

Characterization of the strict convexity of the Besicovitch-Musielak-Orlicz space of almost periodic functions

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Abstract. We introduce the new class of Besicovitch-Musielak-Orlicz almost periodic functions and consider its strict convexity with respect to the Luxemburg norm.

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1. Introduction

We denote by $C^0 a.p.$ the linear set of all continuous almost periodic functions (*u.a.p.*). Let A be the subspace of $C^0 a.p.$ whose elements are the generalized trigonometric polynomials i.e.,

$$A = \left\{ P_n(t) = \sum_{j=1}^n a_j e^{i\lambda_j t}, a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

The class $C^0 a.p.$ is in fact the closure of A in the uniform norm of $C_b(\mathbb{R})$ (the space of continuous and bounded functions on \mathbb{R}).

This topological characterization is used to define widest classes of almost periodic functions as the closure of the linear set A with respect to some specific norms.

The first extension was obtained by A.S. Besicovitch (cf. [2]) in the context of L^p spaces. Namely he defined the $S_{a.p.}^q$, $W_{a.p.}^q$ and $B_{a.p.}^q$ spaces (resp. Stepanoff, Weyl and Besicovitch spaces of almost periodic functions). Later on, T.R. Hillmann (cf. [5]) used a similar approach to obtain an extension in the context of Orlicz spaces.

Most of the Hillmann's work concerns topological and structural properties of the new spaces.

In [9], [10], [11], there are considered the fundamental geometric properties of the Besicovitch-Orlicz spaces of almost periodic functions.

In this paper, we consider the natural extension of almost periodicity to the context of Besicovitch-Musiela-Orlicz spaces, in particular the case when the function φ generating the space depends on a parameter.

The theory of spaces of generalized almost periodic functions was since its beginning a subject of great interest. This was essentially motivated by the development of the theory of differential and partial differential equations with almost periodic terms (cf. [1], [8], [13]).

Actually this interest is still in growth and is enlarged to cover new domains of applications.

2. Preliminaries

In the sequel $\varphi : \mathbb{R} \times [0, +\infty[\rightarrow [0, +\infty[$ will be a continuous function on $\mathbb{R} \times [0, +\infty[$ satisfying:

- (i) For every $t \in \mathbb{R}, \varphi(t, 0) = 0$.
- (ii) For each $t \in \mathbb{R}, \varphi(t, u)$ is convex with respect to $u \in [0, +\infty[$.
- (iii) For every $u \in [0, +\infty[$, $\varphi(t, u)$ is periodic with respect to $t \in \mathbb{R}$, the period τ being fixed and independent of $u \in [0, +\infty[$. Without loss of generality we may suppose that $\tau = 1$.
- (iv) For each $\alpha > 0$, we have $\inf_{t \in \mathbb{R}} \varphi(t, \alpha) = \phi(\alpha) > 0$.

We denote by $M(\mathbb{R})$ the space of all real valued Lebesgue measurable functions. The functional

$$\rho_\varphi : M(\mathbb{R}) \rightarrow [0, +\infty]$$

$$f \mapsto \rho_\varphi(f) = \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi(t, |f(t)|) dt$$

is a convex pseudomodular (cf. [10], [12]).

We define the Besicovitch-Musiela-Orlicz space associated to this pseudomodular by

$$B^\varphi(\mathbb{R}) = \left\{ f \in M(\mathbb{R}) : \lim_{\alpha \rightarrow 0} \rho_\varphi(\alpha f) = 0 \right\}$$

$$= \{ f \in M(\mathbb{R}) : \rho_\varphi(\alpha f) < +\infty, \text{ for some } \alpha > 0 \}.$$

The space $B^\varphi(\mathbb{R})$ is naturally endowed with the pseudonorm

$$\|f\|_\varphi = \inf \left\{ k > 0 : \rho_\varphi \left(\frac{f}{k} \right) \leq 1 \right\}, \quad f \in B^\varphi(\mathbb{R}).$$

Let A be the set of all generalized trigonometric polynomials, i.e.,

$$A = \left\{ P_n(t) = \sum_{j=1}^n a_j e^{i\lambda_j t}, a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

We denote by $\tilde{B}_{a.p.}^\varphi(\mathbb{R})$ (resp. $B_{a.p.}^\varphi(\mathbb{R})$) the closure of A with respect to the pseudomodular ρ_φ (resp. with respect to the pseudonorm $\|\cdot\|_\varphi$), more precisely:

$$\begin{aligned} \tilde{B}_{a.p.}^\varphi(\mathbb{R}) &= \left\{ f \in B^\varphi(\mathbb{R}) : \exists f_n \in A, \exists k_0 > 0, \lim_{n \rightarrow +\infty} \rho_\varphi(k_0(f_n - f)) = 0 \right\}, \\ B_{a.p.}^\varphi(\mathbb{R}) &= \left\{ f \in B^\varphi(\mathbb{R}) : \exists f_n \in A, \forall k > 0, \lim_{n \rightarrow +\infty} \rho_\varphi(k(f_n - f)) = 0 \right\} \\ &= \left\{ f \in B^\varphi(\mathbb{R}) : \exists f_n \in A, \lim_{n \rightarrow +\infty} \|f_n - f\|_\varphi = 0 \right\}. \end{aligned}$$

$\tilde{B}_{a.p.}^\varphi(\mathbb{R})$ and $B_{a.p.}^\varphi(\mathbb{R})$ will be called Besicovitch-Musielak-Orlicz spaces of almost periodic functions.

It is clear that

$$B_{a.p.}^\varphi(\mathbb{R}) \subseteq \tilde{B}_{a.p.}^\varphi(\mathbb{R}) \subseteq B^\varphi(\mathbb{R}).$$

When $\varphi(t, |x|) = |x|$, we denote by $B^1(\mathbb{R})$ and $B^1_{a.p.}(\mathbb{R})$ the respective spaces. The notation ρ_1 is used for the associated pseudomodular.

Recall that the function φ is said to be strictly convex if $\varphi(t, \lambda u + (1 - \lambda)v) < \lambda\varphi(t, u) + (1 - \lambda)\varphi(t, v)$ for almost all $t \in \mathbb{R}$ and for every $0 \leq u < v < +\infty$, $0 < \lambda < 1$.

