Regularity of weak solutions to certain degenerate elliptic equations

Albo Carlos Cavalheiro

Abstract. In this article we establish the existence of higher order weak derivatives of weak solutions of Dirichlet problem for a class of degenerate elliptic equations.

Keywords: degenerate elliptic equations, weighted Sobolev spaces

Classification: Primary 35J70; Secondary 35J25

1. Introduction

In this paper we study the existence of higher order weak derivatives (see Theorem 3.8) of weak solutions of degenerate elliptic equations $Lu = g - \operatorname{div} \vec{f}$, where L is an elliptic operator

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u)(x) - \sum_{i=1}^{n} b_i(x)D_i u(x) \tag{1.1}$$

whose coefficients a_{ij} and b_i are measurable, real-valued functions, and whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies the degenerate ellipticity condition

$$\lambda\omega(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda\omega(x)|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and almost all $x \in \Omega\subset\mathbb{R}^n$ on a bounded open set Ω, ω is a weight function, λ and Λ are positive constants.

2. Definitions and basic results

By a *weight* we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will also be denoted by ω. Thus $\omega(E) = \int_E \omega\, dx$ for measurable sets $E\subset\mathbb{R}^n$.

The author was partially supported by CNPq Grant 476040/2004-03.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open and let ω be a weight. For $1 < p < \infty$, we define $L^p(\Omega, \omega)$, the Banach space of all measurable functions f defined on Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\, dx \right)^{1/p} < \infty.$$

Definition 2.2. Let $1 \le p < \infty$. A weight ω belongs to the Muckenhoupt class A_p if there is a constant $\mathbf{C} = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega\, dx \right) \left(\frac{1}{|B|} \int_B \omega^{-1/(p-1)}\, dx \right)^{p-1} \le \mathbf{C} \quad (\text{if } \ 1 < p < \infty)$$
$$\left(\frac{1}{|B|} \int_B \omega\, dx \right) \left(\operatorname*{ess\,sup}_B \frac{1}{\omega} \right) \le \mathbf{C} \quad (\text{if } \ p = 1),$$

for every ball $B \subset \mathbb{R}^n$, where $|B|$ is the n-dimensional Lebesgue measure of B. The infimum over all constants $\mathbf{C}$ is called " A_p-constant of ω".

Example 2.3. The function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is a weight A_p if and only if $-n < \alpha < n(p-1)$ (see [6, Chapter 15]).

Remark 2.4. If $\omega \in A_p$, $1 \le p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable when $p > 1$, and $1/\omega$ is locally bounded, when $p = 1$, we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω. If Ω is bounded, one obtains in the same way that $L^p(\Omega, \omega) \subset L^1(\Omega)$. It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$. □

Definition 2.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 \le p < \infty$ and k be a nonnegative integer. Suppose that the weight $\omega \in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega, \omega)$ as the set of functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$ for $|\alpha| \le k$. The norm of u in $W^{k,p}(\Omega, \omega)$ is given by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{0 \le |\alpha| \le k} \int_\Omega |D^\alpha u|^p \omega\, dx \right)^{1/p}.$$

We also define $W^{k,p}_0(\Omega, \omega)$ as the closure of $C^\infty_0(\Omega)$ in $W^{k,p}(\Omega, \omega)$.
If $\Omega \subset \mathbb{R}^n$ is open, $k \ge 1$, $1 \le p < \infty$ and $\omega \in A_p$ then $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega, \omega)$ (see Corollary 2.1.6 in [8]). The spaces $W^{k,p}(\Omega, \omega)$ are Banach spaces.

In this paper we use frequently the following two theorems.

Theorem 2.6 (Muckenhoupt Theorem). *Let ω be a weight in $\mathbb{R}^n$ and*

$$[M(f)](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy$$

be the Hardy-Littlewood maximal function. Then for $p > 1$, $M : L^p(\mathbb{R}^n, \omega) \longrightarrow L^p(\mathbb{R}^n, \omega)$ is continuous (that is, $\|Mf\|_{L^p(\mathbb{R}^n,\omega)} \le C_M \|f\|_{L^p(\mathbb{R}^n,\omega)}$) if and only if $\omega \in A_p$. The constant C_M is called Muckenhoupt constant and C_M depends only on n, p and the A_p-constant of ω.

