Some versions of relative paracompactness and their absolute embeddings

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Abstract. Arhangel'skii [Sci. Math. Jpn. **55** (2002), 153–201] defined notions of relative paracompactness in terms of locally finite open partial refinement and asked if one can generalize the notions above to the well known Michael's criteria of paracompactness in [17] and [18]. In this paper, we consider some versions of relative paracompactness defined by locally finite (not necessarily open) partial refinement or locally finite closed partial refinement, and also consider closure-preserving cases, such as 1-*lf*-, 1-*cp*-, α-*lf*, α-*cp*-paracompactness and so on. Moreover, on their absolute embeddings, we have the following results. Theorem 1. A Tychonoff space Y is 1-*lf*- (or equivalently, 1-*cp*-) paracompact in every larger Tychonoff space if and only if Y is Lindelöf. Theorem 2. A Tychonoff space Y is α-*lf*- (or equivalently, α-*cp*-) paracompact in every larger Tychonoff space if and only if Y is compact. We also show that in Theorem 1, "every larger Tychonoff space" can be replaced by "every larger Tychonoff space containing Y as a closed subspace". But, this replacement is not available for Theorem 2.

Keywords: 1-paracompactness of Y in X, 2-paracompactness of Y in X, Aull-paracompactness of Y in X, α-paracompactness of Y in X, 1-*lf*-paracompactness of Y in X, 2-*lf*-paracompactness of Y in X, Aull-*lf*-paracompactness of Y in X, α-*lf*-paracompactness of Y in X, 1-*cp*-paracompactness of Y in X, 2-*cp*-paracompactness of Y in X, Aull-*cp*-paracompactness of Y in X, α-*cp*-paracompactness of Y in X, absolute embedding, compact, Lindelöf

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1. Introduction

Throughout this paper all spaces are assumed to be T_1 topological spaces and the symbol γ denotes an infinite cardinal. The symbol $\mathbb{N}$ denotes the set of all natural numbers. For a subset A of a space X, $\overline{A}^X$ and $\mathrm{Int}_X\, A$ denote the closure and the interior of A in X, respectively.

Let X be a space and Y a subspace of X. Y is *Hausdorff* (respectively, *strongly Hausdorff*) *in* X if for every $y \in Y$ and every $x \in Y$ (respectively, $x \in X$) with $x \neq y$, there exist disjoint open subsets U, V of X such that $x \in U$ and $y \in V$. Y is said to be *regular* (respectively, *strongly regular*) *in* X if for each $y \in Y$ (respectively, $y \in X$) and each closed subset F of X with $y \notin F$, there exist disjoint open subsets U, V of X such that $y \in U$ and $F \cap Y \subset V$. Moreover, Y is *superregular in* X if for every $y \in Y$ and each closed subset F of X with $y \notin F$, there exist disjoint open subsets U, V of X such that $y \in U$ and $F \subset V$ ([1], [2] and [3]).

As relative notions of paracompactness, the following are known. Let X be a space and Y a subspace of X. For $x \in X$, a collection $\mathcal{A}$ of subsets of X is said to be *locally finite at x in X* if there exists a neighborhood of x in X which intersects at most finitely many members of $\mathcal{A}$. In [1], [2] and [3], Y is said to be 1- (respectively, 2-) *paracompact in X* if for every open cover $\mathcal{U}$ of X, there exists a collection $\mathcal{V}$ of open subsets of X with $X = \bigcup \mathcal{V}$ (respectively, $Y \subset \bigcup \mathcal{V}$) such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y in X. Here, $\mathcal{V}$ is said to be a *partial refinement* of $\mathcal{U}$ if each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ containing V. We also say that $\mathcal{V}$ is a *refinement* (respectively, an *open refinement*, a *closed refinement*) of $\mathcal{U}$ if $\mathcal{V}$ is a cover (respectively, an open cover, a closed cover) of X and a partial refinement of $\mathcal{U}$. The term "2-paracompact" is often simply said "paracompact". Moreover, Y is said to be *Aull-paracompact in X* if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection $\mathcal{V}$ of open subsets of X with $Y \subset \bigcup \mathcal{V}$ such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y in X ([2], [4]). The 1-paracompactness and Aull-paracompactness of Y in X need not imply each other ([4]), but each of them clearly implies 2-paracompactness of Y in X. When Y is a closed subspace of X, Y is 2-paracompact in X if and only if Y is Aull-paracompact in X.

Aull [5] defined that Y is *α-paracompact in X* if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection $\mathcal{V}$ of open subsets of X such that $Y \subset \bigcup \mathcal{V}$, $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite in X. Recall that 1- and α-paracompactness do not imply each other in general. But for a regular space X, if Y is α-paracompact in X then Y is 1-paracompact in X, the converse also holds if, in addition, Y is closed ([16, Theorem 1.3], see also Proposition 3.1 below for a generalization).

These notions are central in the study of relative paracompactness and the following relations hold.

$$\begin{array}{c}
Y \text{ is 1-paracompact in } X \\
\downarrow \\
Y \text{ is 2-paracompact in } X \\
\uparrow \\
Y \text{ is Aull-paracompact in } X \\
\uparrow \\
Y \text{ is } \alpha\text{-paracompact in } X
\end{array}$$

Diagram 1

Moreover, absolute embeddings of above relative paracompactness are characterized as follows (see also [13]).

Theorem 1.1 (Lupiañez [14]; Lupiañez-Outerelo [16]). *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(a) *Y is 1- (or equivalently, α-) paracompact in every larger Tychonoff (respectively, regular) space.*
(b) *Y is 1- (or equivalently, α-) paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(c) *Y is compact.*

Theorem 1.2 (Arhangel'skii-Genedi [3]; see also [9], [19]). *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(a) *Y is 2- (or equivalently, Aull-) paracompact in every larger Tychonoff (respectively, regular) space.*
(b) *Y is 2- (or equivalently, Aull-) paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(c) *Y is Lindelöf.*

Arhangel'skii [1, p. 98], [2, p. 174] asked if one can generalize the notions above to the well known Michael's criteria of paracompactness in [17] and [18]. Concerning this problem, Aull [6, Theorem 5] already proved that a subspace Y of a normal space X is α-paracompact if and only if for every cover of Y by open subsets of X has a closure-preserving partial open refinement which covers Y. Moreover, Lupiañez [15, Theorem 1.3] proved that a subspace Y of a regular space X is α-paracompact if and only if every cover $\mathcal{U}$ of Y by open subsets of X has a partial refinement (or equivalently, a closed partial refinement) $\mathcal{A}$ of $\mathcal{U}$ such that $\mathcal{A}$ is locally finite in X and $Y \subset \mathrm{Int}_X(\bigcup \mathcal{A})$.

In Section 2, we introduce notions of relative paracompactness by using locally finite (not necessarily open) partial refinement and locally finite closed partial refinement. We also consider closure-preserving cases.

In Section 3, we discuss locally finite open refinement and closure-preserving open refinement by using the space X_Y, where X_Y is a space obtained from X by letting each point of $X \setminus Y$ be isolated.

In Section 4, we investigate their basic properties and discuss their absolute embeddings. In particular, we have

Theorem 1.3. *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(a) *Y is 1-lf- (or equivalently, 1-cp-) paracompact in every larger Tychonoff (respectively, regular) space.*
(b) *Y is 1-lf- (or equivalently, 1-cp-) paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(c) *Y is Lindelöf.*

Theorem 1.4. *A Tychonoff (respectively, regular) space Y is α-lf- (or equivalently, α-cp-) paracompact in every larger Tychonoff (respectively, regular) space if and only if Y is compact.*

For α-*cp*-paracompact case, a similar statement to (b) in Theorem 1.1 cannot be added to Theorem 1.4. Indeed, we replace "every larger Tychonoff (respectively, regular) space" by "every larger Tychonoff (respectively, regular) space containing Y as a closed subspace" in Theorem 1.4, "Y is compact" is replaced by "Y is paracompact" (see Remark 4.5). In addition, we point out that a Tychonoff (respectively, regular) space Y is 2- (or equivalently, Aull-) *cp*-paracompact in every larger Tychonoff (respectively, regular) space if and only if Y is paracompact (see Theorem 4.6 and Remark 4.7).

