On flat covers in varieties

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Abstract. Flat covers do not exist in all varieties. We give a necessary condition for the existence of flat covers and some examples of varieties where not all algebras have flat covers.

Keywords: flat object, flat cover, variety, cosimplicial object

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1. Introduction

In 2001, L. Bican, R. El Bashir and E. Enochs [2] proved flat cover conjecture saying that each R-module has a flat cover. There is a natural question whether flat covers exist in an arbitrary variety. M. Kilp, U. Knauer and A.V. Mikhalev [6] generalized certain results on projective and flat objects to monoids and acts. In contrast to the classical definition of flat object, J. Rosický [7] used the fact that every module is flat if and only if it is a directed colimit of finitely presented projective modules. We will follow the later approach. The main result of the paper is a criterion for existence of flat covers. By means of the criterion we show that flat covers do not exist in a special variety of unary algebras, as well as in varieties of semigroups, monoids, commutative semigroups or commutative monoids.

Recall that in varieties there are free objects and hence *projective* objects are retracts of free ones. *Finitely presentable* objects can be defined in a purely categorical way but in varieties we can simply say that they are generated by finitely many generators and relations (see [1]). As mentioned, *flat* objects are the directed colimits of finitely presentable projective objects. Given an object M of a variety $\mathcal{V}$, a flat object F together with a morphism $f : F \to M$ is said to be a *flat precover* if any other $f' : F' \to M$ with F' flat factors through f, i.e. there is (not necessarily unique) $g : F' \to F$ such that $fg = f'$. When $F = F'$ and $f = f'$ implies that g must be an automorphism of F then (F, f) is called a *flat cover*.

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The flat covers evidently exist in varieties with all objects flat (those are characterized in [3]) and also in varieties where flat algebras are projective [4].

In the paper we assume that $\mathcal{V}$ is a finitary variety, $\mathcal{P}$ a small full subcategory of $\mathcal{V}$ consisting of representatives of finitely presentable projective objects, and $\mathcal{F}$ the full subcategory of flat objects. But more generally, we could only require that $\mathcal{V}$ is a finitely accessible category, $\mathcal{P}$ some small full subcategory of finitely presentable objects, and $\mathcal{F}$ a closure of $\mathcal{P}$ in $\mathcal{V}$ under directed colimits.

Note that the existence of a flat precover of object M is equivalent to the existence of a weakly terminal object in the comma-category $\mathcal{F} \uparrow M$. From Theorem 2.5 of [8] it follows that in varieties the existence of flat precovers implies the existence of flat covers.

2. δ-objects

From our considerations it follows that for any object M of $\mathcal{V}$ the comma-category $\mathcal{F} \uparrow M$ has (at least) weak coproducts. If there is a family of objects F_i in $\mathcal{F} \uparrow M$ such that each F in $\mathcal{F} \uparrow M$ factorizes through some F_i then the weak coproduct of F_i is a weakly terminal object. Thus the case that the weakly terminal object does not exist implies that we can construct a proper class of objects in $\mathcal{F} \uparrow M$ not factorizing through anything "smaller".

For an arbitrary set X, the poset $(\operatorname{Fin} X, \subseteq)$ of non-empty finite subsets of X is a typical example of directed set. We will represent $\operatorname{Fin} X$ in $\mathcal{P} \uparrow M$ by the following construction.

2.1 Definition. A diagram $D : \Delta_\delta \to \mathcal{P}$, where Δ_δ consists of objects $\Delta_0, \Delta_1, \dots$ and morphisms generated by $\Delta_\delta(\Delta_{n-1}, \Delta_n) = \{\delta_0^{(n)}, \dots \delta_n^{(n)}\}$ satisfying rules

$$\delta_i^{(n+1)} \delta_j^{(n)} = \delta_{j+1}^{(n+1)} \delta_i^{(n)}$$

for each $0 \le i \le j \le n$, is called a *δ-cosimplicial object*, or shortly *δ-object* of $\mathcal{P}$. The letter δ emphasizes that we use only the embeddings (in contrast to the standard definition of cosimplicial object).

The segment from Δ_0 to Δ_n will be denoted by $\Delta_\delta^{(n)}$ and the corresponding diagram $D^{(n)} : \Delta_\delta^{(n)} \to \mathcal{P}$ will be called a *$\delta^{(n)}$-object*.