A normed linear space $(X, \|\cdot\|)$ is strictly convex if $\left\| \frac{x+y}{2} \right\| < 1$ whenever $\|x\| = \|y\| = 1$ and $\|x - y\| > 0$.

We say that φ satisfies the Δ_2 -condition ($\varphi \in \Delta_2$) if there exist $k > 1$ and a measurable nonnegative function h such that $\rho_\varphi(h) < +\infty$ and $\varphi(t, 2u) \leq k\varphi(t, u)$ for almost all $t \in \mathbb{R}$ and all $u \geq h(t)$.

3. Auxiliary results

The space $B_{a.p.}^\varphi(\mathbb{R})$ can be regarded as a subspace of measurable functions on \mathbb{R} with respect to Lebesgue measure. However, the theory of $B_{a.p.}^\varphi(\mathbb{R})$ spaces is different from that of $L^\varphi(\mathbb{R})$ spaces: the usual convergence results of the Lebesgue measure theory are not valid in the $B_{a.p.}^\varphi(\mathbb{R})$ spaces (see [11]).

To handle $B_{a.p.}^\varphi(\mathbb{R})$ spaces as $L^\varphi(\mathbb{R})$ ones, we introduce the set function $\bar{\mu}$.

Let $\Sigma = \Sigma(\mathbb{R})$ be the σ -algebra of all Lebesgue measurable subsets of \mathbb{R} . We denote by $\bar{\mu}$ the set function defined on Σ by

$$\bar{\mu}(A) = \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \chi_A(t) dt = \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \mu(A \cap [-T, +T]),$$

where μ denotes the Lebesgue measure on \mathbb{R} .

It is easily seen that the set function $\bar{\mu}$ is not σ -additive.

A sequence $\{f_n\} \subset B^\varphi(\mathbb{R})$ is said to be $\bar{\mu}$ -convergent to some $f \in B^\varphi(\mathbb{R})$ (in symbol $f_n \xrightarrow{\bar{\mu}} f$) when, for every $\alpha > 0$, we have

$$\lim_{n \rightarrow +\infty} \bar{\mu} \{x \in \mathbb{R} : |f_n(x) - f(x)| > \alpha\} = 0.$$

We give here some technical results that are the key arguments in the proof of the main theorem.

Lemma 1. *Let $\nu(A) = \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi(t, \chi_A(t)) dt$. Then the set function $\bar{\mu}$ is absolutely continuous with respect to ν , i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(3.1) \quad (A \in \Sigma, \nu(A) < \delta) \Rightarrow (\bar{\mu}(A) < \varepsilon).$$

PROOF: Suppose that (3.1) is false. Then for some $\varepsilon_0 > 0$ we will have the following:

for each $n \in \mathbb{N}$, there exists $E_n \in \Sigma$ s.t. $\nu(E_n) < \frac{1}{2n}$ and $\bar{\mu}(E_n) > \varepsilon_0$. Thus

$$\begin{aligned} \nu(E_n) &= \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi(t, \chi_{E_n}(t)) dt \\ &= \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi(t, 1) \chi_{E_n}(t) dt \\ &\geq \phi(1) \bar{\mu}(E_n) \geq \phi(1) \varepsilon_0, \end{aligned}$$

a contradiction. □

Lemma 2. *Let $\{f_n\}_{n \geq 1} \subset B_{a.p.}^\varphi(\mathbb{R})$ be a sequence modular convergent to $f \in B_{a.p.}^\varphi(\mathbb{R})$, i.e., $\lim_{n \rightarrow +\infty} \rho_\varphi(f_n - f) = 0$. Then $f_n \xrightarrow{\bar{\mu}} f$.*

PROOF: Notice first that we have also $\lim_{n \rightarrow +\infty} \rho_\phi(f_n - f) = 0$. Then from a similar result for functions without parameter (cf. [10]) it follows that $f_n \xrightarrow{\bar{\mu}} f$. □

Lemma 3. *Let $h \in B^\varphi(\mathbb{R})$ be such that $\rho_\varphi(h) = a > 0$. Then for every $\bar{\theta} \in (0, 1)$ there exist constants $\beta > 0, T_0 > 0$ and a set $\bar{G} = \{t \in \mathbb{R}, |h(t)| \leq \beta\}$ such that*

$$(3.2) \quad \mu \{\bar{G} \cap [-T, +T]\} \geq \bar{\theta} 2T, \text{ for } T \geq T_0.$$

PROOF: It is clear that $h \in B^\phi(\mathbb{R})$. Then if $\rho_\phi(h) > 0$ the conclusion follows from a similar result for the function ϕ without parameter (cf. [10]). The conclusion is immediate if $\rho_\phi(h) = 0$. □

Lemma 4. *Let $g \in B_{a.p.}^\varphi(\mathbb{R})$. Then for all $\varepsilon > 0$ there exist $\delta > 0$ and $T_0 > 0$ such that $\rho_\varphi(g\chi_Q) \leq \varepsilon$, for all $Q \in \Sigma$ satisfying $\mu\{Q \cap [-T, +T]\} \leq 2\delta T$, $T \geq T_0$.*

PROOF: We may suppose $\rho_\varphi(g) > 0$.

Let $\varepsilon > 0$ and $P_\varepsilon \in A$ be such that $\rho_\varphi(2(g - P_\varepsilon)) < \frac{\varepsilon}{2}$. Using the properties of φ we have $\varphi(t, 2|P_\varepsilon(t)|) \in C^0 a.p.$ (cf. [4]). We then put $M_\varepsilon = \sup_{t \in \mathbb{R}} \varphi(t, 2|P_\varepsilon(t)|)$.

