A p-Laplacian system with resonance and nonlinear boundary conditions on an unbounded domain

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Abstract. We study a nonlinear elliptic system with resonance part and nonlinear boundary conditions on an unbounded domain. Our approach is variational and is based on the well known Landesman-Laser type conditions.

Keywords: quasilinear problem, p-Laplacian system, Landesman-Laser condition, resonance

Classification: 35D05, 35J45, 35J50

1. Introduction and statement of results

Let Ω be an unbounded domain in $\mathbb{R}^N$, $N \geq 3$, with a noncompact and smooth boundary $\partial\Omega$. In this paper we consider the following quasilinear elliptic system

$$(1) \qquad \begin{cases} -\Delta_p u = \lambda_1 a(x)|u|^{p-2}u + \lambda_1 b(x)|u|^\alpha |v|^\beta v + g_1(x,u) - h_1(x), \ x \in \Omega \\ -\Delta_p v = \lambda_1 d(x)|v|^{p-2}v + \lambda_1 b(x)|u|^\alpha |v|^\beta u + g_2(x,u) - h_2(x), \ x \in \Omega \end{cases}$$

subject to the nonlinear boundary conditions

$$(2) \qquad \begin{cases} |\nabla u|^{p-2}\nabla u \cdot \eta + c_1(x)|u|^{p-2}u = 0, \quad x \in \partial\Omega \\ |\nabla v|^{p-2}\nabla v \cdot \eta + c_2(x)|v|^{p-2}v = 0, \quad x \in \partial\Omega \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and η is the unit outward normal vector on $\partial\Omega$. On a single equation level with Ω bounded and Dirichlet boundary conditions, the problem has been studied by Arcoya and Orsina [1] taking into consideration the well known Landesman-Laser type conditions for the resonance part. The extension to the case of a system, again with Ω bounded and Dirichlet boundary conditions, was first considered by Zographopoulos in [7].

In order to confront with our problem we need a suitable space setting which we describe next.

For $\xi \in \mathbb{R}$, we set $w_\xi(x) := \frac{1}{(1+|x|)^\xi}$, and assume that the space $L^r(w_\xi, \Omega) := \{u : \int_\Omega w_\xi(x)|u|^r < +\infty\}$, $r \geq 1$, is supplied with the norm

$$\|u\|_{w_\xi,r} = \left(\int_\Omega w_\xi(x)|u|^r\right)^{1/r}.$$

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)$-functions restricted on Ω. For $p\in(1,+\infty)$, the weighted Sobolev space E_p is the completion of $C_\delta^\infty(\Omega)$ in the norm

$$|||u|||_p=\left(\int_\Omega|\nabla u|^p+\int_\Omega w_p(x)\,|u|^p\right)^{1/p}.$$

By Lemma 2 in [5], we see that if $c(\cdot)$ is a positive continuous function defined on $\mathbb{R}^N$ such that

$$kw_{p-1}(x)\le c(x)\le Kw_{p-1}(x),$$

for some positive constants k and K,
then the norm

$$\|u\|_{1,p}=\left(\int_\Omega|\nabla u|^p+\int_{\partial\Omega}c(x)\,|u|^m\right)^{1/p}$$

is equivalent to $|||\cdot|||_p$.

We will consider our system on the space $E=E_p\times E_p$, supplied with the norm

$$\|(u,v)\|=\|u\|_{1,p}+\|v\|_{1,p}.$$

The following lemma is useful for our compactness arguments.

Lemma 1. (i) *If*

$$p\le r\le\frac{pN}{N-p}\quad\textit{and}\quad N>\alpha\ge N-r\frac{N-p}{p}\,,$$

then the embedding $E\subseteq L^r(w_\alpha,\Omega)$ is continuous. If the upper bound for r in the first inequality and the upper bound for α in the second are strict, then the embedding is compact.

(ii) *If*

$$p\le m\le\frac{p(N-1)}{N-p}\quad\textit{and}\quad N>\beta\ge N-1-m\frac{N-p}{p}\,,$$

then the trace operator $T:E\to L^m(w_\beta,\partial\Omega)$ is continuous. If the upper bound for m in the first inequality and the lower bound for β are strict, then the trace operator is compact.