In the final section, a remark on definitions of relative paracompactness due to Grabner et al. [10], [12] will be given and a gap of a result in [11] will be pointed out.

For general surveys on relative topological properties, see the Arhangel'skii's subsequent articles [1] and [2]. Other undefined notations and terminology are used as in [7].

2. Some versions of relative paracompactness

In this section, we newly define some notions of relative paracompactness and discuss their basic properties.

Let X be a space and Y a subspace of X. We define that Y is 1-*lf-paracompact* (respectively, 1-*lfc-paracompact*) *in* X if every open cover of X has a refinement (respectively, a closed refinement) of $\mathcal{U}$ which is locally finite at each point of Y in X. We also define that Y is 2-*lf-paracompact* (respectively, 2-*lfc-paracompact*) *in* X if for every open cover $\mathcal{U}$ of X there exists a partial refinement (respectively, a closed partial refinement) $\mathcal{V}$ such that $Y \subset \bigcup \mathcal{V}$ and $\mathcal{V}$ is locally finite at each point of Y in X. Furthermore, Y is *Aull-lf-paracompact* (respectively, *Aull-lfc-paracompact*) *in* X if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a partial refinement (respectively, a closed partial refinement) $\mathcal{V}$ of $\mathcal{U}$ such that $Y \subset \bigcup \mathcal{V}$ and $\mathcal{V}$ is locally finite at each point of Y in X. We also say that Y is α-*lf-paracompact* (respectively, α-*lfc-paracompact*) *in* X if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$ there exists a partial refinement (respectively, a closed partial refinement) $\mathcal{V}$ of $\mathcal{U}$ such that $Y \subset \bigcup \mathcal{V}$ and $\mathcal{V}$ is locally finite in X.

Let X be a space and $x \in X$. A collection $\mathcal{A}$ of subsets of X is said to be *closure-preserving at* x in X if for every $\mathcal{A}' \subset \mathcal{A}$ with $x \in \overline{\bigcup \mathcal{A}'}^X$, it holds that $x \in \bigcup \overline{\mathcal{A}'}^X$, where $\overline{\mathcal{A}'}^X = \{\overline{A}^X \mid A \in \mathcal{A}'\}$. The following are known.

Proposition 2.1. *For a collection $\mathcal{A}$ of subsets of a space X and $x \in X$, each of the following statements hold.*

(a) *If $\mathcal{A}$ is locally finite at x in X, then $\mathcal{A}$ is closure-preserving at x in X.*

(b) *$\mathcal{A}$ is locally finite (respectively, closure-preserving) at x in X if and only if $\overline{\mathcal{A}}^X$ is also locally finite (respectively, closure-preserving) at x in X.*
(c) *$\mathcal{A}$ is locally finite at x in X if and only if $\overline{\mathcal{A}}^X$ is point-finite at x and $\mathcal{A}$ is closure-preserving at x in X.*

Hence, we have the following: (a′) If $\mathcal{A}$ is locally finite at each point of Y in X, then $\mathcal{A}$ is closure-preserving at each point of Y in X. (b′) If $\mathcal{A}$ is closure-preserving at each point of Y in X, then $\overline{\mathcal{A}}^X$ is also closure-preserving at each point of Y in X. (c′) For a collection $\mathcal{A}$ of closed subsets of X, $\mathcal{A}$ is locally finite at each point of Y in X if and only if $\mathcal{A}$ is point-finite at each point of Y and closure-preserving at each point of Y in X. Grabner et al. [10], [12] introduced some relative notions related to closure-preserving collections; but their notions do not necessarily satisfy any of (a′), (b′) and (c′) above (for detail, see Section 5).

Let X be a space and Y a subspace of X. We define that Y is 1-*cp-paracompact* (respectively, 1-*cpo-paracompact*, 1-*cpc-paracompact*) *in* X if every open cover of X has a refinement (respectively, an open refinement, a closed refinement) which is closure-preserving at each point of Y in X. We also define that Y is 2-*cp-paracompact* (respectively, 2-*cpo-paracompact*, 2-*cpc-paracompact*) *in* X if for every open cover $\mathcal{U}$ of X there exists a partial refinement (respectively, an open partial refinement, a closed partial refinement) $\mathcal{V}$ such that $Y \subset \bigcup\mathcal{V}$ and $\mathcal{V}$ is closure-preserving at each point of Y in X (see Remark 5.1 below). We say that Y is *Aull-cp-paracompact* (respectively, *Aull-cpo-paracompact*, *Aull-cpc-paracompact*) *in* X if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup\mathcal{U}$ there exists a partial refinement (respectively, an open partial refinement, a closed partial refinement) $\mathcal{V}$ such that $Y \subset \bigcup\mathcal{V}$ and $\mathcal{V}$ is closure-preserving at each point of Y in X. Moreover, we say that Y is *α-cp-paracompact* (respectively, *α-cpo-paracompact*, *α-cpc-paracompact*) *in* X if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup\mathcal{U}$ there exists a partial refinement (respectively, an open partial refinement, a closed partial refinement) $\mathcal{V}$ such that $Y \subset \bigcup\mathcal{V}$ and $\mathcal{V}$ is closure-preserving in X.

Proposition 2.1(b) induces the following.

Proposition 2.2. *Let Y be a subspace of a regular space X. Then, each of the following statements hold.*

(a) *If Y is 1-lf-paracompact in X, then Y is 1-lfc-paracompact in X.*
(b) *If Y is 1-cp-paracompact in X, then Y is 1-cpc-paracompact in X.*

Remark 2.3. If we replace “1-” by “α-”, “2-” or “Aull-” in the statements (a) and (b) of Proposition 2.2, then the condition “X is regular” can be weakened to “Y is strongly regular in X”.

For closed subspaces, we have the following. Here, notice that 2-*cpc*-paracompactness of Y in X induces regularity of Y when Y is closed in X.

Theorem 2.4. *For a closed subspace Y of a space X, the following statements are equivalent.*

(a) *Y is α-lfc-paracompact in X.*
(b) *Y is 2-cpc-paracompact in X.*
(c) *Y is α-lf-paracompact in X and Y is regular.*
(d) *Y is 2-cp-paracompact in X and Y is regular.*
(e) *Y is paracompact Hausdorff.*

Proof: The implications (a) ⇒ (c) ⇒ (d) are obvious. Since the statement (b) induces regularity of Y, the implications (a) ⇒ (b) ⇒ (d) are also obvious. Moreover, since Y is closed in X, the implication (d) ⇒ (e) clearly holds.

(e) ⇒ (a). Suppose that Y is paracompact Hausdorff. Let $\mathcal{U}$ be a collection of open subsets of X with $Y \subset \bigcup \mathcal{U}$. Since Y is paracompact Hausdorff, there exists a locally finite closed refinement $\mathcal{F}$ of $\{U \cap Y \mid U \in \mathcal{U}\}$. Then, $\mathcal{F}$ is a collection of closed subsets of X such that $Y \subset \bigcup \mathcal{F}$ and $\mathcal{F}$ is locally finite in X. Therefore, Y is α-*lfc*-paracompact in X. □

Aull [5] proved that if a subspace Y of a Hausdorff space X is α-paracompact in X then Y is closed in X. We improve this fact as follows.

Lemma 2.5. *Assume that Y is strongly Hausdorff in X. If Y is α-cp-paracompact in X, then Y is closed in X.*

Proof: Let $x \in X \setminus Y$. For each $y \in Y$, there is an open subset U_y of X such that $y \in U_y \subset \overline{U_y} \subset X \setminus \{x\}$. Since Y is α-*cp*-paracompact in X, there exists a partial refinement $\mathcal{V}$ of $\{U_y \mid y \in Y\}$ such that $Y \subset \bigcup \mathcal{V}$ and $\mathcal{V}$ is closure-preserving in X. Then, $G = X \setminus \overline{\bigcup \mathcal{V}}^X$ is an open neighborhood of x in X disjoint from Y. □

The following corollary immediately follows from Theorem 2.4 and Lemma 2.5.

