Non-existence result for quasi-linear elliptic equations with supercritical growth

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Abstract. We obtain a non-existence result for a class of quasi-linear eigenvalue problems when a parameter is small. By using Pohozaev identity and some comparison arguments, non-existence theorems are established for quasi-linear eigenvalue problems under supercritical growth condition.

Keywords: quasi-linear elliptic equations, non-existence, large solution, small solution

Classification: 35J65, 35B25

1. Introduction

In this paper we are concerned with the non-existence of positive solutions of a class of quasi-linear eigenvalue problems

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda f(u(x)) \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $f \in C^1(0,\infty) \bigcap C^0([0,\infty))$, $f(s) > 0$ for $s \geq 0$; $\lambda > 0$, $\Omega = B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball, and $1 < p < N$. By a positive solution u of (1.1)–(1.2) we mean that $u \in C_0^1(\Omega)$, $u > 0$ in Ω, and satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_\Omega f(u)v$$

for any $v \in C_0^\infty(\Omega)$. Thus, solutions are considered in a weak sense. By a small solution u_λ of (1.1)–(1.2) we mean that $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$. By a positive large solution $u_\lambda(r)$ of (1.1)–(1.2) we mean that $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = \infty$.

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Equations of the above form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory ([1], [2]), non-Newtonian filtration ([3]) and the turbulent flow of a gas in a porous medium ([4]). In the non-Newtonian fluid theory, the quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo-plastics. If $p = 2$, they are Newtonian fluids.

For $p = 2$, the problem (1.1)–(1.2) has been studied by many authors, such as Ni and Serrin [5], Gelfand [6], Keller and Cohen [7], Amann [8], Crandall and Rabinowitz [9], Lions [10], Brezis and Nirenberg [11], to name just a few. For $p > 1$, the existence and uniqueness of the positive solutions of (1.1)–(1.2) have been studied by many authors, for example [12]–[17], [20]–[21] and the references therein. When f is strictly increasing on $\mathbb{R}^+$, $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \le \alpha_1 + \alpha_2 s^\mu$, $0 < \mu < p-1$, $\alpha_1, \alpha_2 > 0$, it was shown in [12] that there exist at least two positive solutions for equations (1.1)–(1.2) when λ is sufficiently large. If $\liminf\limits_{s\to 0^+} f(s)/s^{p-1} > 0$, $f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved in [13] that the problem (1.1)–(1.2) has a unique positive solution when λ is sufficiently large. Moreover, it was also shown in [14] that problem (1.1)–(1.2) has a unique positive large solution and at least one positive small solution when λ is large if f is nondecreasing, and there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \le \alpha_1 + \alpha_2 s^\beta$, $0 < \beta < p-1$; $\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0$, and there exist $T, Y > 0$ with $Y \ge T$ such that

$$(f(s)/s^{p-1})' > 0 \quad \text{for} \quad s \in (0, T)$$

and

$$(f(s)/s^{p-1})' < 0 \quad \text{for} \quad s > Y.$$

In contrast to these cases, it seems that very little is known about existence and non-existence of positive solutions and non-small solutions for the problem (1.1)–(1.2) when λ is sufficiently small. Hai [18] considered the case when Ω is an annular domain, and obtained the existence of positive large solutions for the problem (1.1)–(1.2) when λ is sufficiently small. Guo and Yang [22] considered the case when Ω is a bounded smooth domain, and obtained the existence of positive large solutions and small solutions for the problem (1.1)–(1.2) when λ is sufficiently small. In this paper, we shall consider the case when $\Omega = B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball, and establish the non-existence of positive solutions and non-small solutions for the problem (1.1)–(1.2) when λ is sufficiently small.

Our approach depends heavily upon the special properties of the positive radial solutions for the problem (1.1)–(1.2). We expect that such non-existence result of (1.1)–(1.2) are still true for the general domain Ω.

We can find the related non-existence results for $p = 2$ in [19]. When $p = 2$, it is well known that all the positive solutions in $C^2(B_R)$ of the problem

$$\begin{aligned} \triangle u + f(u) = 0 \quad &\text{in} \quad B_R, \\ u(x) = 0 \quad &\text{on} \quad \partial B_R \end{aligned}$$

are radially symmetric solutions for very general f (see [25]). Unfortunately, this result does not apply to the case $p \neq 2$. Kichenassary and Smoller showed that there exist many positive nonradial solutions of the above problem for some f (see [26]). The major stumbling block in the case of $p \neq 2$ is that certain nice features inherent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be found in [12], [13].

2. Non-existence result

In this section we study the non-existence of positive solutions of the problems (1.1)–(1.2). The nonlinear function $f \in C^1(\mathbb{R})$ (or f is in general locally Lipschitz continuous) satisfies the supercritical condition as $u \to \infty$; that is, f satisfies the following conditions:

(H_1) When $p \geq 2$, there are $q > \frac{N(p-1)+p}{N-p}$, $A > 0$ such that $(q+1)F(u) \leq uf(u)$ for $u \geq A$, where $F(u) = \int_0^u f(v)\,dv$ and A is a positive constant with $F(A) > 0$.