We choose $\bar{\theta} \in (0, 1)$ satisfying $M_\varepsilon(1 - \bar{\theta}) < \frac{\varepsilon}{2}$. Then by Lemma 3 there exist $\beta > 0$ and a set $\bar{G} = \{t \in \mathbb{R}, |g(t)| \leq \beta\}$ for which $\mu\{\bar{G} \cap [-T, +T]\} \geq 2\bar{\theta}T$, $\forall T \geq T_0$, for some $T_0 > 0$. Hence, denoting by \bar{G}' the complement of \bar{G} , we will have for all $T \geq T_0$,

$$\begin{aligned}
 (3.3) \quad & \frac{1}{2T} \int_{\bar{G}' \cap [-T, +T]} \varphi(t, |g(t)|) dt \\
 & \leq \frac{1}{2} \left(\frac{1}{2T} \int_{\bar{G}' \cap [-T, +T]} [\varphi(t, 2|g(t) - P_\varepsilon(t)|) + \varphi(t, 2|P_\varepsilon(t)|)] dt \right) \\
 & \leq \frac{\varepsilon}{4} + \frac{1}{4T} M_\varepsilon (1 - \bar{\theta}) 2T \leq \frac{\varepsilon}{2}.
 \end{aligned}$$

We put $\delta = \frac{\varepsilon}{2 \sup_{t \in \mathbb{R}} \varphi(t, \beta)}$ and let $Q \subset \mathbb{R}$ be such that $\mu\{Q \cap [-T, +T]\} \leq 2\delta T$ for $T \geq T_0$.

Then if $Q_1 = Q \cap \bar{G}$ and $Q_2 = Q \cap \bar{G}'$, we will have

$$\begin{aligned}
 \frac{1}{2T} \int_{Q_1 \cap [-T, T]} \varphi(t, |g(t)|) dt & \leq \frac{1}{2T} \int_{Q_1 \cap [-T, T]} \varphi(t, \beta) dt \\
 & \leq \frac{1}{2T} \mu(Q_1) \sup_{t \in \mathbb{R}} \varphi(t, \beta) \\
 & \leq \delta \sup_{t \in \mathbb{R}} \varphi(t, \beta) \leq \frac{\varepsilon}{2}.
 \end{aligned}$$

Similarly using (3.3) we get

$$\frac{1}{2T} \int_{Q_2} \varphi(t, |g(t)|) dt \leq \frac{1}{2T} \int_{\bar{G}' \cap [-T, +T]} \varphi(t, |g(t)|) dt \leq \frac{\varepsilon}{2}.$$

Finally for all $T \geq T_0$, we have

$$\frac{1}{2T} \int_{Q \cap [-T, +T]} \varphi(t, |g(t)|) dt \leq \varepsilon,$$

which means that $\rho_\varphi(g\chi_Q) \leq \varepsilon$. □

Proposition 1. *Let $f \in B_{a.p.}^\varphi(\mathbb{R})$. Then $\varphi(t, |f(t)|) \in B_{a.p.}^1(\mathbb{R})$ and consequently the limit $\lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi(t, |f(t)|) dt$ exists and is finite.*

PROOF: Let $\{f_n\}$ be a sequence of trigonometric polynomials such that $\|f_n - f\|_\varphi \rightarrow 0$. Then using Lemma 2 we have also $f_n \xrightarrow{\bar{\mu}} f$.

Let $\bar{\theta} \in (0, 1)$. In view of Lemma 3, there exist $\beta > 0$ and $T_0 > 0$ for which $\bar{\mu}(\bar{G}) \geq \bar{\theta}$ with $\bar{G} = \{t \in \mathbb{R} : |f(t)| \leq \beta\}$.

Let $\alpha > 0$ and $A_n^\alpha = \{t \in \mathbb{R} : |f_n(t) - f(t)| > \alpha\}$. It is easily seen that $|f_n(t)| \leq \beta + \alpha, \forall t \in \bar{G} \cap (A_n^\alpha)'$.

Now, the function φ being continuous on $\mathbb{R} \times [0, +\infty[$, is also uniformly continuous on $[0, 1] \times [0, \alpha + \beta]$. Moreover, using the periodicity of $\varphi(t, u)$ with respect to $t \in \mathbb{R}$, it follows that φ is uniformly continuous on $\mathbb{R} \times [0, \alpha + \beta]$.

Then for every $\eta > 0$ there exists $\alpha_\eta > 0$ such that

$$\forall t \in \bar{G} \cap (A_n^\alpha)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \implies |f_n(t) - f(t)| > \alpha_\eta.$$

Hence, since $f_n \xrightarrow{\bar{\mu}} f$ we get also

$$\lim_{n \rightarrow +\infty} \bar{\mu} \{t \in \bar{G} \cap (A_n^\alpha)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} = 0.$$

Consequently,

$$\begin{aligned} & \bar{\mu} \{t \in \mathbb{R} : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} \\ & \leq \bar{\mu} \left\{ t \in \bar{G} \cap (A_n^\alpha)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \right\} \\ & \quad + \bar{\mu} \left\{ t \in (\bar{G})' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \right\} \\ & \quad + \bar{\mu} \{t \in A_n^\alpha : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} \\ & \leq \bar{\mu} \left\{ t \in \bar{G} \cap (A_n^\alpha)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \right\} \\ & \quad + \bar{\mu} \left((\bar{G})' \right) + \bar{\mu} (A_n^\alpha) \\ & \leq \bar{\mu} \left\{ t \in \bar{G} \cap (A_n^\alpha)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \right\} \\ & \quad + (1 - \bar{\theta}) + \bar{\mu} (A_n^\alpha). \end{aligned}$$

Letting n tend to infinity, we will have

$$\overline{\lim}_{n \rightarrow +\infty} \bar{\mu} \{t \in \mathbb{R} : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} \leq (1 - \bar{\theta}).$$

Finally, since $\bar{\theta} \in (0, 1)$ is arbitrary, we deduce that for all $\eta > 0$

$$(3.4) \quad \lim_{n \rightarrow +\infty} \bar{\mu} \{t \in \mathbb{R} : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} = 0.$$

On the other hand, using Lemma 4, it is easy to see that given $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following implication holds

$$(Q \in \Sigma, \bar{\mu}(Q) \leq \delta) \implies \max(\rho_\varphi(f\chi_Q), \rho_\varphi(f_n\chi_Q)) \leq \varepsilon.$$

Let $E_n^\varepsilon = \{t \in \mathbb{R} : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \varepsilon\}$. Then since by (3.3), $\bar{\mu}(E_n^\varepsilon) \leq \delta$ for $n \geq n_0$, we get

$$\begin{aligned} & \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| dt \\ & \leq \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{E_n^\varepsilon \cap [-T, T]} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| dt \\ & \quad + \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{(E_n^\varepsilon)' \cap [-T, T]} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| dt \\ & \leq 2\varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Finally by $\varepsilon > 0$ being arbitrary we deduce that

