Topology on ordered fields

Yoshio Tanaka

Abstract. An ordered field is a field which has a linear order and the order topology by this order. For a subfield F of an ordered field, we give characterizations for F to be Dedekind-complete or Archimedean in terms of the order topology and the subspace topology on F.

Keywords: order topology, subspace topology, ordered field, Archimedes' axiom, axiom of continuity

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1. Preliminaries

Let X be a set linearly ordered (or totally ordered) by $\leq$. Then X is called a *linearly ordered topological space* (or *LOTS*) if X has *the order topology* (or *interval topology*) by $\leq$; that is, the topology has a base $\{(\alpha,\beta) : \alpha,\beta \in X\}$, where $(\alpha,\beta) = \{x \in X : \alpha < x < \beta\}$; see [1] etc. As is well-known, every LOTS is normal. For $A \subset X$, A is called a *subspace* of the LOTS X when A has the *subspace topology* (*relative topology, or induced topology*) from X; that is, the topology has a base $\{(\alpha,\beta) \cap A : \alpha,\beta \in X\}$.

Let X be a LOTS with a (linear) order $\leq$. For $A \subset X$, let $\leq_A$ be the restriction of the order $\leq$ to A. Then the order topology on A by $\leq_A$ is coarser than the subspace topology on A. The order topology need not coincide with the subspace topology ([2, 3Q], [3, Remark 3.2], etc.).

For a subset A of a space X, we say that A is *compact*; *connected*; and *discrete in* X if so is A respectively as a subspace of X. Also, A is *closed discrete* in X if A is closed and discrete in X (equivalently, any subset of A is closed in X). For $p \in X$, p is an *accumulation point* of A in X if $p \in \mathrm{cl}(A - \{p\})$. Also, A is *dense* in X if $\mathrm{cl}\, A = X$.

Now, let $\mathbb{R}$; $\mathbb{Q}$; and $\mathbb{N}$ be the usual real number field; rational number field; and the set of natural numbers, respectively.

Let $G = (G,+)$ be an Abelian group (i.e., commutative group which is additive). Let us say that G is an *ordered additive group* ([3], [5]) if G has a linear order $\leq$ such that the order is preserving with respect to addition (i.e., for $a < b$, $a + x < b + x$), and G has the order topology by the order $\leq$. For $x \in G$, define $|x| \in G$ by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$. Then, for $x,y \in G$, $|x+y| \leq |x| + |y|$ holds. For a commutative field $K = (K,+,\times)$ with a linear order $\leq$, we say that K is an *ordered field* if K is an ordered additive group, and the order $\leq$ is moreover preserving with respect to multiplication (i.e., for $a < b$ and $0 < x$, $a \times x < b \times x$). For an ordered field K, K contains a subfield which is isomorphic to $\mathbb{Q}$, so we assume $K \supset \mathbb{Q}$.

Remark 1.1. Obviously, any ordered field has no isolated points. Also, for an ordered additive group G, G has no isolated points iff it is not discrete by the homogeneity of G ([3]).

Let $(K, \leq)$ be an (algebraic) ordered field. A pair $(A|B)$ of non-empty subsets A and B in K is a (Dedekind) *cut* if $K = A \bigcup B$, $A \bigcap B = \emptyset$, and for any $x \in A$, $y \in B$, $x < y$. We recall the following classical *Archimedes' axiom*, and the *axiom of continuity* which is stronger than Archimedes' axiom.

- *Archimedes' axiom*: For each $\alpha, \beta \in K$ with $0 < \alpha < \beta$, there exists $n \in \mathbb{N}$ with $\beta < n\alpha$ (equivalently, for each $\alpha \in K$, there exists $n \in \mathbb{N}$ with $\alpha < n$).
- *Axiom of continuity*: For each cut $(A|B)$ in K, there exists one of $\max A$ and $\min B$ (equivalently, there exists $\max A$ or $\min B$).