$(H_1)'$ When $1 < p < 2$, there are $q + 1 > \frac{2^{(2-p)/(p-1)}Np}{N-p}$, $A > 0$ such that $(q+1)F(u) \leq uf(u)$ for $u \geq A$, where $F(u) = \int_0^u f(v)\,dv$ and A is a positive constant with $F(A) > 0$.

To prove the main theorem, we consider the following initial value problems

$$(\Phi_p(u'))' + \frac{(N-1)}{r}\Phi_p(u') + f(u(r)) = 0, \quad r > 0, \tag{2.1}$$

$$u(0,\alpha) = \alpha > 0, \quad u'(0,\alpha) = 0, \tag{2.2}$$

where $\Phi_p(s) = |s|^{p-2}s$, $p > 1$.

We first recall a Pohozaev identity which was obtained by Ni and Serrin [5], or Mitidieri and Pohozaev [23].

Lemma 2.1. *Let $u(r)$ be a solution of equation (2.1) in $(r_1, r_2) \subset (0,\infty)$ and a be an arbitrary constant. Then, for each $r \in (r_1, r_2)$ we have*

$$\begin{aligned} &\frac{d}{dr}\big[r^N\big\{(1-1/p)|u'|^p + F(u) + \frac{a}{r}uu'|u'|^{p-2}\big\}\big] \\ &\qquad = r^{N-1}\big[NF(u) - auf(u) + (a+1-N/p)|u'|^p\big], \end{aligned} \tag{2.3}$$

where $F(u) = \int_0^u f(s)\,ds$.

Definition 2.2. For each $\alpha \in (0,\infty)$ and $B \geq 0$, let $R(\alpha,B)$ be the first r such that $u(r,\alpha) = B$. If there is no such r, we shall adopt the convention that $R(\alpha,B) = \infty$. We also stipulate that $R(\alpha) = R(\alpha,0)$ and $R_1(\alpha) = R(\alpha,A)$, where A is given in (H_1) or $(H_1)'$.

Definition 2.3. For $p \geq 2$, let $\gamma = \frac{1}{(q+1)(N-p)}[(N-p)(q+1) - Np] > 0$; for $1 < p < 2$, let $\gamma_1 = \frac{1}{(q+1)(N-p)}[(N-p)(q+1) - 2^{(2-p)/(p-1)}Np] > 0$. Define two positive functions $R_*(B)$ and $R^*(B)$ on $[A,\infty]$ by

$$R_*(B)^{p/(p-1)} = M(\overline{B})^{-1/(p-1)}B$$

and

$$R^*(B)^p = \Big(\frac{p}{p-1}\Big)^{p-1}\Big(\frac{NB}{q+1}\Big)^p (F(B))^{-1},$$

where $\overline{B} = [N^{-1/(p-1)}\frac{(p-1)}{p}+1]\gamma^{-1}B$ for $p \geq 2$; $\overline{B} = [2^{\frac{2-p}{p-1}}N^{-\frac{1}{p-1}}\frac{(p-1)}{p}+1]\gamma_1^{-1}B$ for $1 < p < 2$, and $M(\overline{B}) = \max\{f(u) : u \in [0,\overline{B}]\}$.

We shall first prove that for a fixed $B \geq A$, there exist an upper bound and a lower bound for $R(\alpha,B)$.

Lemma 2.4. *Let f satisfy (H_1) for $p \geq 2$ or $(H_1)'$ for $1 < p < 2$. Then for any $B \geq A$ and $\alpha \in (\overline{B},\infty)$, we have*

$$R_*(B) \leq R(\alpha,B) \leq R^*(B), \tag{2.4}$$

and

$$\begin{aligned}\Big(\frac{(q+1)}{N}\frac{F(B)}{B}\Big)^{1/(p-1)} R_*(B)^{1/(p-1)} &\leq -u'(R(\alpha,B),\alpha)\\ &\leq \frac{pN}{(p-1)(q+1)}BR_*(B)^{-1}.\end{aligned} \tag{2.5}$$

Proof: Letting $u(r) = u(r,\alpha)$ and $a = N/(q+1)$ in equation (2.3) and integrating equation (2.3) from 0 to r, from (H_1) or $(H_1)'$ we have

$$\frac{(p-1)}{p}|u'|^p + F(u(r,\alpha)) + \frac{N}{(q+1)}\frac{u(r,\alpha)u'(r,\alpha)|u'(r,\alpha)|^{p-2}}{r} < 0 \tag{2.6}$$

if $u(s,\alpha) > A$ for all $s \in [0,r]$. It is clear that (H_1) or $(H_1)'$ implies $F(u) > 0$ for all $u > A$. Hence, for any $\alpha \in (A,\infty)$, by (2.6) we have $u'(r,\alpha) < 0$ in $(0, R_1(\alpha))$. Furthermore, we have $R_1(\alpha) < \infty$ for all $\alpha \in (A,\infty)$. Indeed, by (H_1) or $(H_1)'$ there is a positive constant m such that