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| dt = 0.$$

It remains to see that $\varphi(t, |f_n(t)|) \in C^0 a.p.$ This follows from the properties of the function φ and the fact that $f_n \in A$ (see [4]). □

Lemma 5. *Let $\{f_n\}_n \subset B_{a.p.}^1(\mathbb{R})$ be such that $f_n \xrightarrow{\bar{\mu}} f \in B_{a.p.}^1(\mathbb{R})$. Suppose there exists $g \in B_{a.p.}^1(\mathbb{R})$ for which $\max(|f_n(t)|, |f(t)|) \leq g(t)$, $t \in \mathbb{R}$. Then $\rho_1(f_n) \rightarrow \rho_1(f)$.*

PROOF: Take $\varepsilon > 0$ and let $\delta > 0$ be associated to g as in Lemma 4. We put $A_n^\varepsilon = \{t \in \mathbb{R} : |f_n(t) - f(t)| \geq \frac{\varepsilon}{2}\}$. Then since $f_n \xrightarrow{\bar{\mu}} f$ it follows that $\bar{\mu}(A_n^\varepsilon) \leq \delta$ for all $n \geq n_0$ and then by Lemma 4

$$\rho_1(|f_n - f| \chi_{A_n^\varepsilon}) \leq \rho_1(2g \chi_{A_n^\varepsilon}) \leq \frac{\varepsilon}{2}.$$

Consequently, for all $n \geq n_0$ we have

$$\begin{aligned} \rho_1(|f_n - f|) & \leq \rho_1(|f_n - f| \chi_{A_n^\varepsilon}) + \rho_1(|f_n - f| \chi_{(A_n^\varepsilon)'}) \\ & \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

i.e., $\lim_{n \rightarrow +\infty} \rho_1(f_n) = \rho_1(f)$. □

Lemma 6. *Let $f \in B_{a.p.}^\varphi(\mathbb{R})$. Then the functional $\lambda \mapsto \rho_\varphi\left(\frac{f}{\lambda}\right)$ is continuous on $]0, +\infty[$.*

PROOF: First, notice that since $f \in B_{a.p.}^\varphi(\mathbb{R})$ we have $\rho_\varphi(\alpha f) < +\infty$ for each $\alpha > 0$. Indeed, f being in $B_{a.p.}^\varphi(\mathbb{R})$ there exists a sequence $\{f_n\}_n \subset A$ such that $\lim_{n \rightarrow \infty} \|f - f_n\|_\varphi = 0$ or equivalently $\lim_{n \rightarrow \infty} \rho_\varphi(\alpha(f - f_n)) = 0$ for every $\alpha > 0$.

Let $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that $\rho_\varphi(2\alpha(f - f_{n_0})) \leq 1$. Then

$$\rho_\varphi(\alpha f) \leq \frac{1}{2}\rho_\varphi(2\alpha(f - f_{n_0})) + \frac{1}{2}\rho_\varphi(2\alpha f_{n_0}),$$

consequently, using the fact that the trigonometric polynomial f_{n_0} is uniformly bounded, it follows that $\rho_\varphi(\alpha f) < +\infty$.

Let now $\lambda_0 \in]0, +\infty[$ and $\{\lambda_n\}$ be a sequence of real numbers which converges to λ_0 . We have

$$\rho_\varphi\left(\frac{f}{\lambda_n} - \frac{f}{\lambda_0}\right) \leq \left|\frac{1}{\lambda_n} - \frac{1}{\lambda_0}\right| \rho_\varphi(f) \text{ for every } n \geq n_0.$$

Then $\lim_{n \rightarrow +\infty} \rho_\varphi\left(\frac{f}{\lambda_n} - \frac{f}{\lambda_0}\right) = 0$.

Now, using Lemma 2 we get $\frac{f}{\lambda_n} \xrightarrow{\bar{\mu}} \frac{f}{\lambda_0}$ and then $\varphi\left(t, \frac{|f(t)|}{\lambda_n}\right) \xrightarrow{\bar{\mu}} \varphi\left(t, \frac{|f(t)|}{\lambda_0}\right)$ (see the proof of Proposition 1). Furthermore

$$\max\left(\varphi\left(t, \frac{|f(t)|}{\lambda_n}\right), \varphi\left(t, \frac{|f(t)|}{\lambda_0}\right)\right) \leq \varphi\left(t, \frac{2}{\lambda_0}|f(t)|\right)$$

and by Proposition 1 we have $\varphi\left(t, \frac{2}{\lambda_0}|f(t)|\right) \in B_{a.p.}^1(\mathbb{R})$. Consequently, using Lemma 5 we deduce

$$\rho_\varphi\left(\frac{f}{\lambda_n}\right) \rightarrow \rho_\varphi\left(\frac{f}{\lambda_0}\right).$$

This means that $\lambda \mapsto \rho_\varphi\left(\frac{f}{\lambda}\right)$ is continuous on $]0, +\infty[$. □

Corollary 1. *Let $f \in B_{a.p.}^\varphi(\mathbb{R})$. Then*

- (1) $\|f\|_\varphi \leq 1$ if and only if $\rho_\varphi(f) \leq 1$;
- (2) $\|f\|_\varphi = 1$ if and only if $\rho_\varphi(f) = 1$.

PROOF: We prove briefly (2), the assertion (1) follows then easily.

Let $f \in B_{a.p.}^\varphi(\mathbb{R})$ with $\|f\|_\varphi = 1$. Then for $\varepsilon > 0$ we will have $\rho_\varphi\left(\frac{f}{1+\varepsilon}\right) \leq 1$ and using Lemma 6 it follows that $\rho_\varphi(f) \leq 1$.

We have also $\rho_\varphi\left(\frac{f}{1-\varepsilon}\right) \geq 1$ and again by Lemma 6 we get $\rho_\varphi(f) \geq 1$. Finally, $\rho_\varphi(f) = 1$.

The converse implication is known for a general modular space. □

Remark 1. We recall that a similar result holds in classical Musielak-Orlicz spaces under the additional Δ_2 -condition. This condition is not necessary in our case since Lemma 6 holds with the restriction $f \in B_{a.p.}^\varphi(\mathbb{R})$.