(iii) *If*

$$1\le q<p\quad\textit{and}\quad\frac{\alpha_1-N}{\alpha_2-N}>\frac{p}{q}\,,$$

then the embedding $L^p(w_{\alpha_1},\Omega)\subseteq L^q(w_{\alpha_2},\Omega)$ is continuous.

Proof: The first and second part of the lemma is Theorem 1 in [5], while the third is a consequence of the following inequality

$$\int_\Omega\frac{1}{(1+|x|)^{\alpha_2}}|u|^q\,dx\le\left(\int_\Omega\frac{1}{(1+|x|)^d}\,dx\right)^{\frac{p-q}{p}}\left(\int_\Omega\frac{1}{(1+|x|)^{\alpha_1}}|u|^p\,dx\right)^{\frac{q}{p}},$$

where $d = \frac{\alpha_2 p - \alpha_1 q}{p-q}$. Note that the integral $\int_\Omega \frac{1}{(1+|x|)^d}\, dx$ converges since $d > N$. $\square$

We study (1)–(2) in connection with the eigenvalue problem

$$
\left\{
\begin{aligned}
-\Delta_p u &= \lambda_1 a(x)|u|^{p-2}u + \lambda_1 b(x)|u|^\alpha |v|^\beta v,\\
-\Delta_p v &= \lambda_1 d(x)|v|^{p-2}v + \lambda_1 b(x)|u|^\alpha |v|^\beta u,
\end{aligned}
\right. \tag{3}
$$

subject to the boundary conditions (2), which was considered in [4] under the following set of assumptions, also needed for the present problem:

(H1) $2 < p < N$, $\alpha, \beta \geq 0$ with $\alpha + \beta = p - 2$ and $\alpha + 1$, $\beta + 1 \leq \frac{pp^*}{N}$, where $p^* = \frac{Np}{N-p}$.

(H2) (i) There exist positive constants α_1, A with $\alpha_1 \in \left(p + \frac{(\beta+1)(N-p)}{p^*}, N\right)$ such that $0 < a(x) \leq A w_{\alpha_1}(x)$ a.e. in Ω.

(ii) There exist positive constants α_2, D with $\alpha_2 \in \left(p + \frac{(\alpha+1)(N-p)}{p^*}, N\right)$ such that

$$0 < d(x) \leq D w_{\alpha_2}(x) \quad \text{a.e. in} \quad \Omega.$$

(iii) $m\{x \in \Omega : b(x) > 0\} > 0$ and

$$0 \leq b(x) \leq B w_s(x) \quad \text{a.e. in} \quad \Omega,$$

where $B > 0$ and $s \in (p,\, N)$.

(H3) $c_1(\cdot)$ and $c_2(\cdot)$ are positive continuous functions defined on R^N with

$$k w_{p-1}(x) \leq c_1(x), c_2(x) \leq K w_{p-1}(x),$$

for some positive constants k and K.

Let

$$I(u,v) = \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p + \frac{\alpha+1}{p}\int_{\partial\Omega} c_1(x)|u|^p + \frac{\beta+1}{p}\int_\Omega |\nabla v|^p + \frac{\beta+1}{p}\int_{\partial\Omega} c_2(x)|v|^p$$

and

$$J(u,v) = \frac{\alpha+1}{p}\int_\Omega a(x)|u|^p + \frac{\beta+1}{p}\int_\Omega d(x)|v|^p + \int_\Omega b(x)|u|^\alpha |v|^\beta uv.$$

Theorem 2 ([4]). *Let Ω be an unbounded domain in R^N, $N \geq 2$, with a non-compact and smooth boundary $\partial\Omega$. Assume that hypotheses* (H1), (H2) *and* (H3) *hold. Then*

(a) *the system* (3) *admits a positive principal eigenvalue λ_1 given by*

$$\lambda_1 = \inf\{I(u,v) : J(u,v) = 1\}.$$

Each component of the associated normalized eigenfunction (u_1, v_1) is positive on $\bar{\Omega}$ and of class $C^{1,\delta}_{\text{loc}}(\Omega)$ for some $\delta \in (0,1)$.

