereditarily s-space. □

Corollary 5.11. *If a Moore space M with $|M| \leq 2^\omega$ can be topologically embedded in a Moore s-space X, then M is a hereditarily s-space.*

In particular, we have:

Corollary 5.12. *If a Moore space M with $|M| \leq 2^\omega$ can be topologically embedded in a Čech-complete Moore space X, then every subspace of M is an s-space.*

Notice that a special case of Corollaries 5.12 and 5.11 is the statement [13] that every metrizable space M with $|M| \leq 2^\omega$ is a hereditarily s-space.

6. Separation of Lindelöf Σ-spaces by continuous maps, with applications to s-spaces

One of basic results on remainders is the theorem that every remainder of any Lindelöf p-space is a Lindelöf p-space [10]. Knowing this, it is natural to conjecture that every remainder of a Lindelöf Σ-space is a Lindelöf Σ-space. However, something almost opposite happens.

Theorem 6.1. *Suppose that X is a space, and X_1, X_2 are disjoint Lindelöf Σ-subspaces of X.*

Then there is a continuous mapping ϕ of X onto a separable metrizable space such that the sets $\phi(X_1)$ and $\phi(X_2)$ are disjoint.

Proof: We may assume that X is compact. Since X_1 and X_2 are Lindelöf Σ-spaces, and $X_1 \cap X_2 = \emptyset$, we can fix countable families η_1 and η_2 of compacta in X such that $\eta_1 \cup \eta_2$ T_0-separates X_1 from $X \setminus X_1$ and X_2 from $X \setminus X_2$. Put $\xi = \{\bigcap \lambda : \lambda \subset (\eta_1 \cup \eta_2), |\lambda| < \omega\}$, that is, ξ is the family of intersections of finite subfamilies of $\eta_1 \cup \eta_2$. Clearly, ξ is a countable family of compacta.

Claim: The family ξ is a Hausdorff separator for (X_1, X_2).

Assume the contrary, and fix $x_i \in X_i$, for $i = 1, 2$, such that x_1 and x_2 are not Hausdorff separated by ξ. Put $\xi_i = \{P \in \xi : x_i \in P\}$, for $i = 1, 2$, and let $\zeta = \{P_1 \cap P_2 : P_i \in \xi_i, i = 1, 2\}$. Since ξ, ξ_1, and ξ_2 are closed under intersections of finite subfamilies, the family ζ is centered. Since ζ consists of compacta, $\bigcap \zeta$ is nonempty. Pick $y \in \bigcap \zeta$. If $y \in X \setminus X_1$, then η_1 does not T_0-separate X_1 from $X \setminus X_1$, a contradiction. If $y \in X \setminus X_2$, then η_2 does not T_0-separate X_2 from $X \setminus X_2$, a contradiction. The Claim is proved.

Denote by $\mathcal{P}$ the set of all disjoint pairs of members of ξ. For each $(P, H) \in \mathcal{P}$, fix a continuous real-valued function $f_{(P,H)} : B \to R$ such that $f_{(P,H)}(P) \subset \{0\}$ and $f_{(P,H)}(H) \subset \{1\}$. This is possible, since members of $\mathcal{P}$ are compact.

For the diagonal product ϕ of the family $\{f_{(P,H)} : (P, H) \in \mathcal{P}\}$, clearly, we have: $\phi(X_1) \cap \phi(X_2) = \emptyset$. Since $\mathcal{P}$ is countable, the space $R^{\mathcal{P}}$ is homeomorphic to R^ω. Hence, $\phi(X)$ is separable and metrizable. □

We say below that a mapping $f : X \to R^\omega$ *cleaves X along a subset Y of X* if $f^{-1}(f(Y)) = Y$.