Corollary 2.6. *Assume that Y is strongly Hausdorff in X. Then, each of the following statements hold.*

(a) *Y is α-lfc-paracompact in X if and only if Y is α-cpc-paracompact in X.*
(b) *Assume that Y is regular. Then, Y is α-lf-paracompact in X if and only if Y is α-cp-paracompact in X.*

Hereafter, the symbol $\mathcal{T}_3$ (respectively, $\mathcal{T}_2$) denotes the class of all regular (respectively, Hausdorff) spaces. Moreover, the symbols SH, R, SuR and StR mean the conditions "Y is strongly Hausdorff in X", "Y is regular in X", "Y is superregular in X" and "Y is strongly regular in X", respectively. The symbol $\mathcal{C}_X$ denotes the family of all closed subsets of X. We denote the condition "Y is $\mathcal{T}_3$-embedded in X" (see Section 3 for definition) by T_3.

The following implications around 1-paracompactness follow from definitions and Proposition 2.2. Here, the implication "Y is 1-*cpo*-paracompact in $X \xrightarrow{\mathsf{R},\mathsf{T}_3} Y$ is 1-paracompact in X" is proved in Section 3 (see Theorem 3.4).

$$\begin{tikzcd}[column sep=large, row sep=huge]
\begin{array}{c} Y \text{ is} \\ 1\text{-paracompact} \\ \text{in } X \end{array} \arrow[r] \arrow[d, shift left] & \begin{array}{c} Y \text{ is} \\ 1\text{-}\mathit{lf}\text{-paracompact} \\ \text{in } X \end{array} \arrow[r, shift left, "\mathsf{X} \in \mathcal{T}_3"] \arrow[d] & \begin{array}{c} Y \text{ is} \\ 1\text{-}\mathit{lfc}\text{-paracompact} \\ \text{in } X \end{array} \arrow[l, shift left] \arrow[d] \\
\begin{array}{c} Y \text{ is} \\ 1\text{-}\mathit{cpo}\text{-paracompact} \\ \text{in } X \end{array} \arrow[r] \arrow[u, shift left, "\begin{array}{c}\mathsf{R} \\ \mathsf{T}_3\end{array}"] & \begin{array}{c} Y \text{ is} \\ 1\text{-}\mathit{cp}\text{-paracompact} \\ \text{in } X \end{array} \arrow[r, shift left, "\mathsf{X} \in \mathcal{T}_3"] & \begin{array}{c} Y \text{ is} \\ 1\text{-}\mathit{cpc}\text{-paracompact} \\ \text{in } X \end{array} \arrow[l, shift left]
\end{tikzcd}$$

Diagram 2

For the α-paracompact case, we have the following implications. The implication "Y is α-cpo-paracompact in $X \xrightarrow{\mathsf{SuR}} Y$ is α-paracompact in X" is proved in Section 3 (see Theorem 3.5). Other implications directly follow from definitions, Corollary 2.6 and Remark 2.3.

$$\begin{tikzcd}[column sep=large, row sep=huge]
\begin{array}{c} Y \text{ is} \\ \alpha\text{-paracompact} \\ \text{in } X \end{array} \arrow[r] \arrow[d, shift left] & \begin{array}{c} Y \text{ is} \\ \alpha\text{-}\mathit{lf}\text{-paracompact} \\ \text{in } X \end{array} \arrow[r, shift left, "\mathsf{StR}"] \arrow[d, shift left] & \begin{array}{c} Y \text{ is} \\ \alpha\text{-}\mathit{lfc}\text{-paracompact} \\ \text{in } X \end{array} \arrow[l, shift left] \arrow[d, shift left] \\
\begin{array}{c} Y \text{ is} \\ \alpha\text{-}\mathit{cpo}\text{-paracompact} \\ \text{in } X \end{array} \arrow[r] \arrow[u, shift left, "\mathsf{SuR}"] & \begin{array}{c} Y \text{ is} \\ \alpha\text{-}\mathit{cp}\text{-paracompact} \\ \text{in } X \end{array} \arrow[r, shift left, "\mathsf{StR}"] \arrow[u, shift left, "\begin{array}{c}\mathsf{SH} \\ \mathsf{Y} \in \mathcal{T}_3\end{array}"] & \begin{array}{c} Y \text{ is} \\ \alpha\text{-}\mathit{cpc}\text{-paracompact} \\ \text{in } X \end{array} \arrow[l, shift left] \arrow[u, shift left, "\mathsf{SH}"]
\end{tikzcd}$$

Diagram 3

Moreover, the following implications hold for 2-paracompact case. These implications follows from definitions, Theorem 2.4 and Remark 2.2. The implication "Y is 2-cpo-paracompact in $X \xrightarrow{\mathsf{R}, \mathsf{Y} \in \mathcal{C}_X} Y$ is 2-paracompact in X" is proved in Section 3 (see Theorem 3.3).

$$
\begin{tikzcd}
\substack{\text{Y is}\\ \text{2-paracompact}\\ \text{in X}} \arrow[r] \arrow[d, shift left] & \substack{\text{Y is}\\ \text{2-\textit{lf}-paracompact}\\ \text{in X}} \arrow[r, shift left, "\mathsf{StR}"] \arrow[d, shift left] & \substack{\text{Y is}\\ \text{2-\textit{lfc}-paracompact}\\ \text{in X}} \arrow[l, shift left] \arrow[d, shift left] \\
\substack{\text{Y is}\\ \text{2-\textit{cpo}-paracompact}\\ \text{in X}} \arrow[r] \arrow[u, shift left, "\substack{\mathsf{R}\\ \mathsf{Y}\in\mathcal{C}_\mathsf{X}}"] & \substack{\text{Y is}\\ \text{2-\textit{cp}-paracompact}\\ \text{in X}} \arrow[r, shift left, "\mathsf{StR}"] \arrow[u, shift left, "\substack{\mathsf{Y}\in\mathcal{T}_3\\ \mathsf{Y}\in\mathcal{C}_\mathsf{X}}"] & \substack{\text{Y is}\\ \text{2-\textit{cpc}-paracompact}\\ \text{in X}} \arrow[l, shift left] \arrow[u, shift left, "\mathsf{Y}\in\mathcal{C}_\mathsf{X}"]
\end{tikzcd}
$$

Diagram 4

Finally, for Aull-paracompact case, we have the following implications. The implication "Y is Aull-*cpo*-paracompact in $X \xrightarrow{\mathsf{R}} Y$ is Aull-paracompact in X" is proved in Section 3 (see Theorem 3.2). Other implications directly follow from definitions, Theorem 2.4 and Remark 2.3.

$$
\begin{tikzcd}
\substack{\text{Y is}\\ \text{Aull-paracompact}\\ \text{in X}} \arrow[r] \arrow[d, shift left] & \substack{\text{Y is}\\ \text{Aull-\textit{lf}-paracompact}\\ \text{in X}} \arrow[r, shift left, "\mathsf{StR}"] \arrow[d, shift left] & \substack{\text{Y is}\\ \text{Aull-\textit{lfc}-paracompact}\\ \text{in X}} \arrow[l, shift left] \arrow[d, shift left] \\
\substack{\text{Y is}\\ \text{Aull-\textit{cpo}-paracompact}\\ \text{in X}} \arrow[r] \arrow[u, shift left, "\mathsf{R}"] & \substack{\text{Y is}\\ \text{Aull-\textit{cp}-paracompact}\\ \text{in X}} \arrow[r, shift left, "\mathsf{StR}"] \arrow[u, shift left, "\substack{\mathsf{Y}\in\mathcal{T}_3\\ \mathsf{Y}\in\mathcal{C}_\mathsf{X}}"] & \substack{\text{Y is}\\ \text{Aull-\textit{cpc}-paracompact}\\ \text{in X}} \arrow[l, shift left] \arrow[u, shift left, "\mathsf{Y}\in\mathcal{C}_\mathsf{X}"]
\end{tikzcd}
$$

Diagram 5

In Diagram 1, the terms "1-", "α-", "2-" and "Aull-" can be replaced by "1-*lf*-", "α-*lf*-", "2-*lf*-" and "Aull-*lf*-", respectively. Moreover, these terms can be replaced by "1-*lfc*-", "α-*lfc*-", "2-*lfc*-" and "Aull-*lfc*-", respectively. Furthermore, the same is available for *cpo*-, *cp*- and *cpc*-.