Proof: See [7] or [4, Corollary 4.3]. □

Theorem 2.7 (Weighted Sobolev inequality). *Let Ω be a bounded open set in $\mathbb{R}^n$, $1 < p < \infty$ and $\omega \in A_p$. Then there exist constants C_Ω and δ positive such that for all $\varphi \in C_0^\infty(\Omega)$ and k satisfying $1 \le k \le \frac{n}{n-1} + \delta$,*

$$\|\varphi\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \||\nabla\varphi|\|_{L^p(\Omega,\omega)}$$

where C_Ω may be taken to depend only on n, the A_p constant of ω, p and the diameter of Ω.

Proof: See Theorem 1.3 of [2]. □

Definition 2.8. We say that $u \in W^{1,2}(\Omega, \omega)$ is a weak solution of the equation

$$Lu = g - \sum_{i=1}^{n} D_i f_i, \quad \text{with } \frac{g}{\omega}, \frac{f_i}{\omega} \in L^2(\Omega, \omega)$$

if

$$\mathcal{B}(u,\varphi) = \sum_{i=1}^{n} \int_\Omega f_i D_i \varphi + \int_\Omega g\varphi\, dx, \quad \forall\, \varphi \in W_0^{1,2}(\Omega,\omega)$$

where

$$\mathcal{B}(u,\varphi) = \int_\Omega \Bigg[\sum_{i,j=1}^{n} a_{ij} D_i u D_j \varphi - \sum_{i=1}^{n} b_i \varphi D_i u \Bigg] dx.$$

Theorem 2.9. *Let L be the operator (1.1) satisfying (1.2) and $|b_i(x)| \le C_1 \omega(x)$ in Ω. Assume that $\psi \in W^{1,2}(\Omega,\omega)$, $g/\omega \in L^2(\Omega,\omega)$, $f_i/\omega \in L^2(\Omega,\omega)$ and $\omega \in A_2$. Then the Dirichlet problem*

$$Lu = g - \sum_{i=1}^{n} D_i f_i$$
$$u - \psi \in W_0^{1,2}(\Omega,\omega)$$

has a unique solution $u \in W^{1,2}(\Omega,\omega)$ and

$$\|u\|_{W^{1,2}(\Omega,\omega)} \le C \left(\|g/\omega\|_{L^2(\Omega,\omega)} + \|f_j/\omega\|_{L^2(\Omega,\omega)} + \|\psi\|_{W^{1,2}(\Omega,\omega)} \right).$$

Proof: It is consequence of the Lax-Milgram Theorem and the proof follows the lines of Theorem 2.2. of [2]. □

3. Differentiability of weak solutions

In this section we prove that weak solutions $u \in W^{1,2}(\Omega,\omega)$ of the equation $Lu = g$ are twice weakly differentiable and $D_{ij}u \in L^2(\Omega',\omega)$ (that is, $u \in W^{2,2}(\Omega',\omega)$, $\forall\, \Omega' \subset\subset \Omega$).

Definition 3.1. Let u be a function on a bounded open set $\Omega \subset \mathbb{R}^n$ and denote by e_i the unit coordinate vector in the x_i direction. We define the difference quotient of u at x in the direction e_i by

$$\Delta_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}, \quad (0 < |h| < \operatorname{dist}(x, \partial\Omega)). \tag{3.1}$$

Lemma 3.2. *Let $\Omega' \subset\subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$. If $u, v \in L^2_{\mathrm{loc}}(\Omega,\omega)$, $\operatorname{supp}(v) \subset \Omega'$ and g is a measurable function with $|g(x)| \le C\omega(x)$, then*

(a) $\Delta_k^h(uv)(x) = u(x + he_k)\Delta_k^h v(x) + v(x)\Delta_k^h u(x)$, *with* $1 \le k \le n$;
(b) $\int_\Omega g(x)u(x)\Delta_k^{-h} v(x)\, dx = -\int_\Omega v(x)\Delta_k^h(gu)(x)\, dx$;
(c) $\Delta_k^h(D_j v)(x) = D_j(\Delta_k^h v)(x)$.

Proof: The proof of this lemma follows trivially from Definition 3.1. □

Definition 3.3. Let ω be a weight in $\mathbb{R}^n$. We say that ω is uniformly A_p in each coordinate if

(a) $\omega \in A_p(\mathbb{R}^n)$;
(b) $\omega_i(t) = \omega(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ is in $A_p(\mathbb{R})$, for $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ a.e., $1 \le i \le n$, with A_p constant of ω_i bounded independently of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$.

Example 3.4. Let $\omega(x,y) = \omega_1(x)\omega_2(y)$, with $\omega_1(x) = |x|^{1/2}$ and $\omega_2(y) = |y|^{1/2}$. We have that ω is uniformly A_2 in each coordinate.