$$f(u) \geq m \ \text{ for all } \ u \geq A. \tag{2.7}$$

From (2.1)–(2.2) and (2.7), for $r \in (0, R_1(\alpha))$ and $\alpha \geq A$, we have

$$r^{N-1}\Phi_p(u'(r,\alpha)) = -\int_0^r s^{N-1} f(u(s,\alpha))\,ds \leq -\frac{m}{N} r^N, \tag{2.8}$$

which implies that

$$R_1(\alpha)^{p/(p-1)} \leq \Big(\frac{N}{m}\Big)^{1/(p-1)} \Big[\frac{p}{(p-1)}(\alpha - A)\Big].$$

Therefore, by (H_1), $(H_1)'$ and (2.6) we obtain

$$\frac{(p-1)}{p}|u'(R(\alpha,B),\alpha)|^p < \frac{N}{(q+1)}\frac{B}{R(\alpha,B)}|u'(R(\alpha,B),\alpha)|^{p-1} \tag{2.9}$$

and

$$F(B) < \frac{N}{(q+1)}\frac{B}{R(\alpha,B)}|u'(R(\alpha,B),\alpha)|^{p-1}. \tag{2.10}$$

Now, (2.9) implies

$$(-u'(R(\alpha,B),\alpha))R(\alpha,B) < \frac{pN}{(p-1)(q+1)}B. \tag{2.11}$$

From (2.10) and (2.11), we obtain an upper bound for $R(\alpha,B)$, that is,

$$R(\alpha,B)^p \leq \Big[(\frac{p}{p-1})^{p-1}(\frac{NB}{q+1})^p\Big]F(B)^{-1} \tag{2.12}$$

for all $\alpha \in (B,\infty)$. This proves the second inequality of (2.4). To prove the first inequality of (2.4), there are two cases to be considered:

(a) $R(\alpha,\overline{B}) \geq R_*(B)$,
(b) $R(\alpha,\overline{B}) < R_*(B)$.

In case (a), since $R(\alpha,B) > R(\alpha,\overline{B})$ we have $R(\alpha,B) > R_*(B)$. In case (b), we need a comparison argument.

Let $v_\alpha(r) \equiv v(r,\alpha,\overline{B})$ be the solution of the initial value problem

$$(\Phi_p(v'))' + \frac{N-1}{r}\Phi_p(v') + \overline{C} = 0 \quad \text{for} \quad r > R(\alpha,\overline{B}), \tag{2.13}$$

$$v(R(\alpha,\overline{B})) = \overline{B}, \tag{2.14}$$

$$v'(R(\alpha,\overline{B})) = u'(R(\alpha,\overline{B}),\alpha), \tag{2.15}$$

where $\overline{C} = M(\overline{B})$.

Then $v_\alpha(r)$ can be solved explicitly as

$$v_\alpha(r)=\overline{B}-\int_{\overline{R}}^{r}\Big[\Big(\frac{\overline{R}}{s}\Big)^{N-1}|u'(\overline{R})|^{p-1}+\frac{\overline{C}}{N}\Big(s-\frac{\overline{R}^N}{s^{N-1}}\Big)\Big]^{1/(p-1)}\,ds, \tag{2.16}$$

where $\overline{R}=R(\alpha,\overline{B})$. We further consider two subcases here: (i) $p\ge 2$ and (ii) $1<p<2$.

In subcase (i), it is obvious that $1/(p-1)\le 1$. Using the inequalities $(1+x)^{1/(p-1)}\le 1+x^{1/(p-1)}$ for $x\ge 0$ and (2.11), we have

$$\begin{aligned}
v_\alpha(r)&\ge \overline{B}-\int_{\overline{R}}^{r}\Big[\Big(\frac{\overline{R}}{s}\Big)^{N-1}|u'(\overline{R})|^{p-1}+\frac{\overline{C}}{N}s\Big]^{1/(p-1)}\,ds\\
&\ge \overline{B}-\int_{\overline{R}}^{r}\Big(\frac{\overline{R}}{s}\Big)^{(N-1)/(p-1)}|u'(\overline{R})|\Big[1+\frac{((\overline{C}/N)s)^{1/(p-1)}}{(\overline{R}/s)^{(N-1)/(p-1)}|u'(\overline{R})|}\Big]\,ds\\
&= \overline{B}-\int_{\overline{R}}^{r}\Big[(\frac{\overline{R}}{s})^{(N-1)/(p-1)}|u'(\overline{R})|+\Big(\frac{\overline{C}}{N}\Big)^{1/(p-1)}s^{1/(p-1)}\Big]\,ds\\
&\ge \overline{B}-\frac{(p-1)}{(N-p)}\overline{R}|u'(\overline{R})|-\Big(\frac{\overline{C}}{N}\Big)^{1/(p-1)}]\int_{\overline{R}}^{r}s^{1/(p-1)}\,ds\\
&\ge \overline{B}-\frac{(p-1)}{(N-p)}\frac{Np}{(p-1)(q+1)}\overline{B}-\Big(\frac{\overline{C}}{N}\Big)^{1/(p-1)}\frac{(p-1)}{p}r^{p/(p-1)}\\
&= \gamma\overline{B}-\Big(\frac{\overline{C}}{N}\Big)^{1/(p-1)}\frac{(p-1)}{p}r^{p/(p-1)}\\
&\ge B
\end{aligned}$$

for all $r\in[R(\alpha,\overline{B}),R_*(B)]$.