Lemma 7. Let $f \in B_{a.p.}^\varphi(\mathbb{R})$ with $\|f\|_\varphi = 1$. Then there exist real numbers $0 < \alpha < \beta$ and $\theta \in (0, 1)$ such that if $G_1 = \{t \in \mathbb{R} : \alpha \leq |f(t)| \leq \beta\}$ we have $\bar{\mu}(G_1) \geq \theta$.

PROOF: Let $\bar{\theta} \in (0, 1)$. Then from Lemma 3 there exist $\beta > 0$ and $T_0 > 0$ such that $\mu\{\bar{G} \cap [-T, +T]\} \geq \bar{\theta}2T, \forall T \geq T_0$, where $\bar{G} = \{t \in \mathbb{R} : |f(t)| \leq \beta\}$.

We claim that the following is also true:

- for each $\delta \in (0, 1)$ there exist $\tilde{\theta} \in (0, 1)$, $T_0 > 0$ and a set $\tilde{G} = \{t \in \mathbb{R}, \varphi(t, |f(t)|) \leq 1 - \delta\}$ such that for $T \geq T_0$

$$(3.5) \quad \mu\{\tilde{G} \cap [-T, +T]\} < \tilde{\theta}2T.$$

For, let $\delta \in (0, 1)$ and P_n be a sequence of trigonometric polynomials approximating f , i.e., $\|f - P_n\|_\varphi \rightarrow 0$. We take P_δ such that $\rho_\varphi(2|f - P_\delta|) < \frac{\delta}{4}$ and put $M = \sup_{t \in \mathbb{R}} \varphi(t, 2P_\delta(t))$.

Let $\varepsilon > 0$ be such that $(\frac{\delta}{4} + M\varepsilon) < \delta$ and suppose that (3.5) is not satisfied. Then taking $\tilde{\theta} = 1 - \varepsilon$, there will exist a sequence $\{T_n\}$ increasing to infinity for which $\mu\{\tilde{G} \cap [-T_n, +T_n]\} \geq \tilde{\theta}2T_n$. We then get

$$\begin{aligned} \frac{1}{2T_n} \int_{-T_n}^{+T_n} \varphi(t, |f(t)|) dt &= \frac{1}{2T_n} \int_{\tilde{G} \cap [-T_n, +T_n]} \varphi(t, |f(t)|) dt \\ &\quad + \frac{1}{2T_n} \int_{(\tilde{G})' \cap [-T_n, +T_n]} \varphi(t, |f(t)|) dt \\ &\leq (1 - \delta) + \frac{1}{2T_n} \int_{(\tilde{G})' \cap [-T_n, +T_n]} \varphi(t, |f(t)|) dt. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\frac{1}{2T_n} \int_{(\tilde{G})' \cap [-T_n, +T_n]} \varphi(t, |f(t)|) dt \\ &\leq \frac{1}{2} \left[\frac{1}{2T_n} \int_{(\tilde{G})' \cap [-T_n, +T_n]} \varphi(t, 2|f(t) - P_\delta(t)|) dt \right. \\ &\quad \left. + \frac{1}{2T_n} \int_{(\tilde{G})' \cap [-T_n, +T_n]} \varphi(t, 2|P_\delta(t)|) dt \right] \\ &\leq \frac{1}{2} \left[\frac{\delta}{4} + M\varepsilon \right] \leq \frac{\delta}{2}. \end{aligned}$$

Then

$$\frac{1}{2T_n} \int_{-T_n}^{+T_n} \varphi(t, |f(t)|) dt \leq 1 - \delta + \frac{\delta}{2} \leq 1 - \frac{\delta}{2}.$$

Hence, letting n tend to infinity we will have $\rho_\varphi(f) \leq 1 - \frac{\delta}{2}$. Finally, using Corollary 1 it follows $\|f\|_\varphi < 1$. This contradicts the fact that $\|f\|_\varphi = 1$.

We now show the statement of the lemma. Let $\delta \in (0, 1)$ and $\alpha > 0$ be such that $\sup_{t \in \mathbb{R}} \varphi(t, \alpha) \leq 1 - \delta$. We choose $\tilde{\theta}$ as in (3.5) and then take $\bar{\theta} > \tilde{\theta}$ as in Lemma 3. If $\beta > \alpha$ is a fixed number we define the set $G_1 = \{t \in \mathbb{R} : \alpha \leq |f(t)| \leq \beta\}$. Then since

$$(G_1)' \cap [-T, T] = \{t \in [-T, T] : |f(t)| \leq \alpha\} \cup \{t \in [-T, T] : f(t) \geq \beta\} \subset \tilde{G} \cup (\bar{G})',$$

it follows that for $T \geq T_0$ we have

$$\begin{aligned} \mu((G_1)' \cap [-T, T]) &\leq \mu(\tilde{G} \cap [-T, T]) + \mu((\bar{G})' \cap [-T, T]) \\ &\leq \tilde{\theta}2T + (1 - \bar{\theta})2T = (1 - (\bar{\theta} - \tilde{\theta}))2T, \end{aligned}$$

or equivalently

$$\mu(G_1 \cap [-T, T]) \geq (\bar{\theta} - \tilde{\theta})2T, \text{ for } T \geq T_0.$$

□

Lemma 8. *Let $\{a_n\}_n, a_n > 0$ be a sequence of real numbers and $\alpha \in (0, 1)$. To each n we associate a measurable set A_n such that*

- (i) $A_i \cap A_j = \emptyset$, for $i \neq j$ and $\bigcup_{n \geq 1} A_n \subset [0, \alpha[$, $\alpha < 1$;
- (ii) $\sum_{n \geq 0} \int_0^1 \varphi(t, a_n \chi_{A_n}(t)) dt < +\infty$.

Consider the function $f = \sum_{n \geq 1} a_n \chi_{A_n}$ on $[0, 1]$ and let \tilde{f} be the periodic extension of f to the whole \mathbb{R} (with period $\tau = 1$). Then $\tilde{f} \in \tilde{B}_{a.p.}^\varphi$.