(b) *the set of eigenfunctions corresponding to λ_1 forms a one dimensional manifold $X \subseteq E$ defined by*

$$X = \{c(u_1, v_1);\ c \in \mathbb{R}\backslash\{0\}\}.$$

(c) *λ_1 is isolated, in the sense that there exists $\eta > 0$ such that the interval $(0, \lambda_1 + \eta)$ does not contain any other eigenvalue than λ_1.*

We make the following assumptions concerning the resonance part:

(H4) (i) $g_1(\cdot,\cdot)$, $g_2(\cdot,\cdot)$ are Caratheodory functions such that $|g_1(x,s)| \leq \dfrac{C_1}{(1+|x|)^{\alpha_3}}$ and $|g_2(x,s)| \leq \dfrac{C_2}{(1+|x|)^{\alpha_4}}$, where $\alpha_3 > N - \frac{N-\alpha_1}{p}$, $\alpha_4 > N - \frac{N-\alpha_2}{p}$, C_1, C_2 are positive constants, and the limits

$$\lim_{s\to\pm\infty} g_i(x,s) = g_i^{\pm}(x), \qquad i = 1, 2,$$

exist for almost every $x \in \Omega$.

(ii) $|h_1(x)| \leq \dfrac{H_1}{(1+|x|)^{\alpha_3}}$ and $|h_2(x)| \leq \dfrac{H_2}{(1+|x|)^{\alpha_4}}$ for some positive constants H_1, H_2.

Furthermore, we will need the following inequalities

$$L^+ < (\alpha+1)\int_\Omega h_1(x)u_1 + (\beta+1)\int_\Omega h_2(x)v_1 < L^-, \tag{4}$$

$$L^- < (\alpha+1)\int_\Omega h_1(x)u_1 + (\beta+1)\int_\Omega h_2(x)v_1 < L^+, \tag{5}$$

where (u_1, v_1) is the normalized eigenfunction of (3)–(2) with positive components and

$$L^+ = (\alpha+1)\int_\Omega g_1^+(x)u_1 + (\beta+1)\int_\Omega g_2^+(x)v_1,$$
$$L^- = (\alpha+1)\int_\Omega g_1^-(x)u_1 + (\beta+1)\int_\Omega g_2^-(x)v_1.$$

Inequalities (4) and (5) are the adaptation to the case of systems of the Landesman-Laser type conditions for scalar equations.

The energy functional of the problem (1)–(2) is

$$\begin{aligned}\Phi(u,v) = \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p + \frac{\alpha+1}{p}\int_{\partial\Omega} c_1(x)|u|^p - \lambda_1 \frac{\alpha+1}{p}\int_\Omega a(x)|u|^p \\ - (\alpha+1)\int_\Omega G_1(x,u) + (\alpha+1)\int_\Omega h_1(x)u \\ + \frac{\beta+1}{p}\int_\Omega |\nabla v|^p + \frac{\beta+1}{p}\int_{\partial\Omega} c_2(x)|v|^p - \lambda_1 \frac{\beta+1}{p}\int_\Omega d(x)|v|^p \\ - (\beta+1)\int_\Omega G_2(x,v) + (\beta+1)\int_\Omega h_2(x)v - \lambda_1 \int_\Omega b(x)|u|^\alpha |v|^\beta uv,\end{aligned}$$

where

$$G_i(x,s) = \int_0^s g_i(x,t)\,dt, \quad i = 1,2.$$

In view of (H1)–(H3), the functional Φ is well defined and continuously differentiable on E. By a *weak solution* of (1)–(2) we mean an element of E which is a critical point of Φ.