An (algebraic) ordered field is *Archimedean*; *Dedekind-complete* if it satisfies Archimedes' axiom; the Axiom of continuity, respectively. The ordered field $\mathbb{Q}$ is Archimedean, but not Dedekind-complete.

For fields (or rings) K and K', $f : K \to K'$ is a *homomorphism* if $f(x+y) = f(x) + f(y), f(xy) = f(x)f(y)$, and $f(1) = 1'$, where 1; $1'$ is the unit in K; K', respectively. Then, a homomorphism is an *isomorphism* if it is a bijection.

For ordered fields $(K, \leq)$ and $(K', \leq')$, $f : (K, \leq) \to (K', \leq')$ is *order-preserving* if for $x < y$, $f(x) <' f(y)$. A homomorphism f is order-preserving iff for $0 < x$, $0 <' f(x)$. The following is well-known; see [2] etc.

Remark 1.2. (1) Any homomorphism from a field is injective.

(2) Let $f : \mathbb{R} \to (K, \leq)$ be a homomorphism. Then f is order-preserving.

(3) For an ordered field K, K is Archimedean iff it is order-preserving isomorphic to a subfield of $\mathbb{R}$; in particular, K is Dedekind-complete iff it is (order-preserving) isomorphic to $\mathbb{R}$.

We assume that spaces are Hausdorff. Let us use the following abbreviated notations in this paper.

Notations. X means a LOTS having an order $\leq$, unless otherwise stated. $A \subset X$ means that the set A has the order $\leq_A$. When X is an ordered field; ordered additive group, we use the symbol K; G respectively, instead of X. A field $A \subset K$ means that A is a subfield of K which has the order $\leq_A$, and also the same meaning for an additive group $A \subset G$.

For $A \subset X$, A^* means a space having the order topology by $\leq_A$. Clearly, A^* is a subspace of X iff the order topology on A coincides with the subspace topology.

$\mathbf{L} \subset G$ means an infinite decreasing sequence having a lower bound 0 in G, and let $\mathbf{L}_0 = \mathbf{L} \cup \{0\}$. In particular, for the decreasing sequence $\{1/n : n \in \mathbb{N}\}$ in K, let $\mathbf{S} = \{1/n : n \in \mathbb{N}\}$ and $\mathbf{S}_0 = \mathbf{S} \cup \{0\}$.

Remark 1.3. If A is compact or connected in X, then A^* is a subspace of X, as is well-known. While, if $A = \mathbb{Q}$, $\mathbb{R}$, or $\mathbf{S}_0 \subset K$, then A^* is the usual subspace in $\mathbb{R}$, so we may put $A^* = A$ (but, A^* need not be a subspace of K; see Theorem 2.2, Example 3.1, or Example 3.3 later). Indeed, this is shown by a well-known fact that $\mathbb{Q}$ and $\mathbb{R}$ have the usual order which is unique as an ordered field, and so does $\mathbf{S}_0$ as a subset of an ordered field, because the set of integers has the unique usual order as an ordered ring. (Every ordered field in $\mathbb{R}$ need not have the unique order; see Example 3.2.)

Remark 1.4. (1) Let $\mathbf{L} \subset G$. Then $\mathbf{L}^*$ is a discrete space (equivalently, discrete subspace of G), but $\mathbf{L}$ need not be closed in G. While, $\mathbf{L}_0^*$ is a compact space, but $\mathbf{L}_0$ need not be compact in G.

(2) For $\mathbf{L} \subset G$, $\mathbf{L}_0$ is compact in $G \Leftrightarrow \mathbf{L}$ converges to 0 in $G \Leftrightarrow \operatorname{cl} \mathbf{L} = \mathbf{L}_0$ in $G \Leftrightarrow \mathbf{L}_0^*$ is a (compact) subspace of G. Also, $\operatorname{cl} \mathbf{L}$ is compact in $G \Leftrightarrow \mathbf{L}$ converges to a point in $G \Leftrightarrow \mathbf{L}$ is not closed (discrete) in G.