2.2 Remark. Given a linearly ordered set X, we assign Δ_n for every $(n+1)$-element subset and $\delta_i^{(n)}$ for the embedding $(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \subseteq (x_0, \dots, x_n)$ where $x_0 < \cdots < x_n$. It gives a functor $E : \operatorname{Fin} X \to \Delta_\delta$. It means that every δ-diagram $\Delta_\delta \to \mathcal{P}$ can be expanded to a directed diagram $\operatorname{Fin} X \to \mathcal{P}$.

2.3 Definition. A $\delta^{(n)}$-object D is called *degenerated* if $D\delta_i^{(k)} = D\delta_j^{(k)}$ for all $0 \le i < j \le k \le n$. A natural transformation $\theta : C \to D$ of two $\delta^{(n)}$-objects

C, D is called a *degeneration* if D is degenerated. A δ-object $D : \Delta_\delta \to \mathcal{P}$ is said to be *finitely degenerable* if for every n there is a degeneration of the restriction $D^{(n)} : \Delta_\delta^{(n)} \to \mathcal{P}$.

2.4 Theorem. *If $\mathcal{F} \uparrow M$ has a weakly terminal object then every δ-object of $\mathcal{P} \uparrow M$ is finitely degenerable.*

Proof: Let $T \to M$ be a weakly terminal object and $C : I \to \mathcal{P}$ expression of T as a directed colimit of elements from $\mathcal{P}$. Let $\lambda = \max\{\operatorname{card} I, \operatorname{card} \mathcal{P}, \aleph_0\}$, D be a δ-object and n a natural number. We will show that $D^{(n)}$ degenerates. Let us put $\kappa = (\exp^{n!} \lambda)^+$ and choose a set X with cardinality κ. By 2.2 D can be expanded to a directed diagram $B : \operatorname{Fin} X \to \mathcal{P}$. Let F be the colimit of B and $t : F \to T$ a morphism to a weakly terminal object T.

We will show that there is a functor $A : \operatorname{Fin} X \to I$ such that t "decomposes" to a natural transformation $\theta : B \to CA$, i.e. $t(BY \to F) = (CAY \to T)\theta_Y$ for each finite Y. Since the elements of $\mathcal{P}$ are finitely presented, for each finite $Y \subseteq X$ the morphism $BY \to F \to T$ factors through some $f_Y : BY \to Ci_Y$. If Y is one-element there is nothing else to do and we put $AY = i_Y$ and $\theta_Y = f_Y$. Assume that we have constructed AY and θ_Y for all at most $(j-1)$-element subsets Y and they work as a natural transformation. Let Y be a j-element set. I is directed, thus we can replace i_Y by an upper bound of all $AZ, \emptyset \neq Z \subseteq Y$. For $\emptyset \neq Z \subseteq Y$ the composites $BZ \to BY \xrightarrow{f_Y} Ci_Y$ and $BZ \xrightarrow{\theta_Z} CAZ \to Ci_Y$ are generally different, but we can prolongate the diagram by some $Ci_Y \to Ci'_Y$ to be equal and improve i_Y by i'_Y and f_Y by $(Ci_Y \to Ci'_Y)f_Y$. In this way we improve i_Y , f_Y for each $\emptyset \neq Z \subseteq Y$ and find AY and θ_Y extending the natural transformation θ.

By a pigeon-hole principle there is subset $X_0 \subseteq X$ of cardinality $\kappa_0 = \kappa$ such that all pairs $A\{x\}, \theta_{\{x\}}$ for $x \in X_0$ are equal. Let us restrict to elements of X_0 and put $\theta_0 = \theta_{\{x\}}$ and $D'\Delta_0 = CA\{x\}$. Now for any two-element subset $\{x, y\} \subseteq X_0$ we consider the image $A\{x\} = A\{y\} \leq A\{x, y\}$ together with morphisms $\theta_{\{x\}} = \theta_{\{y\}}$ and $\theta_{\{x,y\}}$ as a coloring of $\{x, y\}$. By Erdös–Rado theorem

$$(\exp^{n!} \lambda)^+ \to ((\exp^{n!/2!} \lambda)^+)^2_\lambda,$$

thus there is a homogenous subset $X_1 \subseteq X_0$ of cardinality $\kappa_1 = (\exp^{n!/2!} \lambda)^+$, i.e. $\theta_1 = \theta_{\{x,y\}}$ and $D\Delta_1 = CA\{x, y\}$ do not depend on $\{x, y\} \subseteq X_1$. Consequently, $\theta_1 D\delta_0^{(1)} = (CA\{x\} \to CA\{x, y\})\theta_0 = (CA\{y\} \to CA\{x, y\})\theta_0 = \theta_1 D\delta_1^{(1)}$, i.e. we have found a degeneration for the segment $\Delta_\delta^{(1)}$.