In subcase (ii), we have $1/(p-1)>1$. Let $q+1>2^{(2-p)/(p-1)}(Np/(N-p))$. Using the inequalities $(1+x)^{1/(p-1)}\le 2^{(2-p)/(p-1)}(1+x^{1/(p-1)})$ for $x\ge 0$ and (2.13), we have

$$\begin{aligned}
v_\alpha(r)&\ge \overline{B}-\int_{\overline{R}}^{r}\Big[\Big(\frac{\overline{R}}{s}\Big)^{N-1}|u'(\overline{R})|^{p-1}+\frac{\overline{C}}{N}s\Big]^{1/(p-1)}\,ds\\
&\ge \overline{B}-\int_{\overline{R}}^{r}\Big(\frac{\overline{R}}{s}\Big)^{(N-1)/(p-1)}|u'(\overline{R})|2^{(2-p)/(p-1)}\\
&\quad\Big[1+\frac{((\overline{C}/N)s)^{1/(p-1)}}{(\overline{R}/s)^{(N-1)/(p-1)}|u'(\overline{R})|}\Big]\,ds\\
&= \overline{B}-\int_{\overline{R}}^{r}2^{(2-p)/(p-1)}\Big[\Big(\frac{\overline{R}}{s}\Big)^{(N-1)/(p-1)}|u'(\overline{R})|+\Big(\frac{\overline{C}}{N}\Big)^{1/(p-1)}s^{1/(p-1)}\Big]\,ds\\
&\ge \overline{B}-2^{(2-p)/(p-1)}\frac{(p-1)}{(N-p)}\overline{R}|u'(\overline{R})|-2^{(2-p)/(p-1)}
\end{aligned}$$

$$
\begin{aligned}
&\Big(\frac{\overline{C}}{N}\Big)^{1/(p-1)} \int_{\overline{R}}^{r} s^{1/(p-1)}\,ds \\
&\geq \overline{B} - 2^{(2-p)/(p-1)} \frac{(p-1)}{(N-p)} \frac{Np}{(p-1)(q+1)} \overline{B} - 2^{(2-p)/(p-1)} \\
&\quad \Big(\frac{\overline{C}}{N}\Big)^{1/(p-1)} \frac{(p-1)}{p} r^{p/(p-1)} \\
&= \gamma_1 \overline{B} - 2^{(2-p)/(p-1)} \Big(\frac{\overline{C}}{N}\Big)^{1/(p-1)} \frac{(p-1)}{p} r^{p/(p-1)} \\
&\geq B
\end{aligned}
$$

for all $r \in [R(\alpha, \overline{B}), R_*(B)]$. Therefore, (2.4) follows if we can prove that $u(r,\alpha) \geq v_\alpha(r)$ on $[R(\alpha, \overline{B}), R_*(B)]$.

In fact, we have

$$
\text{(2.17)} \qquad \big(r^{N-1}\Phi_p(u')\big)' - \big(r^{N-1}\Phi_p(v_\alpha')\big)' = r^{N-1}\big\{\overline{C} - f(u(r,\alpha))\big\} \geq 0
$$

as long as $u(r,\alpha) > 0$. That is,

$$
\text{(2.18)} \qquad (p-1)\big(r^{N-1}|\xi(r)|^{p-2}(u - v_\alpha)'\big)' \geq 0
$$

as long as $u(r,\alpha) > 0$. Here $\xi(r)$ is between $u'(r)$ and $v_\alpha'(r)$. Integrating (2.18) twice and using (2.14)–(2.15), we obtain $u(r,\alpha) \geq v_\alpha(r)$ on $[R(\alpha, \overline{B}), R_*(B)]$. This proves the first inequality of (2.4).

Finally, (2.5) follows from (2.4), (2.10) and (2.11). The proof is complete. □

Remark 2.5. If the growth of f is critical, then $R(\alpha)$ may tend to 0 as $\alpha \to \infty$. Indeed, let us consider

$$
f(u) = \begin{cases}
\frac{N(N-p)^{p-1}}{p-1} \varepsilon^{p(2-p)} u^{(N(p-1)+p)/(N-p)} & \text{if } u \geq 1,\ p \geq 2 \\
\frac{N(N-p)^{p-1}}{p-1} \varepsilon^{p(2-p)} u^{(N(p2^{(2-p)/(p-1)}-1)+p)/(N-p)} & \text{if } u \geq 1,\ 1 < p < 2 \\
\frac{N(N-p)^{p-1}}{p-1} & \text{if } u \leq 1.
\end{cases}
$$