2. Results

Theorem 2.1. *For an additive group $A \subset G$, if A^* is not discrete, then the following are equivalent.*

(a) *A^* is a subspace of G.*
(b) *A is not closed discrete in G.*
(c) *Any point of A is an accumulation point of A in G.*
(d) *Some point of G is an accumulation point of A in G.*

Proof: For (a)$\Rightarrow$(c), by Remark 1.1 any point of A is an accumulation point of A in A^*, hence in G. (c)$\Rightarrow$(d) is clear, and (b)$\Leftrightarrow$(d) is obvious. For (d)$\Rightarrow$(a), it suffices to show that the subspace topology is coarser than the order topology on A. To see this, let $H = (\alpha, \beta) \cap A$ with $\alpha, \beta \in G$, and let $\gamma \in H$. Let $\delta = \min\{\gamma - \alpha, \beta - \gamma\} > 0$. Let p be an accumulation point of A in G. Then there exist distinct points a, b in A such that $0 < \delta_0 = |a - p| < \delta$, and $0 < |b - p| < \delta - \delta_0$. Put $\sigma = |a - b| > 0$. Then $\sigma \in A$ (thus, $\gamma - \sigma, \gamma + \sigma \in A$), and $\sigma < \delta$ since $\sigma \le |a - p| + |b - p| < \delta$. Let $T = (\gamma - \sigma, \gamma + \sigma)$ be the open interval in A. Then T is an open subset of A^* with $\gamma \in T \subset H$. Hence H is open in A^*. □

Corollary 2.1. *For $A \subset G$, if A is dense in G, A^* is a subspace of G.*

Proof: If A is closed in G, then $A = G$, so let A be not closed in G. Then any interval (α, β) in G is not empty. Indeed, A has an accumulation point in G. Thus, for $\delta = \beta - \alpha > 0$, there exists $\delta_0 \in G$ with $0 < \delta_0 < \delta$. Then $\alpha + \delta_0 \in (\alpha, \beta)$. Thus, for $\gamma \in (\alpha, \beta) \cap A$, we can take $\gamma_1 \in (\alpha, \gamma) \cap A$, and $\gamma_2 \in (\gamma, \beta) \cap A$. Then the open interval $T = (\gamma_1, \gamma_2)$ in A satisfies $\gamma \in T \subset (\alpha, \beta) \cap A$. Hence, A^* is a subspace of G. □

Remark 2.1. (1) If A in Theorem 2.1, or G in Corollary 2.1 is a space, then the result need not hold. Indeed, let $A_0 = [0, 1) \cup [2, 3] \subset \mathbb{R}$, and $A_1 = A_0 \cup \{1\} \subset \mathbb{R}$. Then any point of A_0 is an accumulation point of

A_0 in $\mathbb{R}$, and A_0 is dense in A_1^*. But A_0^* is not a subspace of the space $\mathbb{R}$ or A_1^*.

(2) For a field $A \subset K$, the converse of Corollary 2.1 need not hold (thus, (c) in Theorem 2.1 need not imply that A is dense in G); see Example 3.1.

Theorem 2.2. (1) *The following are equivalent for K (we can omit the parenthetic parts in* (d), (e), *and* (f)).

(a) *K is Archimedean.*
(b) *$\mathbb{Q}$ is dense in K.*
(c) *$\mathbb{Q}$ is a subspace of K.*
(d) *For any field $F \subset K$, F^* is a (dense) subspace of K.*
(e) *For some Archimedean ordered field $F \subset K$, F^* is a (dense) subspace of K.*
(f) *$\mathbf{S}_0$ is a (compact) subspace of K.*

(2) *The following are equivalent for K.*

(a) *K is not Archimedean.*
(b) *$\mathbb{Q}$ is closed discrete in K.*
(c) *Some field $F \subset K$ is closed discrete in K.*
(d) *Any Archimedean ordered field $F \subset K$ is closed discrete in K.*
(e) *$\mathbf{S}_0$ (or $\mathbf{S}$) is closed discrete in K.*