Lemma 3.5. *Let $u \in W^{1,p}(\Omega,\omega)$, $p > 1$, and let ω be a weight uniformly A_p in each coordinate. Then for any $\Omega' \subset\subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, we have*

$$\|\Delta_k^h u\|_{L^p(\Omega',\omega)} \le C\|D_k u\|_{L^p(\Omega,\omega)} \tag{3.2}$$

where $C = 2C_M$, and C_M is the Muckenhoupt constant.

Proof: Case 1. Let us suppose initially that $u \in C^\infty(\Omega)$. We have,

$$\begin{aligned}\Delta_k^h u(x) &= \frac{u(x + he_k) - u(x)}{h} = \frac{1}{h}\int_0^h D_k(x + \zeta e_k)\, d\zeta \\ &= \frac{1}{h}\int_0^h D_k u(x_1, \ldots, x_{k-1}, x_k + \zeta, x_{k+1}, \ldots, x_n)\, d\zeta.\end{aligned}$$

For $1 \le k \le n$, we define the functions

$$G_k(x) = \begin{cases} D_k u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

We have for $x \in \Omega' \subset\subset \Omega$ and h satisfying $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$,

$$\begin{aligned} |\Delta_k^h u(x)| &\le \frac{1}{|h|} \left| \int_0^h |D_k u(x_1, \ldots, x_{k-1}, x_k + \zeta, x_{k+1}, \ldots, x_n)|\, d\zeta \right| \\ &= \frac{1}{|h|} \left| \int_{x_k}^{x_k+h} |D_k u(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)|\, dt \right| \\ &= \frac{1}{|h|} \left| \int_{x_k}^{x_k+h} |G_k(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)|\, dt \right| \\ &\le \frac{1}{|h|} \left| \int_{x_k-h}^{x_k+h} |G_k(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)|\, dt \right| \\ &\le \sup_{h \neq 0} \frac{1}{|h|} \left| \int_{x_k-h}^{x_k+h} |G_k(x_1, \ldots x_{k-1}, t, x_{k+1}, \ldots, x_n)|\, dt \right| \\ &\le 2M(G_k^{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n})(\mathbf{x_k}), \end{aligned}$$

where $G_k^{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n}(\mathbf{x_k}) = G_k(x_1, \ldots, \mathbf{x_k}, \ldots, x_n)$. Consequently, using the notation $\widehat{dx_k} = dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_n$ (where the hat indicates the term that must be omitted in the product) and by Theorem 2.6, we obtain

$$\begin{aligned} &\int_{\Omega'} |\Delta_k^h u(x)|^p \omega(x)\, dx \\ &\le 2^p \int_{\Omega'} [M(G_k^{x_1, \ldots, x_{k-1} x_{k+1}, \ldots, x_n})]^p(\mathbf{x_k}) \omega(x_1, \ldots, x_k, \ldots, x_n)\, dx \\ &\le 2^p \int_{\mathbb{R}^n} [M(G_k^{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n})]^p(\mathbf{x_k}) \omega(x_1, \ldots, x_k, \ldots, x_n)\, dx_1 \ldots dx_k \ldots dx_n \\ &= 2^p \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} [M(G_k^{x_1, \ldots, x_{k-1} x_{k+1}, \ldots, x_n})]^p(\mathbf{x_k}) \omega(x_1, \ldots, \mathbf{x_k}, \ldots x_n) dx_k \right) \widehat{dx_k} \\ &\le 2^p \int_{\mathbb{R}^{n-1}} \left(C_M^p \int_{\mathbb{R}} |G_k^{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n}(\mathbf{x_k})|^p \omega(x_1, \ldots, \mathbf{x_k}, \ldots, x_n)\, dx_k \right) \widehat{dx_k} \\ &= 2^p C_M^p \int_{\mathbb{R}^n} |G_k(x)|^p \omega(x)\, dx \\ &= 2^p C_M^p \int_{\Omega} |D_k u(x)|^p \omega(x)\, dx, \end{aligned}$$

where C_M is independent of $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ because ω is uniformly A_p in each coordinate. Therefore

$$\|\Delta_k^h u\|_{L^p(\Omega', \omega)} \le C \|D_k u\|_{L^p(\Omega, \omega)}, \quad \text{where} \quad C = 2C_M.$$

Case 2. If $u \in W^{1,p}(\Omega,\omega)$ then there exists a sequence $\{u_m\}$, $u_m \in C^\infty(\Omega)$, Cauchy sequence in the norm $\|.\|_{W^{1,p}(\Omega,\omega)}$. By Definition 2.5, we have that

$$u_m \longrightarrow u \text{ and } D_k u_m \longrightarrow D_k u \text{ in } L^p(\Omega,\omega).$$