Similarly, for every finite $Y \subseteq X$ we consider the collection $\{(A_Z, \theta_Z) \mid \emptyset \neq Z \subseteq Y\}$ as a coloring of Y and by induction construct homogenous subsets $X_2, \dots, X_n$ of cardinalities $\kappa_j = (\exp^{n!/j!} \lambda)^+$ due to

$$(\exp^{n!/(j-1)!} \lambda)^+ \to ((\exp^{n!/j!} \lambda)^+)^j_\lambda.$$

At the end we obtain a set X_n of cardinality λ^+ homogenous for all j-element subsets where $1 \leq j \leq n+1$. Thus θ restricted to X_n yields a degeneration of $D^{(n)}$. $\square$

2.5 Remark. If $\mathcal{V}$ is not finitary, we should solve two problems. One is that Δ_δ should be extended by objects and morphisms corresponding to infinite ordinals. The second problem is that the existence of infinitary homogenous sets depends on set theory.

Another open problem is whether the statement of 2.4 can be inverted, that is, whether degeneration of δ-objects implies the existence of a weakly terminal object.

3. Examples

3.1 Proposition. *Let $\mathcal{V}$ be a variety of unary algebras with operations $a_0, a_1, \dots$ satisfying identities*

$$a_i a_j(x) = a_{j+1} a_i(x) \text{ for } i \leq j.$$

Then $\mathcal{V}$ does not have flat covers.

Proof: The variety can be considered also as a variety of S-sets, i.e. sets with action of semigroup S generated by elements $a_0, a_1, \dots$ modulo the above relations. Then $\mathcal{V}$ becomes a finitary (two-sorted) variety. The assignments $* \mapsto a_i(*)$ of a generator $*$ define endomorphisms $f_i : F1 \to F1$ on the free algebra with one generator. Obviously the assignment $D\Delta_i = F1, D\delta_i^{(n)} = f_i$ defines a δ-object. The endomorphisms of $F1$ can be identified with terms on $a_0, a_1, \dots$, thus they are injective. Since every projective P is a retract of free FX and the projection $X \to \{*\}$ extends to a homomorphism $FX \to F1$, we get a composition $F1 \to P \to FX \to F1$ and deduce that all $F1 \to P$ are injective too. Hence no part of the δ-diagram D can be degenerated and $\mathcal{F}$ does not have a weakly terminal object. In other words, the terminal algebra $\{*\}$ of $\mathcal{V}$ does not have a flat precover. $\square$

3.2 Proposition. *Let $\mathcal{V}$ be a variety of all semigroups, commutative semigroups, monoids, or commutative monoids. Then $\mathcal{V}$ does not have flat covers.*

Proof: Put $D\Delta_n = F\{x_0, \dots, x_n\}$ and generate morphisms $D\delta_i^{(n)}$ by

$$D\delta_i^{(n)}(x_k) = \begin{cases} x_k & \text{for } \ i > k, \\ x_k x_{k+1} & \text{for } \ i = k, \\ x_{k+1} & \text{for } \ i < k. \end{cases}$$

Then

$$D\delta_i^{(n+1)} D\delta_j^{(n)}(x_k) = \begin{cases} D\delta_i^{(n+1)}(x_k) = x_k & \text{for } k < i \le j, \\ D\delta_i^{(n+1)}(x_k) = x_k x_{k+1} & \text{for } i = k < j, \\ D\delta_i^{(n+1)}(x_k) = x_{k+1} & \text{for } i < k < j, \\ D\delta_i^{(n+1)}(x_k x_{k+1}) = x_k x_{k+1} x_{k+2} & \text{for } i = k = j, \\ D\delta_i^{(n+1)}(x_k x_{k+1}) = x_{k+1} x_{k+2} & \text{for } i < k = j, \\ D\delta_i^{(n+1)}(x_{k+1}) = x_{k+2} & \text{for } i \le j < k, \end{cases}$$