Let us emphasize the following proposition.

Proposition 2.7. *Let Y be a subspace of a space X. If Y is 2-paracompact in X, then Y is 1-lf-paracompact in X.*

Proof: Let $\mathcal{U}$ be an open cover of X. Since Y is 2-paracompact in X, there is a collection $\mathcal{V}$ of open subsets of X with $Y \subset \bigcup \mathcal{V}$ such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y in X. Put $\mathcal{W} = \mathcal{V} \cup \{\{x\} \mid x \in X \setminus \bigcup \mathcal{V}\}$. Then, $\mathcal{W}$ is a refinement of $\mathcal{U}$ which is locally finite at each point of Y in X. Hence, Y is 1-*lf*-paracompact in X. $\square$

For reverse implications in Diagrams 2, 3, 4, and 5, we have the following examples.

Example 2.8. There exist a Tychonoff space X and its closed subspace Y such that Y is α-*lf*-paracompact in X, but not 1-*cp*-paracompact in X (hence, not 2-paracompact in X).

Proof: Let $X = A(\omega_1) \times (\omega+1) \setminus \{\langle\infty,\omega\rangle\}$, where $A(\omega_1) = D(\omega_1) \cup \{\infty\}$ is the one-point compactification of the discrete space $D(\omega_1)$ of cardinality ω_1. Let $Y = (\{\infty\} \times \omega) \cup (D(\omega_1) \times \{\omega\})$. Then, it is easy to show that Y is α-*lf*-paracompact in X (hence, Y is also Aull-*lf*-paracompact in X and 2-*lf*-paracompact in X). The fact that Y is not 1-*cp*-paracompact in X is proved in Lemma 4.2. By Proposition 2.7, Y is not 2-paracompact in X. □

Example 2.9. There exist a Tychonoff space X and its closed subspace Y such that Y is Aull-paracompact in X, but not 1-paracompact in X (hence, Y is 1-*lf*-paracompact in X, but not α-paracompact in X).

Proof: Let X be the space as in Example 2.8 and $Y = (\{\infty\} \times \omega)$. Then, Y is obviously Aull-paracompact in X. Hence, by Proposition 2.7, Y is 1-*lf*-paracompact in X. Since Y is a closed subspace of a regular space X, Y is not α-paracompact in X. □

3. 1-cpo-, 2-cpo-, Aull-cpo- and α-cpo-paracompactness of a subspace in a space

Y is said to be T_4- (respectively, T_3-) *embedded in* X if for every closed subset F of X disjoint from Y (respectively, $z \in X \setminus Y$), F (respectively, z) and Y are separated by disjoint open subsets of X ([5], see also [13]).

We often use the following proposition.

Proposition 3.1 ([13]; see also [5], [16]). *Let Y be a subspace of a space X. Then, the following statements are equivalent.*

(a) *Y is 1-paracompact in X and T_3-embedded in X.*
(b) *Y is 2-paracompact in X and T_4-embedded in X.*
(c) *Y is Aull-paracompact in X and T_4-embedded in X.*
(d) *Y is α-paracompact in X and satisfies the following condition $(*)$: for every $y \in Y$ and every closed subset F of X with $F \cap Y = \emptyset$, there exists an open subset U of X such that $y \in U \subset \overline{U}^X \subset X \setminus F$.*

As was stated in the previous section, we prove

Theorem 3.2. *Assume that Y is regular in X. Then, Y is Aull-paracompact in X if and only if Y is Aull-cpo-paracompact in X.*

Theorem 3.3. *Assume that Y is a closed subspace of X and Y is regular in X. Then, Y is 2-paracompact in X if and only if Y is 2-cpo-paracompact in X.*

Theorem 3.4. *Assume that Y is regular in X and T_3-embedded in X. Then, Y is 1-paracompact in X if and only if Y is 1-cpo-paracompact in X.*

Theorem 3.5. *Assume that Y is superregular in X (more generally, Y satisfies the condition $(*)$ in Proposition 3.1(d)). Then, Y is α-paracompact in X if and only if Y is α-cpo-paracompact in X.*

Theorem 3.5 is a generalization of [6, Theorem 5] where X is normal.

Let X_Y denote the space obtained from the space X, with the topology generated by a subbase $\{U \mid U$ is open in X or $U \subset X \setminus Y\}$. Hence, points in $X \setminus Y$ are isolated and Y is closed in X_Y. Moreover, X and X_Y generate the same topology on Y ([7]). As is seen [1], the space X_Y is often useful in discussing several relative topological properties. It is easy to see that Y is Hausdorff (respectively, regular) in X if and only if X_Y is Hausdorff (respectively, regular).

Lemma 3.6. *Let Y be a subspace of a space X. Then, Y is Aull-cpo-paracompact in X if and only if every open cover of X_Y has a closure-preserving open refinement.*

Proof: The proof is based on [19]. Assume that every open cover of X_Y has a closure-preserving open refinement. Let $\mathcal{U}$ be a collection of open subsets of X with $Y \subset \bigcup \mathcal{U}$. Then, $\mathcal{U}' = \mathcal{U} \cup \{\{x\} \mid x \in X \setminus \bigcup \mathcal{U}\}$ is an open cover of X_Y. By the assumption, there exists a closure-preserving open (in X_Y) refinement $\mathcal{V}$ of $\mathcal{U}'$. Put $\mathcal{V}' = \{\mathrm{Int}_X V \mid V \in \mathcal{V}, V \cap Y \neq \emptyset\}$. Clearly, $\mathcal{V}'$ is a collection of open subsets of X with $Y \subset \bigcup \mathcal{V}'$ and $\mathcal{V}'$ is a partial refinement of $\mathcal{U}$. To prove that $\mathcal{V}'$ is closure-preserving at each point of Y in X, let $y \in Y$ and $\mathcal{V}'' \subset \mathcal{V}'$ with $y \in \overline{\bigcup \mathcal{V}''}^X$. Then, $y \in \overline{\bigcup \{V \in \mathcal{V} \mid \mathrm{Int}_X V \in \mathcal{V}''\}}^X$. Since $y \in Y$, we have $y \in \overline{\bigcup \{V \in \mathcal{V} \mid \mathrm{Int}_X V \in \mathcal{V}''\}}^{X_Y}$. Since $\mathcal{V}$ is closure-preserving in X_Y, it follows that $y \in \bigcup \{\overline{V}^{X_Y} \mid V \in \mathcal{V}, \mathrm{Int}_X V \in \mathcal{V}''\}$. Hence, there is a $V \in \mathcal{V}$ with $\mathrm{Int}_X V \in \mathcal{V}''$ such that $y \in \overline{V}^{X_Y}$. Since $\overline{V}^{X_Y} \subset \overline{V}^X$ and $y \in Y$, we have $y \in \overline{\mathrm{Int}_X V}^X$. Thus, $y \in \bigcup \overline{\mathcal{V}''}^X$. Therefore, Y is Aull-*cpo*-paracompact in X.