Consequently, since $\omega \in A_p$, there exists a subsequence $\{u_{m_j}\}$ such that $u_{m_j} \longrightarrow u$ a.e. This implies, for $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, that

$$\Delta_k^h u_{m_j} \longrightarrow \Delta_k^h u \text{ a.e.}$$

We have that $\{\Delta_k^h u_{m_j}\}$ is a Cauchy sequence in $L^p(\Omega',\omega)$, for any $\Omega' \subset\subset \Omega$. In fact, using the first case, we have

$$\begin{aligned}\|\Delta_k^h u_{m_r} - \Delta_k^h u_{m_s}\|_{L^p(\Omega',\omega)} &= \|\Delta_k^h (u_{m_r} - u_{m_s})\|_{L^p(\Omega',\omega)} \\ &\le C\|D_k(u_{m_r} - u_{m_s})\|_{L^p(\Omega,\omega)} \\ &= C\|D_k u_{m_r} - D_k u_{m_s}\|_{L^p(\Omega,\omega)} \\ &\longrightarrow 0, \quad \text{as } m_r, m_s \longrightarrow \infty.\end{aligned}$$

Therefore, there exists $g \in L^p(\Omega',\omega)$ such that $\Delta_k^h u_{m_j} \longrightarrow g$ in $L^p(\Omega',\omega)$. Consequently, there exists a subsequence $\Delta_k^h u_{m_{j_r}} \longrightarrow g$ a.e. We can conclude that $\Delta_k^h u = g$ a.e. Hence

$$\Delta_k^h u_{m_j} \longrightarrow \Delta_k^h u \text{ in } L^p(\Omega',\omega).$$

This implies that

$$\begin{aligned}\|\Delta_k^h u\|_{L^p(\Omega',\omega)} &= \lim_{m_j\to\infty} \|\Delta_k^h u_{m_j}\|_{L^p(\Omega',\omega)} \\ &\le C \lim_{m_j\to\infty} \|D_k u_{m_j}\|_{L^p(\Omega,\omega)} \\ &= C\|D_k u\|_{L^p(\Omega,\omega)},\end{aligned}$$

that is, $\|\Delta_k^h u\|_{L^p(\Omega',\omega)} \le C\|D_k u\|_{L^p(\Omega,\omega)}$. □

Lemma 3.6. *Let $u \in L^p(\Omega,\omega)$, $1 < p < \infty$, $\omega \in A_p$ and suppose there exists a constant C such that*

$$\|\Delta_k^h u\|_{L^p(\Omega',\omega)} \le C, \; k = 1, 2, \ldots, n \tag{3.3}$$

for any $\Omega' \subset\subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$ (with C independent of h). Then there exists $v \in L^p(\Omega,\omega)$ such that $D_k u = v$ in the weak sense, that is, $u \in W^{1,p}(\Omega,\omega)$ and $\|D_k u\|_{L^p(\Omega,\omega)} \le C$.

Proof: The proof of this lemma follows the lines of Lemma 7.24 in [5]. □

Remark 3.7. We use the notation

$$\mathcal{D}^k(\Omega,\omega) = \left\{ g \in W^k(\Omega) \ : \ \frac{D^\alpha g}{\omega} \in L^2(\Omega,\omega),\ 0 \le |\alpha| \le k \right\}, \quad \text{for} \quad k = 0,1,2\dots,$$

where $W^k(\Omega)$ denotes the linear space of k times weakly derivative functions. For $k = 0$, we have $g \in \mathcal{D}^0(\Omega,\omega)$ if $g/\omega \in L^2(\Omega,\omega)$. □

We are able now to prove the main result of this paper.

Theorem 3.8. *Let $u \in W^{1,2}(\Omega,\omega)$ be a weak solution of the equation $Lu = g$ in Ω, and assume that*

(a) $g \in \mathcal{D}^0(\Omega,\omega)$;
(b) *ω is a weight uniformly A_2 in each coordinate;*
(c) $|\Delta_k^h a_{ij}(x)| \le C_1\omega(x)$, $x \in \Omega' \subset\subset \Omega$ *a.e.*, $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, *with a constant C_1 independent of Ω' and h;*
(d) $|b(x)| \le C\omega(x)$ *a.e. in Ω, where $b = (b_1,\dots,b_n)$.*

Then for any subdomain $\Omega' \subset\subset \Omega$ we have $u \in W^{2,2}(\Omega',\omega)$ and

$$\|u\|_{W^{2,2}(\Omega',\omega)} \le \mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)} \right) \tag{3.4}$$

for $\mathbf{C} = \mathbf{C}(n, C_M, \lambda, \Lambda, C_1, d')$, and $d' = \operatorname{dist}(\Omega', \partial\Omega)$.