Proof: (2) holds in view of (1) and Theorem 2.1, so we show (1) holds. (a)$\Leftrightarrow$(b) is well-known. We will show the implication (a)$\Rightarrow$(d)$\Rightarrow$(c)$\Rightarrow$(e)$\Rightarrow$(a)$\Leftrightarrow$(f) holds. (d)$\Rightarrow$(c)$\Rightarrow$(e) is obvious. For (a)$\Rightarrow$(d), $\mathbb{Q}$ is dense in K. Thus, F is dense in K. Hence F^* is a (dense) subspace of K by Corollary 2.1. For (e)$\Rightarrow$(a), $\mathbb{Q}$ is dense in F, thus it is a subspace of F^* by Corollary 2.1. While, F^* is a subspace of K. Thus, $\mathbb{Q}$ is a subspace of K. Hence, $\mathbb{Q}$ has an accumulation point in K. Thus, for each $\epsilon > 0$ in K, there exist $p, q \in \mathbb{Q}$ such that $0 < |p-q| < \epsilon$. But, $1/k < |p-q|$ for some $k \in \mathbb{N}$. Then $1/k < \epsilon$. This shows that K is Archimedean. For (a)$\Leftrightarrow$(f), K is Archimedean iff $\mathbf{S}_0$ is compact in K ([4]). Thus the equivalence holds by Remark 1.3. □

Corollary 2.2. (1) *For $\mathbb{Q} \subset K$, $\mathbb{Q}$ is a (dense) subspace of K, or $\mathbb{Q}$ is closed discrete in K.*

(2) *For $\mathbb{R} \subset K$, $K = \mathbb{R}$, or $\mathbb{R}$ is closed discrete in K.*

Proof: (1) holds by Theorem 2.2. For (2), if K is not Archimedean, then $\mathbb{R}$ is closed discrete in K by Theorem 2.2(2). So, let K be Archimedean. Then, $\mathbb{R}$ is a dense subspace of K by Theorem 2.2(1). To show that $\mathbb{R}$ is closed in K, let $p \in \operatorname{cl} \mathbb{R}$. Since K is Archimedean, there exists an infinite sequence L in $\mathbb{R}$ converging to the point p in K by Remark 2.4(2) later. Since L is a Cauchy sequence in $\mathbb{R}$, L converges to a point q in $\mathbb{R}$. But, $\mathbb{R}$ is a subspace of K, hence $p = q \in \mathbb{R}$. Then, $\mathbb{R}$ is closed in K. Thus, $K = \mathbb{R}$ since $\mathbb{R}$ is dense in K. □

Remark 2.2. Related to Theorem 2.2; Corollary 2.2, the following (1); (2) holds respectively in view of Example 3.1.

(1) For some non-Archimedean ordered field K, there exist non-Archimedean ordered fields $K_1, K_2 \subset K$ such that K_1^* is not a subspace of K (equivalently, K_1 is closed discrete in K), while K_2^* is a subspace of K which is not dense in K.

(2) For each ordered field F (in particular, $F = \mathbb{R}$), there exists a non-Archimedean ordered field K such that $F \subset K$ is closed discrete in K.

For spaces X, X', $f : X \to X'$ is *continuous* if $f^{-1}(G)$ is an open subset in X for any open subset G in X'. $f : X \to X'$ is a *homeomorphism* if it a bijection, and f and f^{-1} are continuous. $f : X \to X'$ is a *homeomorphic embedding* if $f : X \to f(X)$ is a homeomorphism to a subspace $f(X)$ of X'.