The main result of this work is the following theorem:

Theorem 3. (i) *Assume that hypotheses* (H1)–(H3) *and inequality* (4) *or* (5) *hold. Then the system* (1)–(2) *admits a weak solution.*

2. The main result

In view of Theorem 2(a), it is clear that $\lambda_1 \le \min\{\lambda_u, \lambda_v\}$, where λ_u, λ_v are the first eigenvalues of the problems $-\Delta_p u = \lambda a(x)|u|^{p-2}u$ and $-\Delta_p v = \lambda d(x)|v|^{p-2}v$, with the boundary conditions (2), respectively. The following lemma shows that this inequality is actually strict.

Lemma 4. $\lambda_1 < \min\{\lambda_u, \lambda_v\}$.

Proof: Let $u_0 > 0$ be an eigenfunction corresponding to λ_u and $v_0 > 0$ an eigenfunction corresponding to λ_v. If $\lambda_u = \lambda_v$, then

$$\lambda_1 \le \frac{I(u_0,v_0)}{J(u_0,v_0)} < \lambda_u,$$

so without loss of generality we may assume that $\lambda_u < \lambda_v$. Let $t > 0$ be such that

$$\frac{\beta+1}{p}\int_\Omega d(x)|v_0|^p < \frac{\lambda_u}{\lambda_v - \lambda_u}\int_\Omega b(x)|tu_0|^\alpha |v_0|^\beta t u_0 v_0. \tag{6}$$

Then, in view of (6),

$$\lambda_1 \leq \frac{I(tu_0, v_0)}{J(tu_0, v_0)} < \lambda_u = \min\{\lambda_u, \lambda_v\}.$$

□

Note that due to assumptions H(1)–H(4), the operators A, N, B, $C : E \to E^*$ given by

$$\begin{aligned}
\langle A(u,v),(\varphi,\psi)\rangle &:= \int_\Omega |\nabla u|^{p-2}u\nabla\varphi + \int_\Omega |\nabla v|^{p-2}\nabla v\nabla\psi,\\
\langle N(u,v),(\varphi,\psi)\rangle &:= \int_\Omega a(x)|u|^{p-2}u\varphi - \int_{\partial\Omega} c_1(x)|u|^{p-2}u\varphi\\
&\quad + \int_\Omega d(x)|v|^{p-2}v\psi - \int_{\partial\Omega} c_2(x)|v|^{p-2}v\psi,\\
\langle B(u,v),(\varphi,\psi)\rangle &:= \int_\Omega b(x)|u|^\alpha|v|^\beta v\varphi + \int_\Omega b(x)|u|^\alpha|v|^\beta u\psi,\\
\langle C(u,v),(\varphi,\psi)\rangle &:= \int_\Omega \left(g_1(x,u) - h_1(x)\right)\varphi + \int_\Omega \left(g_2(x,v) - h_2(x)\right)\psi,
\end{aligned}$$

are well defined. Following standard arguments based on the embeddings given in Lemma 1, we have:

Lemma 5. *The operators A, N, B and C are continuous. Moreover, N, B and C are compact.*

We can now proceed with the proof of the main result:

Proof of Theorem 3: We assume first that (4) holds. We claim that Φ satisfies the PS-condition. Indeed, let $\{(u_n, v_n)\}_{n\in N}$ be a PS-sequence in E. Then

$$-c \leq \Phi(u_n, v_n) \leq c, \tag{7}$$

for some $c > 0$, and there exists a sequence $\{\varepsilon_n\}_{n\in N}$ converging to 0^+, such that

$$-\varepsilon_n\|(u,v)\| \leq \Phi'(u_n,v_n)(u,v) \leq \varepsilon_n\|(u,v)\| \quad \text{for every} \quad (u,v) \in E. \tag{8}$$

We will show that the sequence $\{(u_n, v_n)\}_{n\in N}$ is bounded in E. Assume the contrary, that is $\|(u_n, v_n)\| \to +\infty$. Let

$$\widehat{u}_n := \frac{u_n}{\|(u_n, v_n)\|}, \quad \widehat{v}_n := \frac{v_n}{\|(u_n, v_n)\|}. \tag{9}$$