Then it is well known for any $\varepsilon \in (0,1)$ that

$$
U_\varepsilon(r) = \Big(\frac{\varepsilon}{\varepsilon^2 + r^{p/(p-1)}}\Big)^{(N-p)/p}
$$

is a solution of (2.3)–(2.4) for $U_\varepsilon(r) > 1$, $p \geq 2$. Note that $U_\varepsilon(0) = \varepsilon^{-(N-p)/p} \equiv \alpha$ which tends to ∞ as $\varepsilon \to 0^+$. Let $A = 1$ in (H_1). Then it is easy to verify that

$$
R_1(\alpha)^{p/(p-1)} = \varepsilon - \varepsilon^2
$$

and

$$-u'(R_1(\alpha),\alpha) = \frac{N-p}{p-1}(\varepsilon - \varepsilon^2)^{1/p}\varepsilon^{-1},$$

and so

$$\lim_{\varepsilon\to 0^+} -u'(R_1(\alpha),\alpha)R_1(\alpha) = \frac{N-p}{p-1}$$

which is the contrary of (2.11). Using (2.16), it is easy to see that $R(\alpha)$ behaves like $\alpha^{-\frac{1}{(p-1)N}}$, which tends to 0 as $\alpha \to +\infty$.

Lemma 2.6 ([22]). *Let f be nondecreasing for $0 < s < 1$, and f satisfies*

(i) $f \in C^1(0,\infty) \cup C^0([0,\infty))$;
(ii) $f(s) > 0$ *for* $s \geq 0$ *and* $|f'(s)|$ *is bounded in* $[0,1]$;
(iii) *there exists* $\mu > p-1$ *such that*

$$s^{-\mu} f(s) \to \beta \ \text{ as } \ s \to \infty;$$

(iv) $\limsup_{s\to 0^+} (f(s)/s^{p-1})' < 0$.

Then problem (1.1)–(1.2) *has only one positive small solution for* λ *sufficiently small.*

Lemma 2.7 (Weak comparison principle) [20], [21]. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and $\varphi : (0,\infty) \to (0,\infty)$ is continuous and non-decreasing. Let $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy*

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla\psi\,dx + \int_\Omega \varphi u_1 \psi\,dx \leq \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla\psi\,dx + \int_\Omega \varphi u_2 \psi\,dx$$

for all non-negative $\psi \in W^{1,p}(\Omega)$. Then the inequality

$$u_1 \leq u_2 \ \text{ on } \ \partial\Omega$$

implies that

$$u_1 \leq u_2 \ \text{ in } \ \Omega.$$

Lemma 2.8. *Assume that f satisfies (H_1) for $p \geq 2$ or $(H_1)'$ for $1 < p < 2$, and*

(H_2) $f(u) > 0$ *for* $u > 0$;
(H_3) (i) $f(0) > 0$;
(ii) $f(0) = 0$ *and* $\lim_{s\to 0^+} f(s)/s^{p-1} > 0$.

Then $R(\alpha) < \infty$, for all $\alpha > 0$.

Proof: The hypothesis of the Theorem implies there is an $\epsilon > 0$ such that

$$f(u) \geq \epsilon u^{p-1} \ \text{ for all } \ u \geq 0. \tag{2.19}$$

It is easy to see that $R(\alpha) < \infty$ for all $\alpha > 0$. In fact, consider the problem

$$\begin{aligned} \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + f(u) = 0 \quad &\text{in} \quad B_R, \\ u = 0 \quad &\text{on} \quad \partial B_R. \end{aligned}$$

Let $R = R(\alpha)$, consider the transformation $r = Rs$ and denote $v(s,\alpha) = u(r,\alpha)$. Then v satisfies the problem

$$\operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big) + R^p f(v) = 0 \quad \text{in} \quad B_1, \tag{2.20}$$

$$v = 0 \quad \text{on} \quad \partial B_1. \tag{2.21}$$

Suppose that there exists a sequence $\{(R_n, v_n)\}$ (where $R_n = R(\alpha_n)$, $v_n(s) = v(s,\alpha_n)$) satisfying $R_n \to \infty$ as $n \to \infty$ and v_n is a positive solution of (2.20)–(2.21) for $R = R_n$. Then, $\omega_n(s) = v_n/\|v_n\|_\infty$ solves the problem

$$\begin{aligned} -\operatorname{div}\big(|\nabla \omega_n|^{p-2}\nabla \omega_n\big) = R_n^p \frac{f(v_n)}{\|v_n\|_\infty^{p-1}} \quad &\text{in} \quad B_1, \\ \omega_n(s) = 0 \quad &\text{on} \quad \partial B_1. \end{aligned}$$