Remark 2.3. For $f : X \to X'$, if we take the subspace topology on $f(X) \subset X'$ and $A \subset X$, the following holds: if $f : X \to f(X)$ is continuous, then so is $f : X \to X'$ (the converse also holds), and the restriction $f|A : A \to X'$ is also continuous. However, if we take the order topology, the above need not hold. Indeed, for a non-Archimedean ordered field K, the identity map $1_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{Q}$ (resp. $1_K : K \to K$) is continuous, but the inclusion map (resp. restriction) $i_{\mathbb{Q}} : \mathbb{Q} \to K$ is not continuous, because the range $\mathbb{Q} \subset K$ is closed discrete in K by Theorem 2.2(2), but the domain $\mathbb{Q}$ has no isolated points as an ordered field. Here, we can replace "$\mathbb{Q}$" by any ordered field "F", but use the non-Archimedean ordered field K in Example 3.1, where F is closed discrete in K.

Theorem 2.3. *Let $f : (K, \leq) \to (K', \leq')$ be a homomorphism, and let $F = f(K) \subset K'$. If K is Archimedean, then the following are equivalent.*

(a) *f is continuous.*

(b) *f is a homeomorphic embedding.*

(c) *f is order-preserving, and F^* is a subspace of K'.*

(d) *f is order-preserving, and K' is Archimedean.*

Proof: For (a)$\Rightarrow$(d), since f is a homomorphism (hence, injection by Remark 1.2(1)), for each $n/m \in \mathbb{Q}$, $f(n/m) = n1'/m1'$, thus f is order-preserving on $\mathbb{Q}$. To see f is order-preserving, let $p < q$. Suppose $f(q) <' f(p)$ in K'. Since f is continuous, there exist disjoint open intervals $I_p \ni p$ and $I_q \ni q$ such that any element of $f(I_p)$ is larger than any element of $f(I_q)$. Since K is Archimedean, $\mathbb{Q}$ is dense in K, so take $r_p \in I_p \cap \mathbb{Q}$ and $r_q \in I_q \cap \mathbb{Q}$ such that $r_p < r_q$. Thus $f(r_p) <' f(r_q)$. This is a contradiction. Hence, f is order-preserving. Thus, obviously the field $F \subset K'$ is Archimedean. Suppose K' is not Archimedean. Then F is closed discrete in K' by Theorem 2.2(2). Thus, for $p \in F$, there exists a neighborhood $V(p)$ in K' with $V(p) \cap F = \{p\}$. Thus, $f^{-1}(V(p))(= f^{-1}(p))$ is an isolated point in K since f is injective and continuous. This is a contradiction, for any ordered field has no isolated points by Remark 1.1. Hence, K' is Archimedean. (d)$\Rightarrow$(c) holds by Theorem 2.2(1). The implication (c)$\Rightarrow$(b)$\Rightarrow$(a) is obvious, for F^* is a subspace of K'. □

Remark 2.4. (1) In Theorem 2.3, we cannot delete (*) "F^* is a subspace of K'" in (c); and "K' is Archimedean" in (d), in view of Remark 2.3

(or Example 3.3). While, (a) implies the property (*) in (c) without Archimedes' axiom of K, using Theorem 2.1.

(2) In view of Theorem 2.3 and Remark 1.2(3), K is Archimedean iff K admits an isomorphic and homeomorphic map from K to an ordered field $F \subset \mathbb{R}$ (which is also a subspace of $\mathbb{R}$); in particular, K is Dedekind-complete iff $F = \mathbb{R}$. But, every ordered field isomorphic to a subfield of $\mathbb{R}$ need not be Archimedean; see Example 3.2. Also, every ordered field homeomorphic to a subspace of $\mathbb{R}$ need not be Archimedean. Indeed, take a countable, non-Archimedean ordered field K (as $K = \mathbb{Q}(x)$ in Example 3.1). Since K is countable, it has the obvious countable base, thus K is separable metrizable, as is well-known. Thus, the LOTS K is homeomorphic to a subspace of $\mathbb{R}$ by [1, 6.3.2(c)].