PROOF: Let us first remark that since $\int_0^1 \varphi(t, a_n) dt < +\infty$, for $n \geq 1$ there exists a set $A_n \subset [0, \alpha[$ for which $\int_0^1 \varphi(t, a_n \chi_{A_n}(t)) dt < \frac{1}{n^2}$. It is also clear that we may choose the A_n 's so that the conditions of the lemma are satisfied. Now, for an arbitrary $\varepsilon > 0$ we fix n_0 such that $\sum_{n \geq n_0} \int_0^1 \varphi(t, a_n \chi_{A_n}(t)) dt \leq \frac{\varepsilon}{3}$ and put $f_1 = \sum_{i=1}^{n_0} a_i \chi_{A_i}$ on $[0, 1]$. Let then $M = \max_{i \leq n_0} \sup_{t \in [0, 1]} \varphi(t, 2a_i)$ and $\delta \leq \frac{\varepsilon}{3M}$ (remark that we may suppose $1 - \alpha > \delta$).

Let f_1^r denote the restriction of f_1 to $[0, 1 - \delta]$. Then by Luzin's theorem there exists a continuous function g_ε^r on $[0, 1 - \delta]$ such that

$$\mu \{t \in [0, 1 - \delta] : \varphi(t, |f_1^r(t) - g_\varepsilon^r(t)|) > 0\} \leq \frac{\varepsilon}{3M}.$$

Moreover since f_1 is bounded so is g_ε^r (with the same bound).

Let now g_ε be a linear extension of g_ε^r to $[0, 1]$, more precisely g_ε is such that $g_\varepsilon = g_\varepsilon^r$ on $[0, 1 - \delta]$, g_ε is linear between $1 - \delta$ and 1 and satisfies $g_\varepsilon(1) = g_\varepsilon^r(0)$.

We then get

$$\begin{aligned} & \int_0^1 \varphi \left(t, \frac{|f(t) - g_\varepsilon(t)|}{2} \right) dt \\ & \leq \int_0^1 \varphi \left(t, \frac{|f(t) - f_1(t)| + |f_1(t) - g_\varepsilon(t)|}{2} \right) dt \\ & \leq \frac{1}{2} \int_0^1 \varphi(t, |f(t) - f_1(t)|) dt + \frac{1}{2} \int_0^1 \varphi(t, |f_1(t) - g_\varepsilon(t)|) dt \\ & \leq \frac{1}{2} \int_0^1 \varphi \left(t, \sum_{n \geq n_0} a_n \chi_{A_n}(t) \right) dt \\ & \quad + \frac{1}{2} \int_0^{1-\delta} \varphi(t, |f_1^r(t) - g_\varepsilon^r(t)|) dt + \frac{1}{2} \int_{1-\delta}^1 \varphi(t, |f_1(t) - g_\varepsilon(t)|) dt \\ & \leq \frac{1}{2} \sum_{n \geq n_0} \int_0^1 \varphi(t, a_n \chi_{A_n}(t)) dt + \frac{1}{2} M \frac{\varepsilon}{3M} + \frac{1}{2} M \frac{\varepsilon}{3M} \\ & \leq \frac{\varepsilon}{2}. \end{aligned}$$

Finally, the continuous function $g_\varepsilon : [0, 1] \rightarrow \mathbb{R}$ satisfies

$$g_\varepsilon(0) = g_\varepsilon(1) \quad \text{and} \quad \int_0^1 \varphi \left(t, \frac{|f(t) - g_\varepsilon(t)|}{2} \right) dt \leq \frac{\varepsilon}{2}.$$

Let now \tilde{f} and \tilde{g}_ε be the respective periodic extensions of f and g_ε to the whole \mathbb{R} (with the period $\tau = 1$). Clearly \tilde{g}_ε is *u.a.p.* and then it is also in $B_{a.p.}^\varphi(\mathbb{R})$.

Consequently, there exists $P_\varepsilon \in A$ for which $\rho_\varphi \left(\frac{\tilde{g}_\varepsilon - P_\varepsilon}{2} \right) \leq \frac{\varepsilon}{2}$.

On the other hand \tilde{f} and \tilde{g} being periodic with period $\tau = 1$, using the periodicity of φ (with $\tau = 1$), we get

$$\begin{aligned} \rho_\varphi \left(\frac{\tilde{f} - \tilde{g}_\varepsilon}{2} \right) &= \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi \left(t, \frac{|\tilde{f}(t) - \tilde{g}_\varepsilon(t)|}{2} \right) dt \\ &= \int_0^1 \varphi \left(t, \frac{|f(t) - g_\varepsilon(t)|}{2} \right) dt \leq \frac{\varepsilon}{2}. \end{aligned}$$

Finally,

$$\rho_\varphi \left(\frac{\tilde{f} - P_\varepsilon}{4} \right) \leq \frac{1}{2} \left[\rho_\varphi \left(\frac{\tilde{f} - \tilde{g}_\varepsilon}{2} \right) + \rho_\varphi \left(\frac{\tilde{g}_\varepsilon - P_\varepsilon}{2} \right) \right] \leq \varepsilon,$$

i.e., $\tilde{f} \in \tilde{B}_{a.p.}^\varphi$.

□

4. Results

Lemma 9. *Let $\varphi(t, u)$ be strictly convex with respect to $u \geq 0$ and $f_n, g_n \in B_{a.p.}^\varphi(\mathbb{R})$ be sequences such that, for some $r > 0$, we have*

$$\rho_\varphi(f_n) \leq r, \rho_\varphi(g_n) \leq r \quad \text{and} \quad \lim_{n \rightarrow \infty} \rho_\varphi\left(\frac{f_n + g_n}{2}\right) = r.$$

Then $(f_n - g_n) \xrightarrow{\bar{\mu}} 0$.

PROOF: Suppose that $\lim_{n \rightarrow \infty} (f_n - g_n) \neq 0$ in the $\bar{\mu}$ -convergence sense. Then there exist $\varepsilon > 0, \sigma > 0$ and $n_k \nearrow \infty$ such that if $E_k = \{t \in \mathbb{R} : |f_{n_k}(t) - g_{n_k}(t)| \geq \sigma\}$ we have $\bar{\mu}(E_k) > \varepsilon$.

Take a number $k_\varepsilon > 1$ such that (see Lemma 1) there holds

$$\bar{\mu}(E) \geq \frac{\varepsilon}{4} \Rightarrow \rho_\varphi(\chi_E) > \frac{r}{k_\varepsilon},$$

where $r > 0$ is the constant from the lemma.