Theorem 6.2. *Suppose that X is a Lindelöf p-space, and that Lindelöf Σ-spaces X_1, X_2 are disjoint subspaces of X.*

Then there exists a subspace Y of X such that $X_1 \subset Y$, Y is a Lindelöf p-space, $Y \cap X_2 = \emptyset$, $X \setminus Y$ is a Lindelöf p-space, and there exists a perfect mapping $\phi : X \to R^\omega$ such that $\phi^{-1}(\phi(Y)) = Y$, i.e. ϕ cleaves X along Y.

Proof: Since X is a Lindelöf p-space, there is a perfect mapping g of X onto a separable metrizable space L. By Theorem 6.1, there is a continuous mapping ψ of X onto a separable metrizable space Z such that the sets $\psi(X_1)$ and $\psi(X_2)$ are disjoint. Then the diagonal product ϕ of the mappings g and ψ is a perfect mapping of X onto a separable metrizable space M such that the sets $\phi(X_1)$ and $\phi(X_2)$ are disjoint.

Put $Y = \phi^{-1}(M \setminus \phi(X_2))$. Obviously, $X_1 \subset Y$. Since the mapping ϕ is perfect, Y is a Lindelöf p-space. Clearly, $\phi^{-1}(\phi(Y)) = Y$. Therefore, $\phi^{-1}(\phi(X \setminus Y)) = X \setminus Y$, which implies that $X \setminus Y$ is a Lindelöf p-space as well. □

Corollary 6.3. *Suppose that X is a Lindelöf p-space and $X = X_1 \cup X_2$, where X_1 and X_2 are disjoint Lindelöf Σ-spaces. Then:*

(a) *X is cleavable (over R^ω) along X_i, for $i = 1, 2$, by the same perfect mapping;*
(b) *X_1 and X_2 are p-spaces.*

Corollary 6.4. *Suppose that X is a Lindelöf p-space and $X = X_1 \cup X_2$, where X_1 and X_2 are disjoint Borelian subsets of X of the first type (i.e. $X_1, X_2 \in Bor_1(X)$). Then X_1 and X_2 are Lindelöf p-spaces.*

The next version of corollary 6.3 contains one more assumption.

Corollary 6.5. *Suppose that B is a compact space such that $B = Y \cup Z$, where the subspaces Y and Z satisfy the following conditions:*

(1) *Y and Z are dense in B,*
(2) *Y and Z are disjoint,*
(3) *Y and Z are s-spaces.*

Then Y and Z are Lindelöf p-spaces.

Proof: Since Z is a remainder of Y and Y is an s-space, Theorem 2.7 implies that Z is a Lindelöf Σ-space. Similarly, Y is a Lindelöf Σ-space. By Corollary 6.3(b), Y and Z are Lindelöf p-spaces. □

Since every p-space with a countable network has a countable base [7], it follows from Corollary 6.3 that if a space X with a countable network has a compactification bX such that $bX \setminus X$ is a Lindelöf Σ-space, then X has a countable base. Hence, we have:

Corollary 6.6. *If an s-space X has a countable network, then X has a countable base.*

In the next statement an important concept is decomposed in two components:

Corollary 6.7. *A space X is a Lindelöf p-space if and only if it is a Lindelöf Σ-space and an s-space.*

Example 6.8. Theorem 6.2 does not generalize to the case of disjoint Lindelöf subspaces. This is witnessed by the famous "double arrow" compactum X: the arrows cannot be separated by a continuous mapping of X onto a metrizable space, since the "arrows" are not p-spaces.

Theorem 6.2 has a generalization which resembles the statement: if a compact space X is the union of a countable family γ of spaces with a countable network, then each member of γ is separable and metrizable.