Conversely, assume that Y is Aull-*cpo*-paracompact in X. Let $\mathcal{U}$ be an open cover of X_Y. Then, $\{\mathrm{Int}_X U \mid U \in \mathcal{U}\}$ is a collection of open subsets of X satisfying $Y \subset \bigcup \{\mathrm{Int}_X U \mid U \in \mathcal{U}\}$. By the assumption, there exists a collection $\mathcal{V}$ of open subsets of X with $Y \subset \bigcup \mathcal{V}$ such that $\mathcal{V}$ is a partial refinement of $\{\mathrm{Int}_X U \mid U \in \mathcal{U}\}$ and $\mathcal{V}$ is closure-preserving at each point of Y in X. Put $\mathcal{W} = \mathcal{V} \cup \{\{x\} \mid x \in X \setminus \bigcup \mathcal{V}\}$. Clearly, $\mathcal{W}$ is an open refinement of $\mathcal{U}$. To prove that $\mathcal{W}$ is closure-preserving in X_Y, let $\mathcal{W}' \subset \mathcal{W}$ and $x \in \overline{\bigcup \mathcal{W}'}^{X_Y}$. We first assume $x \in Y$. Then, $x \in \overline{\bigcup \mathcal{W}'}^X$. Since $x \in \bigcup \mathcal{V}$, it is easy to see that $x \in \overline{\bigcup \{W \mid W \in \mathcal{W}' \cap \mathcal{V}\}}^X$. Thus, $x \in \overline{W}^X$ for some $W \in \mathcal{W}' \cap \mathcal{V}$. Since $x \in Y$, we have $x \in \overline{W}^{X_Y} \subset \bigcup \overline{\mathcal{W}'}^{X_Y}$.

Next, assume $x \in X \setminus Y$. Then, $x \in \bigcup \mathcal{W}' \subset \bigcup \overline{\mathcal{W}'}^{X_Y}$. Therefore, $\mathcal{W}$ is closure-preserving in X_Y. This completes the proof. □

Proof Theorem 3.2: Notice that Y is regular in X if and only if X_Y is regular, and Y is Aull-paracompact in X if and only if X_Y is paracompact ([2], [19], see also [13]). [18, Theorem 1] and Lemma 3.6 complete the proof. □

Proof of Theorem 3.3: To prove the "if" part, suppose that Y is 2-paracompact in X. Since Y is closed in X, Y is Aull-paracompact in X. By Theorem 3.2, Y is Aull-*cpo*-paracompact in X. Hence, Y is 2-*cpo*-paracompact in X. The "only if" part is obvious. □

To prove Theorems 3.4 and 3.5, we have the following lemma which improves [16, Lemma 1.2].

Lemma 3.7. *For a subspace Y of a space X, each of the following statements hold.*

(a) *If Y is T_3-embedded in X and 1-cpo-paracompact in X, then Y is T_4-embedded in X.*
(b) *Assume that Y satisfies the condition $(*)$ in Proposition 3.1(d). If Y is α-cpo-paracompact in X, then Y is T_4-embedded in X.*

Proof: (a) Let F be a closed subset of X with $F \cap Y = \emptyset$. For each $x \in F$, there is an open subset U_x of X such that $x \in U_x \subset \overline{U_x}^X \subset X \setminus Y$. Then, $\mathcal{U} = \{U_x \,|\, x \in F\} \cup \{X \setminus F\}$ is an open cover of X. Since Y is 1-*cpo*-paracompact in X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is closure-preserving at each point of Y in X. Put $G = \bigcup \{W \in \mathcal{W} \,|\, W \cap F \neq \emptyset\}$. Then, G is an open subset of X satisfying $F \subset G \subset \overline{G}^X \subset X \setminus Y$. Hence, Y is T_4-embedded in X.

(b) Let F be a closed subset of X with $F \cap Y = \emptyset$. For each $y \in Y$, there is an open subset U_y of X such that $y \in U_y \subset \overline{U_y}^X \subset X \setminus F$. Then, $\mathcal{U} = \{U_y \,|\, y \in Y\}$ is a collection of open subsets of X with $Y \subset \bigcup \mathcal{U}$. Since Y is α-*cpo*-paracompact in X, there exists a collection $\mathcal{V}$ of open subsets of X with $Y \subset \bigcup \mathcal{V}$ such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is closure-preserving in X. Since $\overline{\bigcup \mathcal{V}}^X \cap F = \emptyset$, Y is T_4-embedded in X. □

Proof of Theorem 3.4: To prove the "if" part, suppose that Y is 1-*cpo*-paracompact in X. By Lemma 3.7(a), Y is T_4-embedded in X. Since Y is closed and 2-*cpo*-paracompact in X, Y is Aull-*cpo*-paracompact in X. If follows from Theorem 3.3, Y is 2-paracompact in X. Since Y is T_4-embedded in X, by Theorem 3.1, we have that Y is 1-paracompact in X. The "only if" part is obvious. □

Proof of Theorem 3.5: To prove the "if" part, suppose that Y is α-*cpo*-paracompact in X. By Lemma 3.7(b), Y is T_4-embedded in X. Note that Y is

Aull-*cpo*-paracompact in X. If follows from Theorem 3.1 and Theorem 3.2 that Y is α-paracompact in X. The "only if" part is obvious. □

Corresponding to Proposition 3.1, we have the following result for *cpo*-paracompact cases. This fact follows from Theorems 3.2, 3.3, 3.4 and 3.5, Proposition 3.1 and Lemma 3.7. Notice that if Y is superregular in X, then Y obviously satisfies the condition $(*)$ in Proposition 3.1(d).

Corollary 3.8. *Let Y be a subspace of a space X. Then, the following statements are equivalent.*

(a) *Y is 1-cpo-paracompact in X and T_3-embedded in X.*
(b) *Y is 2-cpo-paracompact in X and T_4-embedded in X.*
(c) *Y is Aull-cpo-paracompact in X and T_4-embedded in X.*

Remark 3.9. In Theorem 3.4, the condition "Y is T_3-embedded in X" cannot be removed. Let X be the space $\Psi = \omega \cup \mathcal{A}$ constructing a m.a.d. family $\mathcal{A}$ of infinite subsets of ω ([8, 5I]) and $Y = \omega$. Then, Y is not 1-paracompact in X (see [13]), but 1-*cpo*-paracompact in X since each point of Y is isolated in X.

At the end of this section, we discuss absolute embeddings of 1-, α-, 2- and Aull-*cpo*-paracompactness. Corollary 3.10 below immediately follows from Theorems 1.1, 3.4 and 3.5.

Corollary 3.10. *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(a) *Y is 1-cpo- (or equivalently, α-cpo-) paracompact in every larger Tychonoff (respectively, regular) space.*
(b) *Y is 1-cpo- (or equivalently, α-cpo-) paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(c) *Y is compact.*

Theorems 1.2, 3.2 and 3.3 induce the following.

Corollary 3.11. *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(a) *Y is 2-cpo- (or equivalently, Aull-cpo-) paracompact in every larger Tychonoff (respectively, regular) space.*
(b) *Y is 2-cpo- (or equivalently, Aull-cpo-) paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.*
(c) *Y is Lindelöf.*

4. More on absolute embeddings

In this section, we discuss absolute embeddings on other versions of relative paracompactness defined in Section 2. The results obtained in this section should be compared with Theorems 1.1 and 1.2.

We actually give characterizations of absolute 1-*lf*- and 1-*cp*-paracompactness as follows.