Then putting

$$A_k = \{t \in \mathbb{R} : |f_{n_k}(t)| > k_\varepsilon\},$$

$$B_k = \{t \in \mathbb{R} : |g_{n_k}(t)| > k_\varepsilon\}$$

we obtain

$$\begin{aligned} r &\geq \rho_\varphi(f_{n_k}) \\ &= \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi(t, |f_{n_k}(t)|) dt \\ &\geq \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{A_k \cap [-T, T]} \varphi(t, k_\varepsilon) dt \\ &\geq k_\varepsilon \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{A_k \cap [-T, T]} \varphi(t, 1) dt = k_\varepsilon \rho_\varphi(\chi_{A_k}). \end{aligned}$$

It follows that $\rho_\varphi(\chi_{A_k}) \leq \frac{r}{k_\varepsilon}$ and then $\bar{\mu}(A_k) \leq \frac{\varepsilon}{4}$.

In the same way we show that $\bar{\mu}(B_k) \leq \frac{\varepsilon}{4}$.

Now, define the set

$$Q = \{(u, v) \in \mathbb{R}^2 / |u| \leq k_\varepsilon, |v| \leq k_\varepsilon, |u - v| \geq \sigma\},$$

and consider the function

$$F(t, u, v) = \frac{2\varphi\left(t, \frac{u+v}{2}\right)}{\varphi(t, u) + \varphi(t, v)}.$$

Since φ is strictly convex we have $F(t, u, v) < 1$, for all $(t, u, v) \in \mathbb{R} \times Q$. Then using the continuity of φ on $\mathbb{R} \times Q$ (where Q is a compact set of \mathbb{R}^2) and its periodicity with respect to t , it follows that

$$\sup_{\mathbb{R} \times Q} F(t, u, v) = 1 - \delta \text{ for some } \delta > 0.$$

More precisely, for $(t, u, v) \in \mathbb{R} \times Q$ we have

$$\varphi\left(t, \frac{u+v}{2}\right) \leq (1 - \delta) \frac{\varphi(t, u) + \varphi(t, v)}{2}.$$

Let now $t \in E_k \setminus (A_k \cup B_k)$. Then $f_{n_k}(t), g_{n_k}(t) \in Q$ and consequently

$$\varphi\left(t, \frac{|f_{n_k}(t) + g_{n_k}(t)|}{2}\right) \leq (1 - \delta) \frac{\varphi(t, |f_{n_k}(t)|) + \varphi(t, |g_{n_k}(t)|)}{2}.$$

Hence

$$\begin{aligned} & r - \rho_\varphi\left(\frac{f_{n_k} + g_{n_k}}{2}\right) \\ & \geq \frac{\rho_\varphi(f_{n_k}) + \rho_\varphi(g_{n_k})}{2} - \rho_\varphi\left(\frac{f_{n_k} + g_{n_k}}{2}\right) \\ & \geq \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{[E_k \setminus (A_k \cup B_k)] \cap [-T, +T]} \\ & \quad \left[\frac{\varphi(t, |f_{n_k}(t)|) + \varphi(t, |g_{n_k}(t)|)}{2} - \varphi\left(t, \frac{|f_{n_k}(t) + g_{n_k}(t)|}{2}\right) \right] dt \\ & \geq \frac{\delta}{2} \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{[E_k \setminus (A_k \cup B_k)] \cap [-T, +T]} [\varphi(t, |f_{n_k}(t)|) + \varphi(t, |g_{n_k}(t)|)] dt \\ & \geq \delta \overline{\lim}_{T \rightarrow +\infty} \frac{1}{2T} \int_{[E_k \setminus (A_k \cup B_k)] \cap [-T, +T]} \varphi\left(t, \frac{|f_{n_k}(t) - g_{n_k}(t)|}{2}\right) dt \\ & \geq \delta \varphi\left(\frac{\sigma}{2}\right) \left(\varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4}\right) = \delta \frac{\varepsilon}{2} \varphi\left(\frac{\sigma}{2}\right). \end{aligned}$$

Finally,

$$r - \rho_\varphi\left(\frac{f_n + g_n}{2}\right) \geq \delta \frac{\varepsilon}{2} \varphi\left(\frac{\sigma}{2}\right) > 0,$$

a contradiction with the hypothesis $\rho_\varphi\left(\frac{f_n + g_n}{2}\right) \rightarrow r$. □

Theorem 1. $\tilde{B}_{a.p.}^\varphi(\mathbb{R})$ is strictly convex if and only if φ is strictly convex and φ satisfies the Δ_2 -condition.

PROOF: Sufficiency. Suppose that φ is strictly convex and satisfies the Δ_2 -condition but $\tilde{B}_{a.p.}^\varphi(\mathbb{R})$ is not strictly convex. Then for some f and $g \in \tilde{B}_{a.p.}^\varphi(\mathbb{R})$ we will have $\|f\|_\varphi = \|g\|_\varphi = 1$ and $\|f - g\|_\varphi > 0$ but $\left\| \frac{f+g}{2} \right\|_\varphi = 1$. From Corollary 1 we will have also $\rho_\varphi(f) = \rho_\varphi(g) = \rho_\varphi\left(\frac{f+g}{2}\right) = 1$. Then from Lemma 9 it follows that for each $\alpha > 0$, $\bar{\mu}\{t \in \mathbb{R} : |f - g| > \alpha\} = 0$. Finally, using Lemma 7 we get $\rho_\varphi(f - g) = 0$. Contradiction.

Necessity. Let $L^\varphi = L^\varphi([0, 1]) = \{f \in M(\mathbb{R}) : \int_0^1 \varphi(t, \lambda|f(t)|) dt < +\infty \text{ for some } \lambda > 0\}$ be the usual Musielak-Orlicz space and $\|\cdot\|_{L^\varphi}$ its associated Luxemburg norm.

We consider the injection map

$$i : L^\varphi \hookrightarrow \tilde{B}_{a.p.}^\varphi(\mathbb{R}), \quad i(f) = \tilde{f},$$

where \tilde{f} is the periodic extension (with period $\tau = 1$) of f to \mathbb{R} . We show first that $i(L^\varphi) \subset \tilde{B}_{a.p.}^\varphi(\mathbb{R})$.