It follows from the above problem that

$$\omega_n(s) = R_n^{p/(p-1)} G_p^1\Big(\frac{f(v_n)}{\|v_n\|_\infty^{p-1}}\Big),$$

where G_p^1 is the inverse of $A_p^1 = -\operatorname{div}(|\nabla\cdot|^{p-2}\nabla\cdot)$ under the Dirichlet boundary condition. By Lemma 2.7 and (2.19) imply that

$$\omega_n(s) \geq \big(\epsilon R_n^p\big)^{1/(p-1)} G_p^1\big(\omega_n^{p-1}\big) = \big(\epsilon R_n^p\big)^{1/(p-1)} \eta_n(s). \tag{2.22}$$

Here η_n satisfies

$$\begin{aligned} -\operatorname{div}\big(|\nabla \eta_n|^{p-2}\nabla \eta_n\big) = \omega_n^{p-1} \quad &\text{in} \quad B_1, \\ \eta_n = 0 \quad &\text{on} \quad \partial B_1. \end{aligned}$$

Since $\omega_n > 0$ and $\|\omega_n\|_\infty = 1$ for any n, the compactness of G_p^1 from $C^0(B_1)$ to $C^1(\overline{B}_1)$ implies that there exists a subsequence of $\{\eta_n(s)\}$ (still denoted by $\{\eta_n(s)\}$ later) such that $\eta_n \to \eta$ in $C^1(\overline{B}_1)$ as $n \to \infty$ and $\eta(s) > 0$ in B_1. Now we easily obtain a contradiction from (2.22) since $R_n \to \infty$ as $n \to \infty$. The proof is complete. □

Theorem 2.9. *Assume that f satisfies (H_1) for $p \geq 2$ or $(H_1)'$ for $1 < p < 2$. If $f(s) > 0$ for $s \geq 0$, then there exists $\lambda_* > 0$ such that there is no positive non-small radially symmetric solution of equations* (1.1)–(1.2) *for any $\lambda \in (0, \lambda_*)$. If $f(0) \leq 0$, then there exists $\lambda_* > 0$ such that there is no positive radially symmetric solution of the problem* (1.1)–(1.2) *for any $\lambda \in (0, \lambda_*)$.*

PROOF: It is easy to see that $(u(\cdot), \lambda)$ is a positive radial solution of equations (1.1)–(1.2) if and only if $u(\cdot, \alpha)$ is a positive solution of equations (2.1)–(2.2) with $u(r) = u(\lambda^{1/p} r, \alpha)$ and $\lambda = R^p(\alpha)$, where $R(\alpha)$ is the first zero of $u(\cdot, \alpha)$. By Lemma 2.8, we have $R(\alpha) < \infty$ for all $\alpha > 0$. Therefore the solution set of (2.1)–(2.2) can be written as $\{(u(\cdot, \alpha), \lambda(\alpha)) : \alpha \in (0, \infty)\}$ with $\lambda(\alpha) = R^p(\alpha)$. Therefore, it is sufficient to study $R(\alpha)$ for $\alpha \in (0, \infty)$.

It is clear that $R(\alpha) > 0$ for $\forall \alpha \in (0, \infty)$. It is also easy to see that $\alpha_k \to \alpha_0 \in (0, \infty)$ and then $R(\alpha_0) > 0$. Hence, by Lemma 2.4, the only possibility for the case where $R(\alpha)$ tends to 0 as $\alpha \to 0^+$. We shall rule out this possibility by considering the following cases: (i) $f(0) = 0$, $\lim_{s \to 0^+} f(s)/s^{p-1} > 0$; (ii) $f(0) = 0$, $\lim_{s \to 0^+} f(s)/s^{p-1} = 0$; (iii) $f(0) = 0$ and $\lim_{s \to 0^+} f(s)/s^{p-1} < 0$ and (iv) $f(0) < 0$. For the case where $f(0) > 0$ and f is nondecreasing for $0 < s < 1$, we know from Lemma 2.6 that there exists a unique positive small solution $u(r, \lambda)$ which will tend to zero uniformly in Ω as $\lambda \to 0^+$. This implies that $u(\cdot, \alpha)$ is a positive small solution if $R(\alpha)$ is sufficiently small.

Case (i). In this case, we shall prove that problem (1.1)–(1.2) has no positive radially symmetric solution u_λ with $\|u_\lambda\|_\infty \to 0$ when λ is sufficiently small.

If $\lim_{s \to 0^+} f(s)/s^{p-1} = \alpha > 0$, suppose that there exists a sequence $\{(\lambda_n, u_n)\}$ satisfying $\lambda_n \to 0$ as $n \to \infty$ and u_n is a radially symmetric positive solution of equations (1.1)–(1.2) for $\lambda = \lambda_n$ such that $\|u_n\|_\infty \to 0$ as $n \to \infty$. Then, $\omega_n(x) = u_n / \|u_n\|_\infty$ satisfies

$$-\operatorname{div}\left(|\nabla \omega_n|^{p-2} \nabla \omega_n\right) = \lambda_n \frac{f(\|u_n\|_\infty \omega_n)}{\|u_n\|_\infty{}^{p-1}} \omega_n^{p-1} \quad \text{in} \quad B_1, \tag{2.23}$$