Since $\|\widehat{u}_n\|_{E_p} \le 1$ and $\|\widehat{v}_n\|_{E_p} \le 1$, by passing to subsequences if necessary, we may assume that $\widehat{u}_n \to \widehat{u}$ and $\widehat{v}_n \to \widehat{v}$ weakly in E_p. Due to our hypotheses on h_1 and g_1 we obtain

$$\lim_{n\to+\infty} \int_\Omega \frac{G_1(x,u_n)}{\|(u_n,v_n)\|^p} = \lim_{n\to+\infty} \int_\Omega \frac{h_1 u_n}{\|(u_n,v_n)\|^p} = 0 \tag{10}$$

and similarly for $G_2(\cdot,\cdot)$ and $h_2(\cdot)$. Dividing (7) by $\|(u_n,v_n)\|^p$ and using (10), we arrive at

$$\begin{aligned}
\limsup_{n\to+\infty} \Bigg[& \tfrac{\alpha+1}{p} \left\{ \int_\Omega |\nabla \widehat{u}_n|^p + \int_{\partial\Omega} c_1(x)|\widehat{u}_n|^p - \lambda_1 \int_\Omega a(x)|\widehat{u}_n|^p \right\} \\
& + \tfrac{\beta+1}{p} \left\{ \int_\Omega |\nabla \widehat{v}_n|^p + \int_{\partial\Omega} c_2(x)|\widehat{v}_n|^p - \lambda_1 \int_\Omega d(x)|\widehat{v}_n|^p \right\} \\
& -\lambda_1 \int_\Omega b(x)|\widehat{u}_n|^\alpha |\widehat{v}_n|^\beta \widehat{u}_n \widehat{v}_n \Bigg] \le 0,
\end{aligned}$$

and Lemma 1 gives

$$\begin{aligned}
\limsup_{n\to+\infty} \Bigg[& \tfrac{\alpha+1}{p} \left\{ \int_\Omega |\nabla \widehat{u}_n|^p + \int_{\partial\Omega} c_1(x)|\widehat{u}_n|^p \right\} \\
& + \tfrac{\beta+1}{p} \left\{ \int_\Omega |\nabla \widehat{v}_n|^p + \int_{\partial\Omega} c_2(x)|\widehat{v}_n|^p \right\} \Bigg] \\
\le \lambda_1 & \left(\tfrac{\alpha+1}{p} \int_\Omega a(x)|\widehat{u}|^p + \tfrac{\beta+1}{p} \int_\Omega d(x)|\widehat{v}|^p + \int_\Omega b(x)|\widehat{u}|^\alpha |\widehat{v}|^\beta \widehat{u}\widehat{v} \right).
\end{aligned}$$

The reverse inequality (with the limsup replaced by liminf) also holds due to the lower semicontinuity of the norms. Thus $(\widehat{u},\widehat{v})$ is a nonzero solution of (3) with $\|(\widehat{u},\widehat{v})\| = 1$. In view of Lemma 4, $\widehat{u} \neq 0$ and $\widehat{v} \neq 0$. By Theorem 2, $\widehat{u}$ and $\widehat{v}$ have the same sign. Suppose that both $\widehat{u}$ and $\widehat{v}$ are positive, the other case can be treated similarly. Thus $\widehat{u} = u_1$ and $\widehat{v} = v_1$. If we replace (u,v) by (u_n,v_n) in (8), write the relation for $-\Phi'$, multiply the members of (7) by p, add memberwise the resulting inequalities, and divide by $\|(u_n,v_n)\|$, we obtain

$$\begin{aligned}
\Bigg| & (\alpha+1)(p-1) \int_\Omega h_1(x)\widehat{u}_n + (\beta+1)(p-1) \int_\Omega h_2(x)\widehat{v}_n \\
& -(\alpha+1)p \int_\Omega \widehat{g}_1(x,u_n)\widehat{u}_n + (\alpha+1) \int_\Omega g_1(x,u_n)\widehat{u}_n - (\beta+1)p \int_\Omega \widehat{g}_2(x,v_n)\widehat{v}_n \\
& +(\beta+1) \int_\Omega g_2(x,v_n)\widehat{v}_n \Bigg| \le \frac{c}{\|(u_n,v_n)\|} + \varepsilon_n,
\end{aligned}$$

where

$$\widehat{g}_i(x,s) := \begin{cases} \frac{G_i(x,s)}{s} & \text{if } s \neq 0, \\ g_i(x,0) & \text{if } s = 0, \end{cases} \quad i = 1,2.$$