Let $f \in L^\varphi([0, 1])$. Then there exists $\lambda > 0$ such that $\varphi(t, \lambda|f(t)|) \in L^1([0, 1])$. From usual arguments of Lebesgue theory we have $\lim_{N \rightarrow +\infty} \mu(V_N) = 0$, where

$$V_N = \{t \in [0, 1] : \varphi(t, \lambda|f(t)|) \geq N\}.$$

Let $E_N = \{t \in [0, 1] : |f(t)| \geq N\}$. Then for $t \in E_N$ we have

$$\varphi(t, \lambda|f(t)|) \geq \varphi(t, \lambda N) \geq \lambda N \varphi(t, 1) \geq \lambda N \phi(1),$$

where $\phi(1) = \inf_{t \in [0, 1]} \varphi(t, 1)$, $\phi(1) > 0$ (we may suppose $\phi(1) = 1$). It follows that $E_N \subset V_{\lambda N}$ and then we get $\lim_{N \rightarrow +\infty} \mu(E_N) = 0$.

Consider the following functions for $N \in \mathbb{N}$,

$$f_N(t) = \begin{cases} f(t) & \text{if } |f(t)| \leq N \\ N & \text{if } |f(t)| \geq N. \end{cases}$$

It is clear that the sequence $\{f_N\}$ is increasing and $f_N \leq f$. Moreover, since $\lim_{N \rightarrow +\infty} \mu(E_N) = 0$ we have $\lim_{N \rightarrow +\infty} \int_{E_N} \varphi(t, \lambda|f(t)|) dt = 0$.

Then for a given $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that

$$\int_0^1 \varphi(t, \lambda|f(t) - f_{N_\varepsilon}(t)|) dt \leq \int_{E_{N_\varepsilon}} \varphi(t, \lambda|f(t)|) dt \leq \varepsilon.$$

Now for f_{N_ε} being bounded there exists a sequence of simple functions $(S_{N_\varepsilon})_n$ uniformly convergent to f_{N_ε} . In particular, there exists a simple function S_{N_ε} such that $\sup_{t \in [0,1]} |\lambda(f_{N_\varepsilon}(t) - S_{N_\varepsilon}(t))| \leq \varepsilon$ and then

$$\begin{aligned} & \int_0^1 \varphi\left(t, \frac{\lambda}{2} |f(t) - S_{N_\varepsilon}(t)|\right) dt \\ & \leq \frac{1}{2} \int_0^1 \varphi(t, \lambda |f(t) - f_{N_\varepsilon}(t)|) dt + \frac{1}{2} \int_0^1 \varphi(t, \lambda |f_{N_\varepsilon}(t) - S_{N_\varepsilon}(t)|) dt \leq \varepsilon. \end{aligned}$$

We denote by \tilde{f} , $\tilde{f}_{N_\varepsilon}$ and $\tilde{S}_{N_\varepsilon}$ the respective periodic extensions (with period $\tau = 1$) of the functions f , f_{N_ε} and S_{N_ε} . We have from the periodicity properties of φ , \tilde{f} , $\tilde{f}_{N_\varepsilon}$ and $\tilde{S}_{N_\varepsilon}$:

$$\begin{aligned} \rho_\varphi\left(\frac{\lambda}{2} (\tilde{f} - \tilde{S}_{N_\varepsilon})\right) &= \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} \varphi\left(t, \frac{\lambda}{2} |\tilde{f}(t) - \tilde{S}_{N_\varepsilon}(t)|\right) dt \\ &= \int_0^1 \varphi\left(t, \frac{\lambda}{2} |f(t) - S_{N_\varepsilon}(t)|\right) dt \leq \varepsilon. \end{aligned}$$

Moreover, from Lemma 8 we have $\tilde{S}_{N_\varepsilon} \in \tilde{B}_{a.p.}^\varphi(\mathbb{R})$. Then there exists $P_\varepsilon \in A$ for which $\rho_\varphi\left(\frac{1}{4}(\tilde{S}_{N_\varepsilon} - P_\varepsilon)\right) \leq \varepsilon$ (see the proof of Lemma 8).

Finally, putting $\alpha = \min(\lambda, \frac{1}{4})$ we get

$$\rho_\varphi\left(\frac{\alpha}{2} (\tilde{f} - P_\varepsilon)\right) \leq \frac{1}{2} \left\{ \rho_\varphi\left(\frac{\lambda}{2} (\tilde{f} - \tilde{S}_{N_\varepsilon})\right) + \rho_\varphi\left(\frac{1}{4} (\tilde{S}_{N_\varepsilon} - P_\varepsilon)\right) \right\} \leq \varepsilon.$$

This means that $\tilde{f} \in \tilde{B}_{a.p.}^\varphi(\mathbb{R})$.

Now, since $i : L^\varphi([0, 1]) \hookrightarrow \tilde{B}_{a.p.}^\varphi(\mathbb{R})$ is an isometry, the strict convexity of $\tilde{B}_{a.p.}^\varphi(\mathbb{R})$ implies the strict convexity of $L^\varphi([0, 1])$.

Consequently $\varphi(t, u)$, $t \in [0, 1]$, $u \geq 0$ is strictly convex and satisfies the Δ_2 -condition for Musielak-Orlicz spaces (see [6], [7]) i.e., there exist $k \geq 1$ and $h \geq 0$ with $\int_0^1 h(t) dt < \infty$ such that $\varphi(t, 2u) \leq k\varphi(t, u) + h(t)$ for all $u \geq 0$ and almost all $t \in [0, 1]$. The periodically (with $\tau = 1$) extended functions $\varphi(t, u)$, $t \in \mathbb{R}$, $u \geq 0$ and $\tilde{h}(t)$, $t \in \mathbb{R}$ satisfy the conditions $\tilde{h} \in B^1(\mathbb{R})$ and $\varphi(t, 2u) \leq k\varphi(t, u) + \tilde{h}(t)$ for $u \geq 0$ and almost all $t \in \mathbb{R}$.

Now, putting $f(t) = \sup\{u \geq 0 : \varphi(t, u) \leq \tilde{h}(t)\}$ it follows that f is measurable and $\varphi(t, f(t)) = \tilde{h}(t)$ for $t \in \mathbb{R}$. Finally, we get

$$\varphi(t, 2u) \leq k\varphi(t, u) + \tilde{h}(t) \leq (k + 1)\varphi(t, u)$$

for $u \geq f(t)$ and almost all $t \in \mathbb{R}$, i.e., φ satisfies the Δ_2 -condition for Besicovitch-Musielak-Orlicz spaces.

□

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