Theorem 4.1. *For a Tychonoff* (*respectively, regular*) *space Y, the following statements are equivalent.*

(a) *Y is 1-lfc-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(b) *Y is 1-cpc-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(c) *Y is 1-lf-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(d) *Y is 1-cp-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(e) *Y is Lindelöf.*

In the statements from (a) *to* (d) *above, "every larger Tychonoff* (*respectively, regular*) *space" can be replaced by "every larger Tychonoff* (*respectively, regular*) *space containing Y as a closed subspace".*

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.2. *Let $X = A(\omega_1) \times (\omega+1) \setminus \{\langle\infty, \omega\rangle\}$ and $Y = (\{\infty\} \times \omega) \cup (D(\omega_1) \times \{\omega\})$. Then, Y is not 1-cp-paracompact in X.*

Proof: We may assume $D(\omega_1) \cap \omega = \emptyset$. Put $U_\alpha = \{\alpha\} \times (\omega+1)$ for each $\alpha \in D(\omega_1)$, and let $U_n = A(\omega_1) \times \{n\}$ for each $n < \omega$. Then, $\mathcal{U} = \{U_\alpha \,|\, \alpha \in D(\omega_1)\} \cup \{U_n \,|\, n < \omega\}$ is an open cover of X. Suppose that $\mathcal{V}$ is a refinement of $\mathcal{U}$ which is closure-preserving at each point of Y in X. Let $n < \omega$ be fixed. Then notice that

$$|\{\alpha \in D(\omega_1) \,|\, (\exists V \in \mathcal{V})\langle\alpha, n\rangle \in V \subset U_\alpha\}| < \omega,$$

because $\mathcal{V}$ is closure-preserving at $\langle\infty, n\rangle$ and $V \cap V' = \emptyset$ if $V, V' \in \mathcal{V}, V \subset U_\alpha, V' \subset U_\beta, \alpha, \beta \in D(\omega_1)$ and $\alpha \neq \beta$. Hence, there exists $\alpha_n \in D(\omega_1)$ such that $V \subset U_n$ for every $\alpha > \alpha_n$ and every $V \in \mathcal{V}$ with $\langle\alpha, n\rangle \in V$. Now, let $\alpha^* = \sup\{\alpha_n \,|\, n < \omega\}$. Here, notice that $V \cap V' = \emptyset$ if $V, V' \in \mathcal{V}, V \subset U_m, V' \subset U_n, m, n \in \omega$ and $m \neq n$. Then $\mathcal{V}$ is not closure-preserving at $\langle\alpha^* + 1, \omega\rangle$ in X, a contradiction. Therefore, Y is not 1-*cp*-paracompact in X. □

Now, we give the proof of Theorem 4.1. Here, a space X is said to be ω_1-*compact* if every uncountable subset of X has an accumulation point in X. It is well-known that a space X is Lindelöf if and only if X is paracompact and ω_1-compact.

Proof of Theorem 4.1: The equivalence (a)⇔(c) and (b)⇔(d) follow from Proposition 2.2. The implications (a)⇒(b) and (c)⇒(d) are trivial. Moreover, the implication (e)⇒(c) follows from Theorem 1.2 and Proposition 2.7.

To prove (d) ⇒ (e), suppose that Y is 1-*cp*-paracompact in every larger Tychonoff (respectively, regular) space but not Lindelöf. Since Y is paracompact in itself, Y is not ω_1-compact. Hence, Y contains an uncountable closed discrete subset D of cardinality ω_1. Enumerate $D = \{y_\alpha \mid \alpha \in D(\omega_1)\} \cup \{y_n \mid n \in \omega\}$, where $D(\omega_1) \cap \omega = \emptyset$.

Let $Z = A(\omega_1) \times (\omega + 1) \setminus \{\langle \infty, \omega \rangle\}$ as in Lemma 4.2. Let X be the quotient space obtained from $Y \oplus Z$ by identifying y_α with $\langle \alpha, \omega \rangle$ for each $\alpha \in D(\omega_1)$ and y_n with $\langle \infty, n \rangle$ for each $n \in \omega$. Then X is a larger Tychonoff (respectively, regular) space. Using Lemma 4.2, it is easy to see that Y is not 1-*cp*-paracompact in X. □

Example 4.3. There exist a Tychonoff space X and an open subspace Y of X such that Y is Aull-paracompact in X and 1-*cpo*-paracompact in X, but neither 1-paracompact in X nor α-*cp*-paracompact in X.

Proof: Let X be the space $\Psi = \omega \cup \mathcal{A}$ as in Remark 3.9. Then, Y is Aull-paracompact in X and 1-*cpo*-paracompact in X, but not 1-paracompact in X (see Remark 3.9). To prove that Y is not α-*cp*-paracompact in X, consider $\mathcal{U} = \{\{n\} \mid n \in \omega\}$. Then, $\mathcal{U}$ is a collection of open subsets of X with $Y = \bigcup \mathcal{U}$ but any open partial refinement of $\mathcal{U}$ covering Y is closure-preserving at no point of $X \setminus Y$. □

For absolute α-*lf*- or α-*cp*-paracompactness, we have

Theorem 4.4. *For a Tychonoff* (*respectively, regular*) *space Y, the following statements are equivalent.*

(a) *Y is α-lfc-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(b) *Y is α-cpc-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(c) *Y is α-lf-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(d) *Y is α-cp-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(e) *Y is compact.*

Proof: The implications (a) ⇒ (c) ⇒ (d) and (a) ⇒ (b) ⇒ (d) are obvious. It is clear that (e) implies (a).

(d) ⇒ (e). Suppose that Y is not compact. Since Y is paracompact in itself, Y is not countably compact. Hence, Y has a countable closed discrete subset $\{y_n \mid n \in \mathbb{N}\}$. Let $\Psi = \omega \cup \mathcal{A}$ be the space as in Example 4.3. Let X be the quotient space obtained from $Y \oplus \Psi$ by identifying y_n with n for each $n \in \mathbb{N}$. Note that X is a larger Tychonoff (respectively, regular) space of Y. Using Example 4.3, it is easy to see that Y is not α-*cp*-paracompact in X. □

Remark 4.5. Notice that in Theorems 4.4, "every larger Tychonoff (respectively, regular) space" cannot be replaced by "every larger Tychonoff (respectively, regular) space containing Y as a closed subspace". Indeed, for a Tychonoff (respectively, regular) space Y, the following statements are equivalent.

(a) Y is α-*lfc*-paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.
(b) Y is α-*cp*-paracompact in every larger Tychonoff (respectively, regular) space containing Y as a closed subspace.
(c) Y is paracompact.

In the statements (a) and (b) above, "α-*lfc*-" (or equivalently, "α-*cp*-") can be replaced by "α-*lf*-" (or "α-*cpc*-").

Moreover, we characterize absolute embeddings of relative paracompactness of 2- or Aull-paracompactness types as follows.

Theorem 4.6. *For a Tychonoff* (*respectively, regular*) *space* Y, *the following statements are equivalent.*

(a) Y *is Aull-lfc-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(b) Y *is* 2-*cp-paracompact in every larger Tychonoff* (*respectively, regular*) *space.*
(c) Y *is paracompact.*

In the statements (a) *and* (b) *above, "every larger Tychonoff* (*respectively, regular*) *space" can be replaced by "every larger Tychonoff* (*respectively, regular*) *space containing* Y *as a closed subspace".*

Proof: The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial.

To prove (c) $\Rightarrow$ (a), suppose that Y is paracompact. Let X be a space with $Y \subset X$ and $\mathcal{U}$ a collection of open subsets of X with $Y \subset \bigcup \mathcal{U}$. Since Y is regular, for each $y \in Y$ there exist an open subset V_y of X and a $U_y \in \mathcal{U}$ such that $y \in V_y \subset \overline{V_y}^X \subset U_y$. Put $\mathcal{V} = \{V_y \mid y \in Y\}$. Since Y is paracompact in itself and regular, there exists a locally finite closed (in Y) cover $\mathcal{W}$ of Y such that $\mathcal{W}$ refines $\mathcal{V}$. Notice that $\mathcal{W}$ is locally finite at each point of Y in X. Then $\overline{\mathcal{W}}^X$ is a collection of closed subsets of X with $Y \subset \bigcup \overline{\mathcal{W}}^X$ such that $\overline{\mathcal{W}}^X$ is a partial refinement of $\mathcal{U}$. Moreover, by Proposition 2.1(b), $\overline{\mathcal{W}}^X$ is locally finite at each point of Y in X. Hence, Y is Aull-*lfc*-paracompact in X. $\square$

Remark 4.7. In Theorem 4.6, "Aull-*lfc*-paracompact" can be replaced by "Aull-*cpc*-paracompact", "Aull-*lf*-paracompact" and "Aull-*cp*-paracompact". Moreover in Theorem 4.6, "2-*cp*-paracompact" can be replaced by "2-*lfc*-paracompact", "2-*cpc*-paracompact" and "2-*lf*-paracompact".