By letting $n \to +\infty$, we get

$$\begin{aligned} \lim_{n\to+\infty} \Big\{ (\alpha+1) \int_\Omega [g_1(x,u_n)\widehat{u}_n - p\widehat{g}_1(x,u_n)\widehat{u}_n] \\ +(\beta+1) \int_\Omega [g_2(x,v_n)\widehat{v}_n - p\widehat{g}_2(x,v_n)\widehat{v}_n] \Big\} \\ = (\alpha+1)(1-p) \int_\Omega h_1(x)\widehat{u} + (\beta+1)(1-p) \int_\Omega h_2(x)\widehat{v}. \end{aligned} \tag{11}$$

By (9), $u_n(x)$ and $v_n(x)$ tend to $+\infty$, so

$$g_1(x,u_n) \to g_1^+(x) \text{ and } g_2(x,v_n) \to g_2^+(x) \quad \text{a.e. in } \ \Omega.$$

Therefore

$$\lim_{n\to+\infty} \int_\Omega [g_1(x,u_n)\widehat{u}_n - p\widehat{g}_1(x,u_n)\widehat{u}_n] = (1-p) \int_\Omega g_1^+(x)\widehat{u}, \tag{12}$$

with a similar relation holding for $g_2(\cdot,\cdot)$ as well. In view of (11) and (12), we have

$$(\alpha+1) \int_\Omega g_1^+(x)u_1 + (\beta+1) \int_\Omega g_2^+(x)v_1 = (\alpha+1) \int_\Omega h_1(x)u_1 + (\beta+1) \int_\Omega h_2(x)v_1,$$

contradicting (4). Thus $\{(u_n,v_n)\}_{n\in N}$ is bounded. Therefore, up to subsequences, $u_n \to u_0$ and $v_n \to v_0$ weakly in E_p and strongly in $L^p(w_{\alpha_1},\Omega)$ and $L^p(w_{\alpha_2},\Omega)$, respectively. By taking $(u,v) = (u_n,v_n) - (u_0,v_0)$ in (8), and using Lemma 1, we derive that

$$\begin{aligned} (\alpha+1) \Big\{ \int_\Omega \Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0 \Big) (\nabla u_n - \nabla u_0) \\ + \int_{\partial\Omega} c_1 \Big(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0 \Big) (u_n - u_0) \Big\} \\ + (\beta+1) \Big\{ \int_\Omega \Big(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v_0|^{p-2}\nabla v_0 \Big) (\nabla v_n - \nabla v_0) \\ + \int_{\partial\Omega} c_1 \Big(|v_n|^{p-2}v_n - |v_0|^{p-2}v_0 \Big) (v_n - v_0) \Big\} \to 0, \end{aligned}$$

which, in view of inequality 2.5 in [2], implies that $(u_n, v_n) \to (u_0, v_0)$ in E.
We show next that Φ is coercive. Indeed, if this were not the case, there would exist a sequence $\{(u_n, v_n)\}_{n\in N}$ with $\|(u_n, v_n)\| \to +\infty$ and

$$|\Phi(u_n, v_n)| \le M, \quad \text{for some} \quad M > 0. \tag{13}$$

Working as before, we get that

$$\begin{aligned}\lim_{n\to+\infty} \Bigg[&\tfrac{\alpha+1}{p} \left\{ \int_\Omega |\nabla \widehat{u}_n|^p + \int_{\partial\Omega} c_1(x)|\widehat{u}_n|^p - \lambda_1 \int_\Omega a(x)|\widehat{u}_n|^p \right\} \\ &+ \tfrac{\beta+1}{p} \left\{ \int_\Omega |\nabla \widehat{v}_n|^p + \int_{\partial\Omega} c_2(x)|\widehat{v}_n|^p - \lambda_1 \int_\Omega d(x)|\widehat{v}_n|^p \right\} \\ &-\lambda_1 \int_\Omega b(x)|\widehat{u}_n|^\alpha |\widehat{v}_n|^\beta \widehat{u}_n \widehat{v}_n \Bigg] = 0,\end{aligned}$$