Proof: Since $u \in W^{1,2}(\Omega,\omega)$ is a weak solution of the equation $Lu = g$, we have by

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j v(x)\, dx - \int_\Omega \sum_{i=1}^n b_i(x) D_i u(x) v(x)\, dx \\ &= \int_\Omega g(x) v(x)\, dx \end{aligned} \tag{3.5}$$

for all $v \in W_0^{1,2}(\Omega,\omega)$ (in particular for $v \in C_0^\infty(\Omega)$). Hence

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j v(x)\, dx \\ &= \int_\Omega [g(x)v(x) + \sum_{i=1}^n b_i(x) D_i u(x) v(x)]\, dx. \end{aligned}$$

In (3.5) let us replace v by $\Delta_k^{-h} v$ $(1 \le k \le n)$, with $v \in C_0^\infty(\Omega)$, $\operatorname{supp} v \subset\subset \Omega$ and let $|2h| < \operatorname{dist}(\operatorname{supp} v, \partial\Omega)$. We then obtain

$$
\begin{aligned}
&-\left(\int_\Omega [g(x) + b_i(x)D_iu(x)]\Delta_k^{-h}v(x)\,dx\right)\\
&= -\int_\Omega a_{ij}(x)D_iu(x)D_j(\Delta_k^{-h}v(x))\,dx\\
&= -\int_\Omega a_{ij}(x)D_iu(x)\Delta_k^{-h}D_jv(x),dx \qquad \text{(by Lemma 3.2(b))}\\
&= \int_\Omega \Delta_k^h(a_{ij}D_iu)(x)D_jv(x)\,dx \qquad \text{(by Lemma 3.2(a))}\\
&= \int_\Omega \Big(a_{ij}(x+he_k)\Delta_k^hD_iu(x) + D_iu(x)\Delta_k^h a_{ij}(x)\Big)D_jv(x)\,dx\\
&= \int_\Omega \Big([h\Delta_k^h a_{ij}(x) + a_{ij}(x)]\Delta_k^hD_iu(x) + D_iu(x)\Delta_k^ha_{ij}(x)\Big)D_jv(x)\,dx.
\end{aligned}
$$

Consequently,

(3.6)

$$
\begin{aligned}
&\int_\Omega a_{ij}(x)D_jv(x)\Delta_k^hD_iu(x)\,dx = -\bigg(\int_\Omega [g(x)+b_i(x)D_iu(x)]\Delta_k^{-h}v(x)\,dx\\
&\quad + \int_\Omega \Delta_k^ha_{ij}(x)D_iu(x)D_jv(x)\,dx + \int_\Omega h\Delta_k^ha_{ij}(x)\Delta_k^hD_iu(x)D_jv(x)\,dx\bigg)\\
&\le \int_\Omega |g(x)+b_i(x)D_iu(x)||\Delta_k^{-h}v(x)|\,dx + \int_\Omega |\Delta_k^ha_{ij}(x)||D_iu(x)||D_jv(x)|\,dx\\
&\quad + |h|\int_\Omega |\Delta_k^ha_{ij}(x)||\Delta_k^hD_iu(x)||D_jv(x)|\,dx\\
&= \mathrm{I} + \mathrm{II} + C_1|h|\int_\Omega \omega(x)|\Delta_k^hD_iu(x)||D_jv(x)|\,dx.
\end{aligned}
$$

Let us estimate the integrals I and II. Considering $f = g + b_iD_iu$, by (a) and (d), we have

$$
\begin{aligned}
\mathrm{I} &= \int_\Omega |f||\Delta_k^{-h}v|\,dx\\
&= \int_\Omega \left(\frac{|f|}{\omega}\right)\omega^{1/2}|\Delta_k^{-h}v|\omega^{1/2}\,dx\\
&\le \left(\int_\Omega \left(\frac{f}{\omega}\right)^2\omega\,dx\right)^{1/2}\left(\int_{\operatorname{supp}(v)} |\Delta_k^{-h}v|^2\omega\,dx\right)^{1/2}\\
&\le C_M\|f/\omega\|_{L^2(\Omega,\omega)}\left(\int_\Omega |D_kv|^2\omega\,dx\right)^{1/2} \quad \text{(by Lemma 3.5)}\\
&= C_M\Big(\|g/\omega\|_{L^2(\Omega,\omega)} + C_1\|u\|_{W^{1,2}(\Omega,\omega)}\Big)\|D_kv\|_{L^2(\Omega,\omega)}.
\end{aligned}
$$