5. Concluding remarks

In this section, we give some related remarks to relative paracompactness discussed in the previous sections. Let Y be a subspace of a space X and $\mathcal{F}$ a collection of subsets of X. In [10] and [12], Grabner et al. introduced the following two relative notions of closure-preserving collections. It is defined in [12] that $\mathcal{F}$ is *closure preserving with respect to* Y if for every $\mathcal{F}' \subset \{F \in \mathcal{F} \mid F \cap Y \neq \emptyset\}$ either $Y \subset \bigcup \mathcal{F}'$ or $\bigcup \mathcal{F}'$ is closed in X. Moreover, $\mathcal{F}$ is *weakly closure preserving with respect to* Y if for every $\mathcal{F}' \subset \{F \in \mathcal{F} \mid F \cap Y \neq \emptyset\}$, it holds that $(\bigcup \mathcal{F}') \cap Y = \overline{\bigcup \mathcal{F}'}^X \cap Y$. In [10], they assume that $\mathcal{F}$ is a collection of closed subsets of X in the above definitions. As was mentioned in Section 2, the notion of closure preserving collections with respect to Y above does not satisfy the statements (a$'$), (b$'$) and (c$'$) stated below Proposition 2.1. Actually, there exists a collection $\mathcal{A}$ of closed subsets of X such that $\mathcal{A}$ is locally finite at each point of Y in X, but not closure preserving with respect to Y (consider $X = \omega + 1$, $Y = \omega$ and $\mathcal{A} = \{\{n\} \mid n < \omega\}$). There exists a collection $\mathcal{A}$ of subsets of X such that $\mathcal{A}$ is closure preserving with respect to Y, but $\overline{\mathcal{A}}^X$ is not closure preserving with respect to Y (consider, $X = (\omega+1)^2 \setminus (\{\omega\} \times \omega)$, $Y = (\omega + 1) \times \{\omega\}$ and $\mathcal{A} = \{\{n\} \times \omega \mid n < \omega\}$). Moreover, there exists a collection $\mathcal{A}$ of closed subsets of X which is point-finite at each point of Y and closure preserving with respect to Y, but not locally finite at some point of Y in X (consider $X = \omega + 1$, $Y = \{\omega\}$ and $\mathcal{A} = \{\{n\} \mid n < \omega\}$).

Remark 5.1. In [10], Grabner et al. defined that Y is *weakly cp-paracompact in* X if for every open cover $\mathcal{U}$, there is a closed partial refinement $\mathcal{F}$ such that $Y \subset \bigcup \mathcal{F}$ and $\mathcal{F}$ is weakly closure preserving with respect to Y. In [12], Grabner et al. modified the definition of weak cp-paracompactness in X as follows: Y is *weakly cp-paracompact in* X if for every open cover $\mathcal{U}$, there is a (not necessarily closed) partial refinement $\mathcal{F}$ such that $Y \subset \bigcup \mathcal{F}$ and $\mathcal{F}$ is weakly closure preserving with respect to Y. They commented in [12] that the new definition of weak cp-paracompactness in X appears to be weaker. Note that Y is 2-*cpc*-paracompact in X if and only if Y is weakly cp-paracompact in X (in the sense in [10]). Moreover, Y is 2-*cp*-paracompact in X if Y is weakly cp-paracompact in X (in the sense of revised definition in [12]). Assuming Y is strongly regular in X, these notions are equivalent as in Diagram 4.

Remark 5.2. In [11, Lemma 2.2], Grabner et al. assert that if a closed collection $\mathcal{F}$ is weakly closure preserving with respect to Y and A is a subset of Y then $A \subset X \setminus \overline{\bigcup(\mathcal{F} \setminus \{F \in \mathcal{F} \mid F \cap Y \neq \emptyset\})}^X$. However, this contains a gap. For, consider $X = \omega + 1, Y = A = \{\omega\}$ and $\mathcal{F} = \{\{n\} \mid n \in \omega\}$. The referee suggests us to point out that the lemma is correct and its application remains the same, if the notion of weakly closure preserving with respect to Y is replaced with closure preserving at each point of Y.

To discuss the notions by Grabner et al. and our notions defined in Section 2, let us introduce some other notions relative paracompactness. We define that Y is *α'-paracompact* (respectively, *α'-lf-paracompact*, *α'-lfc-paracompact*) *in* X if for every open cover $\mathcal{U}$ of X there exists an open partial refinement (respectively, a partial refinement, a closed partial refinement) $\mathcal{V}$ of $\mathcal{U}$ such that $Y \subset \bigcup \mathcal{V}$ and $\mathcal{V}$ is locally finite in X.

We also say that Y is *α'-cpo-paracompact* (respectively, *α'-cp-paracompact*, *α'-cpc-paracompact*) *in* X if for every open cover $\mathcal{U}$ of X there exists an open partial refinement (respectively, a partial refinement, a closed partial refinement) $\mathcal{V}$ such that $Y \subset \bigcup \mathcal{V}$ and $\mathcal{V}$ is closure-preserving in X. Notice that it is easy to see that a subspace Y of a space X is α'-*cpc*-paracompact in X if and only if Y is *cp*-paracompact in X in the sense of Grabner et al. [10]; this fact is pointed out in [12] assuming that X is Hausdorff. But, in Proposition 5.3 below, we show that α'-*lfc*-paracompactness is coincident with α'-*cpc*-paracompactness without any additional condition.

The notion of α'-paracompactness is intermediate between α- and 2-paracompactness, and is independent from 1-paracompactness. It is obvious that α'-paracompactness is equivalent to α-paracompactness for closed subspaces. On the other hand, there exist a Tychonoff space X and its subspace Y such that Y is α'-paracompact in X, but not α-paracompact in X (consider $X = \omega + 1$ and $Y = \omega$). Moreover, there exist a Tychonoff space X and its subspace Y such that Y is 1-paracompact in X, but not α'-paracompact in X (consider $X = A(\omega_1) \times (\omega + 1) \setminus \{\langle \infty, \omega \rangle\}$ and $Y = D(\omega_1) \times \omega$).

In the rest of this section, we prove the following Proposition 5.3.

Proposition 5.3. *For a subspace Y of a space X, the following statements are equivalent.*

(a) *Y is α'-lfc-paracompact in X.*
(b) *Y is α'-cpc-paracompact in X.*
(c) *Y is α'-lf-paracompact in X and $\overline{Y}^X$ is regular.*
(d) *Y is α'-cp-paracompact in X and $\overline{Y}^X$ is regular.*
(e) *$\overline{Y}^X$ is paracompact Hausdorff.*

Grabner et al. [10, Theorem 35] (respectively, [12, Theorem 8]) proved that the statements (b) and (e) in Proposition 5.3 above are equivalent assuming that X is regular (respectively, Hausdorff).

Lemma 5.4. *Let Y be a subspace of a space X. Then, the following statements are equivalent.*

(a) *Y is α'-lf- (respectively, α'-cp-) paracompact in X and $\overline{Y}^X$ is regular.*
(b) *$\overline{Y}^X$ is α'-lfc- (respectively, α'-cpc-) paracompact in X.*
(c) *Y is α'-lfc- (respectively, α'-cpc-) paracompact in X.*

Proof: (a) $\Rightarrow$ (b). Assume that $\overline{Y}^X$ is regular. The "if" part is obvious.