and

$$D\delta_{j+1}^{(n+1)} D\delta_i^{(n)}(x_k) = \begin{cases} D\delta_{j+1}^{(n+1)}(x_k) = x_k & \text{for } k < i \le j, \\ D\delta_{j+1}^{(n+1)}(x_k x_{k+1}) = x_k x_{k+1} & \text{for } i = k < j, \\ D\delta_{j+1}^{(n+1)}(x_{k+1}) = x_{k+1} & \text{for } i < k < j, \\ D\delta_{j+1}^{(n+1)}(x_{k+1}) = x_k x_{k+1} x_{k+2} & \text{for } i = k = j, \\ D\delta_{j+1}^{(n+1)}(x_k x_{k+1}) = x_{k+1} x_{k+2} & \text{for } i < k = j, \\ D\delta_{j+1}^{(n+1)}(x_{k+1}) = x_{k+2} & \text{for } i \le j < k, \end{cases}$$

thus D is a δ-diagram. Let $S = \{0, \frac{1}{2}, 1\}$ be a three-element chain considered as a meet-semilattice, thus S is a commutative monoid with unit 1. (In case of semigroups or commutative semigroups S can be considered without 1.)

Let us consider homomorphisms $f_n : D\Delta_n \to S$ generated by

$$f_n(x_k) = \begin{cases} 0 & \text{for } k = 0, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Thus for any term t we have $f_n(t) = 0$ iff t contains x_0. On the other hand, for any morphism δ in Δ_δ we have that $D\delta(t)$ contains x_0 iff t does so. Hence the morphisms f_n are compatible with those of D and they together form a δ-object in $\mathcal{P} \uparrow S$. It remains to show that it is not finitely degenerable and we will do that for $D^{(1)}$. Assume that $g(x_0x_1) = gD\delta_0^{(0)}(x_0) = gD\delta_1^{(0)}(x_0) = g(x_0)$ for some homomorphism $g : D\Delta_1 \to P$ with P projective and there is $h : P \to S$ such that $hg = f_1$. Since a projective object is a retract of a free one, P can be considered free. Then the equality $g(x_0x_1) = g(x_0)$ can be interpreted as a solution of equation $x_0x_1 = x_0$ in a free object. But such a solution in (commutative) semigroups does not exist, while in (commutative) monoids there is just a trivial solution with $g(x_1) = 1$ contradictory to $hg(x_1) = f_1(x_1) = \frac{1}{2}$. □

3.3 Remark. (1) The comma-categories of R-modules are not longer additive but they are affine in sense that for given morphisms $f, g, h : M \to N$ in $\mathcal{V}$ the morphism $f - g + h : M \to N$ belongs to $\mathcal{V}$. This leads to a notion of naturally Malcev variety (see [5]). The author can prove that in a naturally Malcev variety every δ-object is finitely degenerable. The basic idea is that

$$\begin{aligned}(D\delta_0^{(2)} - D\delta_1^{(2)} + D\delta_2^{(2)})D\delta_0^{(1)} &= D(\delta_0^{(2)}\delta_0^{(1)}) - D(\delta_1^{(2)}\delta_0^{(1)}) + D(\delta_2^{(2)}\delta_0^{(1)})\\ &= D(\delta_2^{(2)}\delta_0^{(1)})\\ &= D(\delta_0^{(2)}\delta_1^{(1)})\\ &= (D\delta_0^{(2)} - D\delta_1^{(2)} + D\delta_2^{(2)})D\delta_1^{(1)},\end{aligned}$$

i.e. $D\delta_0^{(2)} - D\delta_1^{(2)} + D\delta_2^{(2)}$ is a degeneration for $D^{(1)}$. The degeneration is more complicated in higher dimension.

(2) Abelian groups are $\mathbb{Z}$-modules and hence flat covers exist. An open question is if it holds in the variety of all groups.

(3) The semigroup S in proof of 3.2 is not a group but it is a band (i.e. idempotent semigroup) and hence an inverse semigroup. It is interesting to see that in a free inverse semigroup the equation $x_0x_1 = x_0$ may have non-trivial solutions, e.g. $g(x_0) = ab, g(x_1) = b^{-1}b$, and consequently it is possible to show that D is finitely degenerable. Thus existence of flat covers in inverse semigroups is also open and it seems to be a promising variety for further studying. Natural questions are how much general our example is and whether our problem reduces to a specific equational problem in free algebras.

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