$$\begin{aligned}
\text{II} &= \int_\Omega |\Delta_k^h a_{ij}||D_i u||D_j v|\,dx \le \int_\Omega C_1 \omega |D_i u||D_j v|\,dx \\
&= C_1 \int_\Omega |D_i u|\omega^{1/2}|D_j v|\omega^{1/2}\,dx \\
&\le C_1 \Big(\int_\Omega |D_i u|^2 \omega\,dx\Big)^{1/2} \Big(\int_\Omega |D_j v|^2 \omega dx\Big)^{1/2} \\
&\le C_1 \|u\|_{W^{1,2}(\Omega,\omega)} \|D_j v\|_{L^2(\Omega,\omega)}.
\end{aligned}$$

Replacing the estimates of I and II in (3.6), we get the estimate

$$\begin{aligned}
&\int_\Omega a_{ij}(x)\Delta_k^h D_i u(x) D_j v(x)\,dx \\
&\le C\Big(\|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}\Big)\|Dv\|_{L^2(\Omega,\omega)} \qquad (3.7)\\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||D_j v(x)|\,dx.
\end{aligned}$$

We denote by $a = \|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}$.
Let $\Omega' \subset\subset \Omega$. To proceed further let us take a function $\psi \in C_0^\infty(\Omega)$, satisfying $0 \le \psi \le 1$, $\psi \equiv 1$ in Ω' and with $\|D\psi\|_\infty \le 2/d'$, where $d' = \mathrm{dist}(\Omega', \partial\Omega)$, and set $v = \psi^2 \Delta_k^h u$ (with $|2h| < \mathrm{dist}(\operatorname{supp}\psi, \partial\Omega)$). We have

$$D_j v = (2\psi D_j \psi)\Delta_k^h u + \psi^2 D_j(\Delta_k^h u).$$

Then we obtain

$$\begin{aligned}
&\int_\Omega \Big(a_{ij}(x)\psi^2 D_j(\Delta_k^h u) D_i(\Delta_k^h u) + 2a_{ij}(x)\psi D_j\psi \Delta_k^h u \Delta_k^h D_i u\Big)\,dx \\
&\le Ca\|2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)\|_{L^2(\Omega,\omega)} \\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx \\
&\le Ca\Big(\|\psi D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + \|\psi^2 D_j(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\Big) \\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_\Omega a_{ij}(x)[\psi D_i(\Delta_k^h u)][\psi D_j(\Delta_k^h u)]\,dx \\
&\le Ca\Big(\|\psi D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + \|\psi^2 D_j(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\Big) \\
&\quad + 2\int_\Omega |a_{ij}(x)||\psi D_i \Delta_k^h u||D_j\psi\Delta_k^h u|\,dx \\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx.
\end{aligned}$$

By (1.2) we have $|a_{ij}(x)| \le C\omega(x)$, and we can estimate the integral on the right hand side by

$$\begin{aligned}
&\int_\Omega |a_{ij}(x)||\psi D_i(\Delta_k^h u)||D_j\psi\Delta_k^h u|\,dx \\
&\le C\int_\Omega |\psi D_i(\Delta_k^h u)||D_j\psi\Delta_k^h u|\omega\,dx \\
&\le C\left(\int_\Omega |\psi D_i(\Delta_k^h u)|^2\omega\,dx\right)^{1/2}\left(\int_\Omega |D_j\psi\Delta_k^h u|^2\omega\,dx\right)^{1/2} \\
&= C\|\psi D_i(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\int_\Omega a_{ij}(x)[\psi D_j(\Delta_k^h u)][\psi D_i(\Delta_k^h u)]\,dx \\
&\le Ca\|\psi D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} \\
&\quad + Ca\|\psi^2 D_j(\Delta_k^h u)\|_{L^2(\Omega,\omega)} \\
&\quad + 2C\|\psi D_i(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} \\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx.
\end{aligned}\tag{3.8}$$

Finally, the integral on the right hand side in (3.8) can be estimated

$$\begin{aligned}
&\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx \\
&\le \int_\Omega 2\omega(x)|\Delta_k^h D_i u(x)||\psi D_j\psi\Delta_k^h u| + \int_\Omega \omega(x)|\Delta_k^h D_i u||\psi^2 D_j(\Delta_k^h u)|\,dx \\
&= 2\int_\Omega \omega(x)|\psi\Delta_k^h D_i u||D_j\psi\Delta_k^h u|\,dx + \int_\Omega \omega(x)|\psi\Delta_k^h D_i u||\psi D_j(\Delta_k^h u)|\,dx \\
&\le 2\|\psi\Delta_k^h D_i u\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)} \\
&\quad + \|\psi\Delta_k^h D_i u\|_{L^2(\Omega,\omega)}\|\psi\Delta_k^h D_j u\|_{L^2(\Omega,\omega)}.
\end{aligned}$$