Corollary 2.3. *Let $f : K \to K'$ be a homomorphism with $f(K) = K'$. If K is Archimedean, then the following are equivalent.*

(a) *f is continuous.*
(b) *f is a homeomorphism.*
(c) *f is order-preserving.*

Theorem 2.4. *The following are equivalent for K.*

(a) *K is Dedekind-complete.*
(b) *K is homeomorphic to $\mathbb{R}$ (or, K is a continuous image of $\mathbb{R}$).*
(c) *Some field $F \subset K$ is isomorphic to $\mathbb{R}$, and K is Lindelöf (i.e., every open cover of K has a countable subcover).*
(d) *Some field $S \subset K$ is isomorphic to $\mathbb{R}$, and S^* is a subspace of K.*
(e) *Some subset A of K with $|A| \geq 2$ is connected in K.*
(f) *Some (or any) closed interval $[a, b]$ $(a < b)$ in K is compact in K.*
(g) *For any decreasing sequence L in K having a lower bound, L has a limit point in K.*
(h) *For any $\mathbf{L} \subset K$, $\mathrm{cl}\,\mathbf{L}$ is compact in K.*

Proof: (a), (e), and (f) are equivalent (see [4], [6], etc.). (a)$\Leftrightarrow$ (g) is well-known. (g)$\Leftrightarrow$ (h) holds by Remark 1.4(2), here K is an ordered field, so we can put $L = \mathbf{L}$ in (g). We show the implication (a)$\Rightarrow$ (b)$\Rightarrow$ (c)$\Rightarrow$ (d)$\Rightarrow$ (a) holds. (a)$\Rightarrow$ (b) holds by Remark 2.4(2). For (b)$\Rightarrow$ (c), obviously K is Lindelöf. Also, K is connected, so K is Dedekind-complete (by (e)), thus it is isomorphic to $\mathbb{R}$ by Remark 1.2(3). For (c)$\Rightarrow$ (d), $F \subset K$ is Archimedean by Remark 1.2(2). If K is not Archimedean, F is closed discrete in K by Theorem 2.2(2). Then F is countable since it is Lindelöf. This is a contradiction, for F is uncountable. Thus K is Archimedean. Then F^* is a subspace of K by Theorem 2.2(1). Hence (d) holds. For (d)$\Rightarrow$ (a), S is Dedekind-complete by Remark 1.2(3), hence S^* is homeomorphic to $\mathbb{R}$ (by (b)). Thus, S^* is connected in K, then (e) holds. Hence (a) holds. □

Remark 2.5. (1) In Theorem 2.4, we cannot delete the Lindelöf property in (c), in view of the Remark 2.3 (last sentence) or Example 3.3.

(2) We recall that for G being non-discrete, G is metrizable iff some countable subset of G has an accumulation point in G ([5]). Then the following holds in view of Remarks 1.1 and 1.4.

For G being non-discrete, G is metrizable $\Leftrightarrow$ some $\mathbf{L} \subset G$ has a limit point in $G \Leftrightarrow$ (h) in Theorem 2.4 holds for G, but replace "any" by "some".

The following holds by Theorems 2.2 & 2.4, and Remarks 1.4(2) & 2.5(2).

Corollary 2.4. (1) *K is Archimedean, but not Dedekind-complete iff $\mathbf{S}_0$ is a (compact) subspace of K, but some $\mathbf{L} \subset K$ is closed discrete in K.*

(2) *The following are equivalent for K.*

(a) *K is metrizable, but not Archimedean (resp. not Dedekind-complete).*

(b) *Some $\mathbf{L} \subset K$ has a limit point in K, but $\mathbf{S}$ (resp. some $\mathbf{L}' \subset K$) has no limit points in K. Here, $\mathbf{L}' \subset K$ is an infinite decreasing sequence having a lower bound 0 in K.*

(c) *Some $\mathbf{L}_0^*$ is a (compact) subspace of K, but $\mathbf{S}$ (resp. some $\mathbf{L}' \subset K$) is closed discrete in K.*