where $\widehat{u}_n$ and $\widehat{v}_n$ are defined in (9). Thus $(\widehat{u}_n, \widehat{v}_n) \to (u_1, v_1)$ or $(\widehat{u}_n, \widehat{v}_n) \to -(u_1, v_1)$ in E. If $(\widehat{u}_n, \widehat{v}_n) \to (u_1, v_1)$, by (13) and the variational characterization of λ_1, we obtain

$$(\alpha+1) \int_\Omega g_1^+(x)u_1 + (\beta+1) \int_\Omega g_2^+(x)v_1 \ge (\alpha+1) \int_\Omega h_1(x)u_1 + (\beta+1) \int_\Omega h_2(x)v_1,$$

while if $(\widehat{u}_n, \widehat{v}_n) \to -(u_1, v_1)$, we get

$$(\alpha+1) \int_\Omega g_1^-(x)u_1 + (\beta+1) \int_\Omega g_2^-(x)v_1 \le (\alpha+1) \int_\Omega h_1(x)u_1 + (\beta+1) \int_\Omega h_2(x)v_1,$$

contradicting (4). We can now use Theorem 4.7 in [3] to get a weak solution of (1)–(2).

Assume next that (5) holds. We split E as the direct sum of the eigenspace X and $Y = \{(u, v) \in E : \int_\Omega u u_1^{p-1} + \int_\Omega v v_1^{p-1} = 0\}$. Then Φ has a saddle point geometry, i.e.,

(i) $\Phi(t(u_1, v_1)) \to -\infty$ if $|t| \to +\infty$, and
(ii) Φ is bounded from below on Y.

Indeed, since

$$\begin{aligned}\Phi(t(u_1, v_1)) &= (\alpha + 1) \left[\int_\Omega h_1(x)tu_1 - \int_\Omega G_1(x, tu_1) \right] \\ &\quad + (\beta + 1) \left[\int_\Omega h_2(x)tv_1 - \int_\Omega G_2(x, tv_1) \right] \\ &= (\alpha + 1)t \left[\int_\Omega h_1(x)u_1 - \int_\Omega \frac{G_1(x, tu_1)}{tu_1} u_1 \right] \\ &\quad + (\beta + 1)t \left[\int_\Omega h_2(x)v_1 - \int_\Omega \frac{G_2(x, tv_1)}{tv_1} v_1 \right],\end{aligned}$$

by taking the limit as $|t| \to \infty$ and working as in the first part of the proof, we can use (5) to get (i). To prove (ii) we exploit the isolation of λ_1, see Theorem 2, to derive that there exists $\widehat{\lambda} > \lambda_1$ such that

$$\widehat{\lambda} < \frac{I(u,v)}{J(u,v)}$$

for every $(u,v) \in Y$. If $(u,v) \in Y$, in view of Lemma 1,

$$\begin{aligned}\Phi(u,v) &= I(u,v) - \lambda_1 J(u,v) + (\alpha+1)\left[\int_\Omega h_1(x)u - \int_\Omega G_1(x,u)\right] \\ &\quad + (\beta+1)\left[\int_\Omega h_2(x)v - \int_\Omega G_2(x,v)\right] \\ &> \left(1 - \frac{\lambda_1}{\widehat{\lambda}}\right) I(u,v) - (\alpha+1)c_1\|u\|_{1,p} - (\beta+1)c_2\|v\|_{1,p},\end{aligned}$$

for some $c_1, c_2 > 0$. Consequently, Φ is bounded from below on Y. An application of the saddle point theorem, see [6], provides a weak solution of (1)–(2). □

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