Applying this result in (3.8), we obtain

$$\begin{aligned}
&\int_\Omega a_{ij}(x)[\psi D_j\Delta_k^h u][\psi D_i\Delta_k^h u]\,dx \\
&\le Ca\|\psi D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)} \\
&\quad + Ca\|\psi^2 D_j\Delta_k^h u\|_{L^2(\Omega,\omega)}
\end{aligned}$$

$$
\begin{aligned}
&+ 2C\|\psi D_i\Delta_k^h u\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)}\\
&+ 2C_1|h|\|\psi\Delta_k^h D_i u\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)}\\
&+ C_1|h|\|\psi\Delta_k^h D_i u\|_{L^2(\Omega,\omega)}\|\psi\Delta_k^h D_j u\|_{L^2(\Omega,\omega)}.
\end{aligned}
$$

Consequently, by condition (1.2), we then have

$$\int_\Omega a_{ij}(x)[\psi D_j(\Delta_k^h u)][\psi D_i(\Delta_k^h u)]\,dx \ge \lambda\int_\Omega |\psi D(\Delta_k^h u)|^2\omega\,dx.$$

Denoting $b = \|\psi D(\Delta_k^h u)\|_{L^2(\Omega,\omega)}$, we have

$$
\begin{aligned}
\lambda b^2 \le{}& Ca\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + Cab + 2Cb\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)}\\
&+ 2C_1|h|b\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)} + C_1|h|b^2.
\end{aligned}
$$

Using the Young's inequality

$$AB = (\varepsilon^{-1}A)(\varepsilon B) \le \frac{1}{2}[(\varepsilon^{-1}A)^2 + (\varepsilon B)^2], \quad \forall\,\varepsilon > 0$$

to estimate ab and $b\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)}$, we obtain

$$
\begin{aligned}
\lambda b^2 \le{}& Ca\ \|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + \frac{C}{2}\varepsilon^{-2}a^2 + \frac{C}{2}\varepsilon^2 b^2\\
&+ 2C\varepsilon^2 b^2 + C\frac{\varepsilon^{-2}}{2}\|D_j\psi\Delta_k^h u\|^2_{L^2(\Omega,\omega)}\\
&+ 2C_1|h|b\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + C_1|h|b^2\\
\le{}& Ca\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)} + \frac{C\varepsilon^{-2}}{2}a^2 + \frac{C\varepsilon^2}{2}b^2\\
&+ 2C^2\varepsilon^2 b^2 + C\frac{\varepsilon^{-2}}{2}\|D_j\psi\Delta_k^h u\|^2_{L^2(\operatorname{supp}\psi,\omega)}\\
&+ C_1\varepsilon^2|h|^2 b^2 + C_1\frac{\varepsilon^{-2}}{2}\|D_j\psi\Delta_k^h u\|^2_{L^2(\operatorname{supp}\psi,\omega)} + C_1|h|b^2.
\end{aligned}
$$

Choose $\varepsilon > 0$ and h such that

$$\frac{C}{2}\varepsilon^2 + 2C\varepsilon^2 \le \lambda/4 \quad\text{and}\quad |h| < \lambda/8C_1.$$

Then

$$\left(\frac{C}{2}\varepsilon^2 + 2C\varepsilon^2 + C_1|h|^2 + C_1|h|\right) \le \frac{\lambda}{2}$$

and we can use Lemma 3.5 to get

$$\begin{aligned}\lambda b^2 &\le Ca\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)}\\ &\quad+\frac{C}{2}\varepsilon^{-2}a^2+\frac{\lambda}{2}b^2+C\varepsilon^{-2}\|D_j\psi\Delta_k^h u\|^2_{L^2(\operatorname{supp}\psi,\omega)}\\ &\le Ca\|D_j\psi\|_\infty\|\Delta_k^h u\|_{L^2(\operatorname{supp}v,\omega)}+\frac{C}{2}\varepsilon^{-2}a^2\\ &\quad+\frac{\lambda}{2}b^2+C\varepsilon^{-2}\|D_j\psi\|^2_\infty\|\Delta_k^h u\|^2_{L^2(\operatorname{supp}v,\omega)}\\ &\le Ca\|D_j\psi\|_\infty\|D_k u\|_{L^2(\Omega,\omega)}+\frac{C}{2}\varepsilon^{-2}a^2+\frac{\lambda}{2}b^2\\ &\quad+C\varepsilon^{-2}\|D_j\psi\|^2_\infty\|D_k u\|^2_{L^2(\Omega,\omega)}.\end{aligned}$$