Theorem 6.9. *Suppose that X is a Lindelöf p-space, and that $X = \bigcup\{X_n : n \in \omega\}$, where $\gamma = \{X_n : n \in \omega\}$ is a countable disjoint family of Lindelöf Σ-spaces. Then:*

(a) *there exists a continuous mapping of bX to R^ω that cleaves X along X_n for every $n \in \omega$ simultaneously;*

(b) *X_n is a p-space, for each $n \in \omega$; even more, for every $A \subset \omega$, $\bigcup\{X_n : n \in \omega\}$ is a Lindelöf p-space.*

Proof: Let $B = bX$ be any compactification of X. Observe that the remainder $B \setminus X$ of X in B is a Lindelöf p-space, since X is a Lindelöf p-space [10]. For each $k \in \omega$, put $Y_k = \bigcup\{X_n : n \in \omega, n \neq k\} \cup (B \setminus X)$. Then $B = X_k \cup Y_k$, and X_k, Y_k are disjoint. Observe that Y_k is a Lindelöf Σ-space, as the union of a countable family of Lindelöf Σ-spaces. Therefore, by Theorem 6.2, X_k is a p-space, and we can fix a continuous mapping f_k of bX to R^ω such that $X_k = f_k^{-1}(f_k(X_k))$, for $k \in \omega$. The diagonal product f of the family $\{f_k : k \in \omega\}$ is a continuous mapping of B to R^ω that cleaves X along X_k, for every $k \in \omega$. Conclusion (b), obviously, also holds. □

7. Hereditarily s-spaces, compactness, and cardinal invariants

The next theorem shows that p-spaces and s-spaces have a strong property in common:

Theorem 7.1. *If X is an s-space, then $w(X) = nw(X)$.*

Proof: Put $\tau = nw(X)$. We have to show that $w(X) \leq \tau$. Fix a compactification bX of X. By Theorem 2.7, the remainder $Y = bX \setminus X$ is a Lindelöf Σ-space. Therefore, we can fix a countable closed source $\mathcal{S}$ for Y in bX. Since $\tau = nw(X)$, we can also fix a closed source $\mathcal{S}_1$ for X in bX such that $|\mathcal{S}_1| \leq \tau$. Put $\mathcal{F} = \mathcal{S} \cup \mathcal{S}_1$, and let ξ be the family of intersections of arbitrary finite subfamilies of $\mathcal{F}$. Then ξ is a family of compact subspaces of bX, and $|\mathcal{F}| \leq \tau$.

Claim: The family ξ is a Hausdorff separator for (X, Y).

The proof of the Claim and the rest of the proof of the theorem are practically the same as in the proof of Theorem 6.2. □

Corollary 7.2. *If an s-space X is the union of a countable family of separable metrizable spaces, then X is also separable and metrizable.*

Corollary 7.3. *If $f : X \to Y$ is a continuous mapping of a space X onto an s-space Y, then $w(Y) \leq w(X)$.*

If X is a space such that every subspace of X is a Lindelöf p-space, then X has a countable base (see [22], [8]). It is not unnatural to ask at this point whether every compact hereditarily s-space is metrizable. The answer is "no".

Example 7.4. Let B be the Alexandroff compactification by one point of an uncountable discrete space. Thus, $B = A \cup \{p\}$, where the subspace A is discrete. Every subspace Y of B is either compact or discrete. Hence, B is hereditarily

Čech-complete. By Proposition 2.1, this implies that B is a hereditarily s-space. However, B is neither metrizable, nor first-countable.

The last example is complemented by the next result:

Proposition 7.5. *Every hereditarily s-space X is Fréchet-Urysohn.*

PROOF: Every subspace of X is of countable type, by Proposition 2.11. Hence, every subspace of X is a k-space. However, every hereditarily k-space is Fréchet-Urysohn [6]. □

Remark. We could refer above to a more general theorem of G. Grabner and A. Szymanski proved by the same argument in [20]: if a space X is hereditarily of point-countable type, then X is Fréchet-Urysohn.

Theorem 7.6. *If a topological group G is a hereditarily s-space, then G is metrizable.*

PROOF: There exists a nonempty compact subspace F of G with a countable base of open neighbourhoods in G, by Proposition 2.11. Now, in a standard way, we can find a compact subgroup H of G such that $H \subset F$ and H has a countable base of open neighbourhoods in G. By Proposition 7.5, G and H are Fréchet-Urysohn spaces. Therefore, H is metrizable, since every compact topological group of countable tightness is metrizable [9]. Since H has a countable base of open neighbourhoods in G, the space G is first-countable [4]. Hence, G is metrizable, since G is a topological group. □

The situation with compact hereditarily s-spaces changes drastically when we add one more natural restriction. We call a space X *dense-in-itself* if none of the points of X is isolated.