To prove the "only if" part, let $\mathcal{U}$ be an open cover of X. Since $\overline{Y}^X$ is regular, for each $x \in \overline{Y}^X$ there exist an open subset V_x of $\overline{Y}^X$ and a $U_x \in \mathcal{U}$ such that $x \in V_x \subset \overline{V_x}^{\overline{Y}^X} \subset U_x$. Then, let W_x be an open subset of X such that $W_x \cap \overline{Y}^X = V_x$. Put $\mathcal{V} = \{W_x \mid x \in \overline{Y}^X\} \cup \{X \setminus \overline{Y}^X\}$. Since Y is α'-*lf*-paracompact in X, there exists a partial refinement $\mathcal{A}$ of $\mathcal{V}$ such that $Y \subset \bigcup \mathcal{A}$ and $\mathcal{A}$ is locally finite in X. Put $\mathcal{F} = \{\overline{A \cap Y}^{\overline{Y}^X} \mid A \in \mathcal{A}, A \cap Y \neq \emptyset\}$. Then, $\mathcal{F}$ is a closed (in Y) partial refinement of $\mathcal{U}$ such that $\overline{Y}^X = \bigcup \mathcal{F}$ and $\mathcal{F}$ is locally finite in X by Proposition 2.1(b). Hence $\overline{Y}^X$ is α'-*lfc*-paracompact in X.

The implication (b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). Since α'-*cpc*-paracompactness induces regularity of $\overline{Y}^X$, it is clear. $\square$

Therefore, if $\overline{Y}^X$ is regular, α'-*lf*- (respectively, α'-*cp*-) paracompactness is equivalent to α'-*lfc*- (respectively, α'-cpc-) paracompactness.

For a closed subspace Y of a space X, we have that Y is α'-*lf*- (respectively, α'-*lfc*-, α'-*cp*-, α'-*cpc*-) paracompact in X if and only if Y is α-*lf*- (respectively, α-*lfc*-, α-*cp*-, α-*cpc*-) paracompact in X. By using these facts, we prove Proposition 5.3.

Proof of Proposition 5.3: The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are obvious.

(a) $\Rightarrow$ (c). Obviously, Y is α'-*lf*-paracompact in X. Then, by Lemma 5.4, $\overline{Y}^X$ is α'-*lfc*-paracompact in X. Hence, by Theorem 2.4, $\overline{Y}^X$ is paracompact Hausdorff. Therefore, $\overline{Y}^X$ is regular. (b) $\Rightarrow$ (d) can be proved similarly.

(d) $\Rightarrow$ (e). By Lemma 5.4, $\overline{Y}^X$ is α'-*cp*-paracompact in X. Hence, $\overline{Y}^X$ is clearly paracompact Hausdorff.

(e) $\Rightarrow$ (a). Since $\overline{Y}^X$ is paracompact Hausdorff, by Theorem 2.4, $\overline{Y}^X$ is α'-*lfc*-paracompact in X. Hence, by Lemma 5.4, Y is α'-*lfc*-paracompact in X. $\square$

Proposition 5.3 and Lemma 5.4 induce the following.

Corollary 5.5. *Assume that $\overline{Y}^X$ is regular. If Y is α'-cp-paracompact, then Y is α'-lf-paracompact in X.*

Proof: Assume that $\overline{Y}^X$ is regular. Since Y is α'-*cp*-paracompact in X and $\overline{Y}^X$ is regular, by Lemma 5.4, $\overline{Y}^X$ is α'-*cpc*-paracompact in X. By Proposition 5.3, $\overline{Y}^X$ is α'-*lfc*-paracompact in X. Hence, Y is α'-*lf*-paracompact in X. $\square$

Moreover, by applying Theorem 3.5, we have

Corollary 5.6. *Assume that Y is closed in X and Y satisfies the condition $(*)$ in Proposition* 3.1(d). *If Y is α'-cpo-paracompact in X, then Y is α'-paracompact in X.*

Proof: Assume that Y is closed in X and Y satisfies the condition $(*)$ in Proposition 3.1(d). Since Y is α'-*cpo*-paracompact in X, Y is α-*cpo*-paracompact in X. Then, by Theorem 3.5, Y is α-paracompact in X. Hence, Y is obviously α'-paracompact in X. $\square$

We conclude this paper by the following implications among α'-cases. These implications directly follow from definitions, Proposition 5.3, Corollaries 5.5 and 5.6. Here, the symbol $(*)$ denotes the condition $(*)$ in Proposition 3.1(d).

$$\begin{tikzcd}
\begin{array}{c} Y \text{ is} \\ \alpha'\text{-paracompact} \\ \text{in } X \end{array} \arrow[r] \arrow[d, shift left] & \begin{array}{c} Y \text{ is} \\ \alpha'\text{-}lf\text{-paracompact} \\ \text{in } X \end{array} \arrow[r, shift left, "\overline{Y}^X \in \mathcal{T}_3"] \arrow[d, shift left] & \begin{array}{c} Y \text{ is} \\ \alpha'\text{-}lfc\text{-paracompact} \\ \text{in } X \end{array} \arrow[l, shift left] \arrow[d, leftrightarrow] \\
\begin{array}{c} Y \text{ is} \\ \alpha'\text{-}cpo\text{-paracompact} \\ \text{in } X \end{array} \arrow[r] \arrow[u, shift left, "{(*)\ Y \in \mathcal{C}_X}"] & \begin{array}{c} Y \text{ is} \\ \alpha'\text{-}cp\text{-paracompact} \\ \text{in } X \end{array} \arrow[r, shift left, "\overline{Y}^X \in \mathcal{T}_3"] \arrow[u, shift left, "\overline{Y}^X \in \mathcal{T}_3"] & \begin{array}{c} Y \text{ is} \\ \alpha'\text{-}cpc\text{-paracompact} \\ \text{in } X \end{array} \arrow[l, shift left]
\end{tikzcd}$$

Diagram 6

References

[1] Arhangel'skii A.V., *Relative topological properties and relative topological spaces*, Topology Appl. **70** (1996), 87–99.

[2] Arhangel'skii A.V., *From classic topological invariants to relative topological properties*, Sci. Math. Jpn. **55** (2002), 153–201.

[3] Arhangel'skii A.V., Genedi H.M.M., *Beginnings of the theory of relative topological properties*, in: General Topology. Spaces and Mappings, MGU, Moscow, 1989, pp. 3–48.

[4] Arhangel'skii A.V., Gordienko I.Ju., *Relative symmetrizability and metrizability*, Comment. Math. Univ. Carolin. **37** (1996), 757–774.

[5] Aull C.E., *Paracompact subsets*, Proc. Second Prague Topological Symposium, Academia, Prague, 1966, pp. 45–51.

[6] Aull C.E., *Paracompact and countably paracompact subsets*, Proc. Kanpur Topological Conference, 1968, pp. 49–53.

[7] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[8] Gillman L., Jerison M., *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[9] Gordienko I.Ju., *A characterization of relative Lindelöf property by relative paracompactness*, General Topology. Spaces, mappings and functors, MUG, Moscow, 1992, pp. 40–44.

[10] Grabner E.M., Grabner G.C., Miyazaki K., *On properties of relative metacompactness and paracompactness type*, Topology Proc. **25** (2000), 145–177.

[11] Grabner E.M., Grabner G.C., Miyazaki K., Tartir J., *Relative collectionwise normality*, Appl. Gen. Topol. **5** (2004), 199–212.

[12] Grabner E.M., Grabner G.C., Miyazaki K., Tartir J., *Relationships among properties of relative paracompactness type*, Questions Answers Gen. Topology **22** (2004), 91–104.

[13] Kawaguchi S., Sokei R., *Some relative properties on normality and paracompactness, and their absolute embeddings*, Comment. Math. Univ. Carolin. **46** (2005), 475–495.

[14] Lupiañez F.G., *On covering properties*, Math. Nachr. **141** (1989), 37–43.
[15] Lupiañez F.G., *α-paracompact subsets and well-situated subsets*, Czechoslovak Math. J. **38(113)** (1988), 191–197.
[16] Lupiañez F.G., Outerelo E., *Paracompactness and closed subsets*, Tsukuba J. Math. **13** (1989), 483–493.
[17] Michael E., *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4** (1953), 831–838.
[18] Michael E., *Another note on paracompact spaces*, Proc. Amer. Math. Soc. **8** (1957), 822–828.
[19] Yamazaki K., *Aull-paracompactness and strong star-normality of subspaces in topological spaces*, Comment. Math. Univ. Carolin. **45** (2004), 743–747.

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