Since $\|D_k u\|_{L^2(\Omega,\omega)}\le a$, we have

$$\begin{aligned}\frac{\lambda}{2}b^2 &\le C\|D_j\psi\|_\infty a^2+\frac{C}{2}\varepsilon^{-2}a^2+C\varepsilon^{-2}\|D_j\psi\|^2_\infty a^2\\ &\le\left(C\|D_j\psi\|_\infty+\frac{C}{2}\varepsilon^{-2}+C\varepsilon^{-2}\|D_j\psi\|^2_\infty\right)a^2\\ &=Ca^2.\end{aligned}$$

Consequently, we obtain

$$b\le\left(\frac{2C}{\lambda}\right)^{1/2}a.$$

Using $\psi\equiv 1$ in Ω', we conclude that

$$\|\Delta_k^h(Du)\|_{L^2(\Omega',\omega)}\le\mathbf{C}a,\ \forall\,k,\ 1\le k\le n,\forall\,\Omega'\subset\subset\Omega$$

with $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$ and $h<\lambda/8C_1$. By Lemma 3.6 we obtain $Du\in W^{1,2}(\Omega',\omega)$. Therefore we have that $u\in W^{2,2}(\Omega',\omega)$ and

$$\|u\|_{W^{2,2}(\Omega',\omega)}\le\mathbf{C}a=\mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega)}+\|g/\omega\|_{L^2(\Omega,\omega)}\right).$$

□

By a straightforward induction argument, we can then conclude the following extension of Theorem 3.8.

Theorem 3.9. *Let $u\in W^{1,2}(\Omega,\omega)$ be a weak solution of the equation $Lu=g$ in Ω, and assume that*

(a) *ω is a weight uniformly A_2 in each coordinate;*

(b) $g \in \mathcal{D}^k(\Omega, \omega)$, $k \in \mathbb{N}$, $k \geq 1$;
(c) *there exist* $D^\alpha a_{ij}$ *a.e. and* $|\Delta_p^h D^\alpha a_{ij}(x)| \leq C_1 \omega(x)$, $x \in \Omega' \subset\subset \Omega$, $0 \leq |\alpha| \leq k$, $1 \leq p \leq n$, $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, *with constant* C_1 *independent of* Ω' *and* h;
(d) *there exist* $D^\alpha b_i$ *a.e.*, $0 \leq |\alpha| \leq k-1$, *and* $|D^\alpha b_i(x)| \leq C_2 \omega(x)$, $x \in \Omega' \subset\subset \Omega$.

Then for any subdomain $\Omega' \subset\subset \Omega$, *we have* $u \in W^{k+2,2}(\Omega', \omega)$ *and*

$$\|u\|_{W^{k+2,2}(\Omega',\omega)} \leq \mathbf{C}\bigg(\|u\|_{W^{1,2}(\Omega,\omega)} + \sum_{0 \leq |\alpha| \leq k} \|D^\alpha g/\omega\|_{L^2(\Omega,\omega)}\bigg)$$

for $\mathbf{C} = \mathbf{C}(n, \lambda, \Lambda, C_M, C_1, C_2, d')$, *and* $d' = \operatorname{dist}(\Omega', \partial\Omega)$.

References

[1] Fabes E., Jerison D., Kenig C., *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier (Grenoble) **32** (1982), no. 3, 151–182.

[2] Fabes E., Kenig C., Serapioni R., *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.

[3] Franchi B., Serapioni R., *Pointwise estimates for a class of strongly degenerate elliptic operators: a geometrical approach*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), no. 4, 527–568.

[4] Garcia-Cuerva J., Rubio de Francia J., *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies 116, North-Holland, Amsterdam, 1985.

[5] Gilbarg D., Trudinger N., *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin-New York, 1977.

[6] Heinonen J., Kilpeläinen T., Martio O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford University Press, New York, 1993.

[7] Muckenhoupt B., *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[8] Turesson B.O., *Nonlinear potential theory and weighted Sobolev spaces*, Lecture Notes in Math. **1736**, Springer, Berlin, 2000.

State University of Londrina (Universidade Estadual de Londrina), Department of Mathematics (Departamento de Matemática), 86051-990 Londrina - PR - Brasil

E-mail: accava@gmail.com

(*Received* December 21, 2005, *revised* June 20, 2006)