3. Examples

Example 3.1. Let F be an ordered field. Let $K = F(x_1, x_2)$ be the field of all rational functions in the variables (independent indeterminates) x_i ($i = 1, 2$) with coefficients in F. We give a linear order $\leq$ on K as follows: Arrange any monomial $x_1^{m_1} \cdot x_2^{m_2}$ ($m_1, m_2 \in \mathbb{N}$) in K by $x_2^{m_2} \cdot x_1^{m_1}$. For distinct monomials $u = x_{i_1}^{m_1} \cdot x_{i_2}^{m_2}$ and $v = x_{j_1}^{p_1} \cdot x_{j_2}^{p_2}$ (possibly, $u = x_{i_1}^{m_1}$ etc.), define $u \prec v$ lexicographically; that is, $u \prec v$ if one of the following holds: $(i_1 < j_1); (i_1 = j_1, m_1 < p_1); (i_1 = j_1, m_1 = p_1, i_2 < j_2)$; $(i_1 = j_1, m_1 = p_1, i_2 = j_2, m_2 < p_2)$. Consider $1 \in F$ as an "empty monomial" x_i^0, and let $1 \prec u$ for any other monomial u. Then, for $u \prec v$ and any monomial w, $wu \prec wv$ (by the arrangement and the order among the monomials). We arrange any non-zero polynomial $w = \alpha_1 w_1 + \cdots + \alpha_m w_n$ ($n \leq 4$) in K by $w_1 \prec w_2 \prec \cdots \prec w_n$, here $\alpha_i \in F - \{0\}$, and w_i are monomials (containing the empty monomial) in K, and let $0u = 0$ for any monomial u. Let us define a linear order $\leq$ in K. For $\eta \in K$, let $\eta = \pm(g/f)$, where $f = a_1 u_1 + \cdots + a_m u_m$ and $g = b_1 v_1 + \cdots + b_n v_n$ are polynomials with $a_m, b_n > 0$ in F. Define $\eta > 0$ if the sign of the fraction is "+", and $\eta < 0$ if " $-$ ". For $\eta,\ \xi \in K$, define $\eta < \xi$ if $0 < \xi - \eta$. Let $K = (K, \leq)$. Let $K_1 = F(x_1), K_2 = F(x_2)$, and $K_1, K_2 \subset K$. Then it is routinely shown that K is an ordered field. The following hold for fields $F, K_1, K_2 \subset K$. (For (i) and (ii), cf. [3].)

(i) K, K_1^*, and K_2^* are metrizable, but any of them is not Archimedean.

(ii) F is closed discrete in K, K_1^*, and K_2^* (but, F^* need not be metrizable).

(iii) K_1 is closed discrete in K.

(iv) K_2^* is a subspace of K, but K_2 is not dense in K.

Proof: For (i), note that $n < x_1 < x_2$ for all $n \in \mathbb{N}$. Then any of K, K_1, and K_2 is not Archimedean. We show that K is metrizable. The decreasing sequence