$$\omega_n(x) = 0 \quad \text{on} \quad \partial B_1. \tag{2.24}$$

Since $\omega_n > 0$, $\|\omega_n\|_\infty = 1$ for any n and $\frac{f(\|u_n\|_\infty \omega_n)}{(\|u_n\|_\infty \omega_n)^{p-1}} \to \alpha$ as $n \to \infty$, the compactness of G_p^1 from $C^0(B_1)$ to $C_0^1(\overline{B_1})$ (see [12]) implies that there exists a subsequence of $\{\omega_n\}$ (still denoted by $\{\omega_n\}$ later) and $\overline{\omega} \in C_0^1(\overline{B_1})$ such that $\omega_n \to \overline{\omega}$ in $C^1(\overline{B_1})$. Thus, $\overline{\omega}$ is a bounded solution of

$$-\operatorname{div}\left(|\nabla \overline{\omega}|^{p-2} \nabla \overline{\omega}\right) = 0 \quad \text{in} \quad B_1,$$
$$\overline{\omega} = 0 \quad \text{on} \quad \partial B_1.$$

This implies that $\overline{\omega} \equiv 0$ in B_1. This contradicts the facts that $\omega_n \to \overline{\omega}$ in $C^1(\overline{B_1})$ and $\|\omega_n\|_\infty = 1$.

If $\lim_{s\to 0^+} f(s)/s^{p-1} = +\infty$, suppose that there exists a sequence $\{(\lambda_n, u_n)\}$ satisfying $\lambda_n \to 0$ as $n \to \infty$ and u_n is a radial positive solution of equations (1.1)–(1.2) for $\lambda = \lambda_n$ such that $\|u_n\|_\infty \to 0$ as $n \to \infty$. Then $\omega_n(x) = u_n/\|u_n\|_\infty$ satisfies

$$\begin{aligned} -(r^{N-1}\Phi_p(\omega_n'))' &= \lambda_n r^{N-1}\|u_n\|_\infty^{(p-1)} f(\|u_n\|_\infty \omega_n) \ \text{ in } \ (0,1), \\ \omega_n'(0) &= 0, \ \omega_n(1) = 0 \end{aligned} \tag{2.25}$$

and $\omega_n(0) = 1$. First, we shall prove that $\tau_n = \lambda_n \|u_n\|_\infty^{(p-1)}$ is uniformly bounded. Suppose that $\tau_n \to \infty$ as $n \to \infty$. Let $y_n = \tau_n^{1/p} r, \widetilde{\omega}_n(y_n) = \omega_n(r)$. Then $\widetilde{\omega}_n$ satisfies

$$\begin{aligned} -\operatorname{div}\left(|\nabla\widetilde{\omega}_n|^{p-2}\nabla\widetilde{\omega}_n\right) &= f(\|u_n\|_\infty \widetilde{\omega}_n) \ \text{ in } \ B_n, \\ \widetilde{\omega}_n &= 0 \ \text{ on } \ \partial B_n. \end{aligned}$$

Here B_n is B_1 under the change of variables. Since $\|u_n\|_\infty \to 0$ as $n \to \infty$ and $f(0) = 0$, we have that $\widetilde{\omega}_n \to \widetilde{\omega}$ in $C^1_{\text{loc}}(0,\infty)$ as $n \to \infty$ and $\widetilde{\omega}(r)$ is a bounded solution of

$$-\operatorname{div}\left(|\nabla\widetilde{\omega}|^{p-2}\nabla\widetilde{\omega}\right) = 0 \ \text{ in } \ \mathbb{R}^N$$

with $\|\widetilde{\omega}\|_\infty = 1$. This implies that $\widetilde{\omega} \equiv 0$ in $\mathbb{R}^N$. This contradicts the fact that $\|\widetilde{\omega}\|_\infty = 1$. Thus, $\{\tau_n\}$ is uniformly bounded. Then, equation (2.25) and $\|\omega_n\|_\infty = 1$ imply that there exists a subsequence of $\{\omega_n\}$ and $\omega \in C^1_0(\overline{B_1})$ such that $\omega_n \to \omega$ in $C^1(\overline{B_1})$. Then ω is a bounded solution of the problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla\omega|^{p-2}\nabla\omega\right) &= 0 \ \text{ in } \ B_1, \\ \omega &= 0 \ \text{ on } \ \partial B_1 \end{aligned}$$

with $\|\omega\|_\infty = 1$. This implies that $\omega \equiv 0$. This contradicts the fact that $\|\omega\|_\infty = 1$.