Theorem 7.7. *Suppose that B is a dense-in-itself compact space such that every subspace of B is an s-space. Then B is separable metrizable.*

This theorem follows from a more general result obtained below after a few simple steps.

Theorem 7.8. *Suppose that an s-space X is a dense subspace of a Lindelöf Σ-space Z. Then the subspace $Y = Z \setminus X$ is a Lindelöf Σ-space as well.*

PROOF: Fix a compactification bZ of Z. Clearly, bZ is also a compactification of X. By Theorem 2.7, the remainder $Y_1 = bZ \setminus X$ is a Lindelöf Σ-space. We have: $Y = Z \cap Y_1$. Therefore, Y is a Lindelöf Σ-space, since Z and Y_1 are Lindelöf Σ-spaces. □

Proposition 7.9. *Every subset Y of a dense-in-itself hereditarily s-space X can be represented as the union of two subsets with empty interiors.*

PROOF: If $U = Int(Y) = \emptyset$, then we have nothing to prove. So we assume that U is nonempty. By Proposition 7.5, X and U are Fréchet-Urysohn spaces. Clearly, U is dense-in-itself. By a theorem of N.V. Veličko from [26, Theorem 2],

$U = A_1 \cup A_2$, where $A_1 \cap A_2 = \emptyset$ and the interiors of A_1 and A_2 are empty. Put $Y_1 = (Y \setminus U) \cup A_1$ and $Y_2 = (Y \setminus U) \cup A_2$. It follows from $U = Int(Y)$ that $Int(Y_1) = \emptyset = Int(Y_2)$ and $Y = Y_1 \cup Y_2$. □

Theorem 7.10. *Suppose that X is a dense-in-itself space satisfying the following conditions:*

(a) *X is a Lindelöf Σ-space;*
(b) *X is a hereditarily s-space.*

Then X has a countable base.

Proof: *Claim 1*: Every subspace Z of X is a Lindelöf Σ-space.

By Proposition 7.9, we can assume that $Int(Z) = \emptyset$, since the union of two Lindelöf Σ-spaces is a Lindelöf Σ-space. Then the subspace $Y = X \setminus Z$ is dense in X. Since Y is an s-space and X is a Lindelöf Σ-space, it follows that $Z = X \setminus Y$ is also a Lindelöf Σ-space, by Theorem 7.8. Claim 1 is proved.

By a theorem of R. Hodel [22], it follows from Claim 1 that X has a countable network. Hence, by Theorem 7.1, X has a countable base. □

Of course, assumption (a) in the last theorem cannot be dropped, since every metrizable space with $w(X) \leq 2^\omega$ is a hereditarily s-space. However, E.G. Pytkeev has shown in [25] that the next statement holds (see also [20]): if a dense-in-itself space X is a hereditarily s-space, then it is first-countable. Hence, we have:

Theorem 7.11. *Every dense-in-itself hereditarily s-space is first-countable.*

The next result directly follows from Theorems 7.10 and 5.8:

Theorem 7.12. *Suppose that X is a dense-in-itself Lindelöf Σ-space with a sequence $\mathcal{S} = \{\gamma_n : n \in \omega\}$ of families of open subsets of X satisfying the following conditions for every $n \in \omega$:*

(i) $|\gamma_n| \leq 2^\omega$,
(ii) *γ_n is of uniformly bounded pointwise order on X,*
(iii) *for any $x \in X$ and any open neighbourhood Ox of x, there exists $k \in \omega$ such that $x \in St_{\gamma_k}(x) \subset Ox$.*

Then X is a separable metrizable space.