$\{1/x_2^n : n \in \mathbb{N}\}$ in K_2 converges to 0 in K (indeed, let $\eta \in K$ with $\eta > 0$. We may assume that $x_2^m \cdot x_1^n$ is the largest monomial in the denominator of η. Then, $\eta > 1/x_2^k$ for $k \in \mathbb{N}$ with $k > m$). Thus, K is metrizable by Remark 2.5(2). Similarly, K_i^* (i = 1,2) are metrizable (because, the sequence $\{1/x_i^n : n \in \mathbb{N}\}$ in K_i converges to 0 in K_i^*). For (ii), let $\eta \in K$, and $H(\eta) = (\eta - 1/x_1, \eta + 1/x_1)$. Then $H(\eta)$ is a neighborhood of η in K with $|H(\eta) \cap F| \leq 1$ (indeed, if $H(\eta) \cap F$ contains α, β, then $|\alpha - \beta| < 2/x_1$, so $\alpha = \beta$). Thus, (ii) holds in K. Similarly, (ii) holds in K_1^* and K_2^*. For the parenthetic part, note that every ordered field need not be metrizable; see Example 3.3 below (or, [3], [5], etc.). For (iii), let $\eta \in K$, and $V(\eta) = (\eta - 1/2x_2, \eta + 1/2x_2)$. Then $V(\eta)$ is a neighborhood of η in K with $|V(\eta) \cap K_1| \leq 1$ (indeed, suppose $V(\eta) \cap K_1$ contains η_1, η_2 with $\eta_1 < \eta_2$. Then $0 < \eta' = \eta_2 - \eta_1 < 1/x_2$. But, $x_1^m < x_2 < x_2 \cdot x_1^n$ for any $m, n \in \mathbb{N}$. Then, since $\eta' \in K_1$, $\eta' > 1/x_2$, a contradiction). For (iv), K_2 has an accumulation point 0 in K by the proof of (i). Thus K_2^* is a subspace of K by Theorem 2.1. To see K_2 is not dense in K, let $W = (1/3x_1, 1/x_1)$. Then W is a neighborhood of $1/2x_1$ in K, but $W \cap K_2 = \emptyset$ (indeed, suppose W contains an element $\eta = (b_0 + b_1x_2 + \cdots + b_nx_2^n)/(a_0 + a_1x_2 + \cdots + a_mx_2^m)$ $(a_m, b_n > 0)$ in K_2. We assume $m, n \geq 1$. Since $1/3x_1 < \eta$, $m \leq n$. But, $1/x_1 > \eta$, so $m > n$, a contradiction). □

Example 3.2. Let $K = (\mathbb{Q}(x), \leq)$ be a non-Archimedean ordered field defined in Example 3.1. For a transcendental real number c ($c = \pi$ etc.), define an ordered field $K' = \mathbb{Q}(c) \subset \mathbb{R}$ by replacing "x" by "c" in $\mathbb{Q}(x)$. Note that for every polynomial $f \in K$, if $f(c) = 0$, then $f = 0$ since c is a transcendental real number. Define $h : K \to K'$ by $h((g/f)) = g(c)/f(c)$. Then h is an isomorphism. Thus, the following hold ((iii) holds by Corollary 2.3).

(i) K is isomorphic to the field $K' \subset \mathbb{R}$, but K is not Archimedean.
(ii) K' is Archimedean, but K' is not Archimedean with respect to an order $\preceq$ defined by $a \prec b$ iff $h^{-1}(a) < h^{-1}(b)$.
(iii) The identity map from K' to $(K', \preceq)$ is not continuous.

Example 3.3. For a completely regular space X, let $C(X)$ be the collection of all continuous functions from X into $\mathbb{R}$. For a maximal ideal M of the ring $C(X)$, the residue class field $K = C(X)/M$ is an ordered field. In view of [2, Theorem 5.5], the field K contains a subfield F which is the image under an order-preserving isomorphism h from $\mathbb{R}$ into K. Thus, we can assume $\mathbb{R} \subset K$. The ordered field K is called *real* if it is isomorphic to $\mathbb{R}$, and K is called *hyper-real* if it is not real ([2]). For example, the ordered fields $C(\mathbb{N})/M$, $C(\mathbb{Q})/M$, and $C(\mathbb{R})/M$ are hyper-real; see [2] (or [5]). The field $K = C(X)/M$ is real (resp. hyper-real) iff K is Archimedean (resp. non-Archimedean); see [2, 5.6]. Thus, for the field K the following hold by Theorems 2.2 and 2.4, here see [3] or [5] for (i).

(i) K is real $\Leftrightarrow$ K is homeomorphic to $\mathbb{R}$ $\Leftrightarrow$ K is Lindelöf $\Leftrightarrow$ K is metrizable.
(ii) K is hyper-real $\Leftrightarrow$ the field F (or $\mathbb{R}$) $\subset K$ is closed discrete in K $\Leftrightarrow$ the function h into K is not continuous $\Leftrightarrow$ any non-constant function from $\mathbb{R}$ into K is not continuous.

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Department of Mathematics, Tokyo Gakugei University, Koganei, Tokyo, 184-8501, Japan

E-mail: ytanaka@u-gakugei.ac.jp

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