Case (ii). In this case, we shall prove that $\lim_{\alpha\to 0^+} R(\alpha) = \infty$. We observe that $u(\cdot,\alpha)$ satisfies the following equation:

$$u(r,\alpha) = \alpha - \int_0^r \Big(\int_0^s (\frac{z}{s})^{N-1} f(u(z))\,dz\Big)^{1/(p-1)} ds. \tag{2.26}$$

Since $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$, for any $\epsilon > 0$ there exists $\delta > 0$ such that $f(u) \le \epsilon u^{p-1}$ for $u \in (0,\delta)$. Therefore, if $u(r,\alpha) \in (0, 2\alpha) \subset (0,\delta)$ then

$|f(u(r,\alpha))| \le 2^{p-1}\epsilon\alpha^{p-1}$. Now, it is easy to verify that

$$\begin{aligned} &\Big|\int_0^r \Big(\int_0^s (\frac{z}{s})^{N-1} f(u(z,\alpha))\,dz\Big)^{1/(p-1)} ds\Big| \\ &\le \int_0^r \Big(\int_0^s (\frac{z}{s})^{N-1} |f(u(z,\alpha))|\,dz\Big)^{1/(p-1)} ds \\ &\le 2\alpha\epsilon^{1/(p-1)}\Big(\int_0^r s^{(1-N)/(p-1)}\Big(\int_0^s z^{N-1}\,dz\Big)^{1/(p-1)} ds\Big) \qquad (2.27)\\ &= 2\alpha\epsilon^{1/(p-1)}\Big(\frac{1}{N}\Big)^{1/(p-1)}\Big(\int_0^r s^{1/(p-1)}\,ds\Big) \\ &= \Big(\frac{1}{N}\Big)^{1/(p-1)} 2\alpha\epsilon^{1/(p-1)} \frac{(p-1)}{p} r^{p/(p-1)} \end{aligned}$$

as far as $u(s,\alpha) \in (0,2\alpha)$ for all $s \in (0,r)$. Hence, by (2.26)–(2.27), and for $\alpha \in (0,\delta/2)$ and $r \in (0, (\frac{p}{2(p-1)})^{(p-1)/p}(N/\epsilon)^{1/p})$, we have

$$|u(r,\alpha)| \le \alpha + \Big|\int_0^r \Big(\int_0^s (\frac{z}{s})^{N-1} f(u(z))\,dz\Big)^{1/(p-1)} ds\Big| \le 2\alpha,$$

so $u(r,\alpha) \in (0,2\alpha)$. This implies $\lim_{\alpha\to 0^+} R(\alpha) = \infty$.

Case (iii). In this case, there are positive constants m and δ such that $-mu^{p-1} \le f(u) \le 0$ on $[0,\delta]$. Therefore, if $u(s,\alpha) \in [0,\delta]$ for all $s \in (0,r)$, then by (2.26) we have

$$\begin{aligned} u(r,\alpha) &\le \alpha + m^{1/(p-1)} \int_0^r \Big(\int_0^s (\frac{z}{s})^{N-1} u^{p-1}(z,\alpha)\,dz\Big)^{1/(p-1)} ds \qquad (2.28)\\ &\le \alpha + m^{1/(p-1)} u(r,\alpha) \frac{(p-1)}{p}\Big(\frac{1}{N}\Big)^{1/(p-1)} r^{p/(p-1)}. \end{aligned}$$

Hence, if $u(R(\alpha,\delta),\alpha) = \delta$, then (2.26) implies that
$R^{p/(p-1)}(\alpha,\delta) \ge \frac{p(\delta-\alpha)N^{1/(p-1)}}{\delta(p-1)m^{1/(p-1)}}$ and so $R(\alpha)$ has a positive lower bound for $\alpha \in (0,\delta/2)$.

Case (iv). In this case, there are $\epsilon > 0$ and $\delta > 0$ such that $f(u) \le -\epsilon$ on $[0,\delta]$. Let $\overline{C} = -\epsilon$ in (2.13), $R(\alpha,\overline{B}) = 0$, $\overline{B} = \alpha$ in (2.14), and $u'(0,\alpha) = 0$ in (2.15). Then (2.18) becomes $v_\alpha(r) = \alpha + (\frac{\epsilon}{N})^{1/(p-1)}(\frac{p-1}{p})r^{p/(p-1)}$ which implies that

$$u(r,\alpha) \ge v_\alpha(r) = \alpha + \Big(\frac{\epsilon}{N}\Big)^{1/(p-1)}\Big(\frac{p-1}{p}\Big) r^{p/(p-1)} \qquad (2.29)$$

as long as $u(r,\alpha) \in [0,\delta]$. In particular, $R(0) > 0$. The continuous dependence of $u(\cdot,\alpha)$ in α and (2.29) imply that there is a positive lower bound for $R(\alpha)$ for all $\alpha \in [0,\delta]$. The proof of Theorem 2.9 is complete. □

Remark 2.10. It is worth remarking that the validity of Theorem 2.9 relies on the topology of the domain Ω. Indeed when Ω is an annular domain, i.e., $\Omega = \{x \in \mathbb{R}^N : a < |x| < b\}$, $N \geq 2$, and $f(u)$ is continuous and $\lim_{u\to\infty} \frac{f(u)}{|u|^{p-2}u} = \infty$ (f is superlinear) uniformly for $t \in [a, b]$, there is at least one positive non-small solution for each $\lambda \in (0, \lambda^*)$, see [18], [24] and the reference therein.

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