In connection with the last two theorems, it is appropriate to consider the next example:

Example 7.13. Let $\mathcal{R}_M$ be the Michael line, that is, $\mathcal{R}_M$ is the set of reals with the Michael line topology $\mathcal{T}_M$. We also take the discrete space $D = \{0, 1\}$. Below $\mathcal{Q}$ is the set of rationals, and $\mathcal{P}$ is the set of irrationals.

In the product space $X = \mathcal{R}_M \times D$ we identify $(q, 0)$ with $(q, 1)$, for every rational number q. The resulting quotient mapping and quotient space are denoted by f and Y, respectively. Then the mapping f is perfect, $|X| = 2^\omega$, and X has a σ-disjoint base. Therefore, the product mapping f^ω of X^ω onto Y^ω is also perfect, X^ω has a σ-disjoint base, and $|X^\omega| = 2^\omega$. It follows that X^ω and Y^ω are hereditarily s-spaces. Now observe that none of the spaces X^ω, Y^ω has isolated points, and that both of them are non-metrizable. In fact, the space Y^ω is not even submetrizable. This was established in [24], where X, f, and Y were defined by V.V. Popov to show that submetrizability is not preserved in general by perfect mappings.

We also have the following characterization of separable metrizable spaces:

Theorem 7.14. *A space X has a countable base if and only if every subspace of X is a Lindelöf s-space.*

Proof: Suppose that every subspace of X is a Lindelöf s-space. Then X is perfect. By Proposition 2.1, every subspace of X is a p-space. Therefore, X has a countable base, since every subspace of X is a Lindelöf p-space (see [8], [22]). □

Another class of hereditarily s-spaces is identified in the next statement:

Theorem 7.15. *Every σ-discrete metrizable space X is a hereditarily s-space.*

Proof: Fix a Čech-complete metrizable space Z such that X is a dense subspace of Z. Fix also a compactification bZ of Z. Then bZ is also a compactification of X. Now let A be any discrete subspace of X (we do not assume that A is closed in X).

Claim 1: A is a G_δ-subset of bZ.

We denote by F the closure of A in bZ. Clearly, A is open in F, since A is discrete. Hence, there exists an open subset U of bZ such that $U \cap F = A$. The set $P = F \cap Z$ is closed in Z. Since Z is perfect, $F \cap Z = Z \cap (\bigcap\{V_n : n \in \omega\})$, for some countable family $\eta = \{V_n : n \in \omega\}$ of open subsets of bZ. Finally, since Z is a G_δ-set in bZ, we have $Z = \bigcap\{W_n : n \in \omega\}$, where W_n is open in bZ. Then $U \cap (\bigcap\{V_n : n \in \omega\}) \cap (\bigcap\{W_n : n \in \omega\}) = A$. Claim 1 is verified.

It follows from Claim 1 that every σ-discrete subspace of X is the union of a countable family of G_δ-subsets of bZ. Thus, X has a countable open source in bZ, which implies that X is an s-space. Since every subspace of X is also metrizable and σ-discrete, we conclude that X is a hereditarily s-space. □

Corollary 7.16. *Every metrizable space has a dense hereditarily s-subspace.*

Using the last statement, it is easy to see that a dense-in-itself hereditarily s-space may have as large cardinality as we wish.

Problem 7.17. *Suppose that X is a metacompact Moore space such that $w(X) \leq 2^\omega$. Does it follow that X is an s-space? Equivalently, are remainders of X Lindelöf Σ-spaces?*

Even the answer to the following question is unknown:

Problem 7.18. *Suppose that X is a separable Moore space. Does it follow that X is an s-space? Equivalently, are remainders of X Lindelöf Σ-spaces?*

Problem 7.19. *Is every Moore space X with $w(X) \leq 2^\omega$ an s-space?*

Problem 7.20. *Is every separable first-countable σ-space X an s-space?*

Problem 7.21. *Is every first-countable σ-space X with $w(X) \leq 2^\omega$ an s-space?*

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