The rank of the diagonal and submetrizability

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Abstract. Several topological properties lying between the submetrizability and the G_δ-diagonal property are studied. We are mostly interested in their relationship to each other and to the submetrizability. The first example of a Tychonoff space with a regular G_δ-diagonal but without a zero-set diagonal is given. The same example shows that a Tychonoff separable space with a regular G_δ-diagonal need not be submetrizable. We give a necessary and sufficient condition for submetrizability of a regular separable space. The rank 5-diagonal plays a crucial role in this criterion. Every closed bounded subset of a Tychonoff space with a G_δ-diagonal is shown to be Čech-complete. Under a slightly stronger condition, any such subset is shown to be a Moore space. We also establish that every closed bounded subset of a Tychonoff space with a regular G_δ-diagonal is metrizable by a complete metric and, therefore, has the Baire property. Some further results are obtained, and new open problems are posed.

Keywords: G_δ-diagonal, rank k-diagonal, submetrizability, condensation, regular G_δ-diagonal, zero-set diagonal, Čech-completeness, pseudocompact space, Moore space, Mrowka space, bounded subset, extent, Souslin number

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1. Introduction

Condensations are one-to-one continuous mappings onto. A space is *submetrizable* if it condenses onto a metrizable space. An important ingredient of submetrizability is the G_δ-diagonal property. Below we consider a series of properties between these two. First of all, we consider how the properties are related to each other and to the submetrizability. In particular, the first example of a Tychonoff space with a regular G_δ-diagonal that is not a zero-set diagonal is given (Example 2.9). This solves Problem 24 from [2]. The same example shows that not every Tychonoff separable space with a regular G_δ-diagonal is submetrizable. This provides an answer to Problem 16 from [2]. A necessary and sufficient condition for submetrizability of a regular separable space is given; rather unexpectedly, it turned out that the rank 5-diagonal plays a crucial role in that. Every closed bounded subset of a Tychonoff space with a G_δ-diagonal is shown to be Čech-complete, and, under a slightly stronger assumption, any such subset is shown to be a Moore space. Several new open problems are identified.

All spaces are assumed to be topological T_1-spaces. In terminology we follow [7] and [2]. If A is a subset of X and γ is a family of subsets of X, then $\mathrm{St}(A, \gamma) =$

$\bigcup\{U \in \gamma : U \cap A \neq \emptyset\}$. We also put $\mathrm{St}^0(A,\gamma) = A$ and, for a natural number n, $\mathrm{St}^{n+1}(A,\gamma) = \mathrm{St}(\mathrm{St}^n(A,\gamma),\gamma)$. If $A = \{x\}$, for some $x \in X$, then we write x instead of $\{x\}$.

A *diagonal sequence* of rank k on a space X, where $k \in \omega$, is a countable family $\{\gamma_n : n \in \omega\}$ of open coverings of X such that $\{x\} = \bigcap\{\mathrm{St}^k(x,\gamma_n) : n \in \omega\}$, for every $x \in X$. A space X has a *rank k-diagonal*, where $k \in \omega$, if there is a diagonal sequence $\{\gamma_n : n \in \omega\}$ on X of rank k. The diagonal k-sequences of open covers were introduced by T. Ishii in [11]. Ph. Zenor has dealt with the case $k = 3$ in [16], and A. Bella [4] has considered this notion for the case $k = 2$. R.F. Gittings has considered diagonal k-sequences of open covers, and some special versions of them, in the context of a classification of p-spaces he offered in [8], [9].

A space has a G_δ-diagonal if and only if it has a rank 1-diagonal [7]. The *rank* of the diagonal of X is defined as the greatest natural number n such that X has a rank n-diagonal, if such a number n exists. The rank of the diagonal of X is infinite, if X has a rank n-diagonal for every $n \in \omega$. Clearly, every submetrizable space has a diagonal of infinite rank.

Proposition 1.1. *Every Moore space X has a rank 2-diagonal.*

PROOF: Indeed, fix a development $\{\gamma_n : n \in \omega\}$ of X, and let a, b be any two distinct points of X. We have to show that $b \notin \mathrm{St}^2(a,\gamma_n)$, for some $n \in \omega$.

Assume the contrary. Then $\mathrm{St}(a,\gamma_n) \cap \mathrm{St}(b,\gamma_n) \neq \emptyset$, for each $n \in \omega$, and we can fix $x_n \in \mathrm{St}(a,\gamma_n) \cap \mathrm{St}(b,\gamma_n) \neq \emptyset$. Since the family $\{\mathrm{St}(a,\gamma_n) : n \in \omega\}$ forms a base at a, the sequence $s = \{x_n : n \in \omega\}$ converges to a. For a similar reason, s must converge to b. But this is impossible, since $a \neq b$ and the space X is Hausdorff. □

A space X has a *regular G_δ-diagonal* [16] if there is a countable family $\{U_n : n \in \omega\}$ of open neighbourhoods of the diagonal Δ_X in the square $X \times X$ such that $\Delta_X = \bigcap\{\overline{U_n} : n \in \omega\}$.

Proposition 1.2 (Ph. Zenor)**.** *If the rank of the diagonal of a space X is at least 3, then X has a regular G_δ-diagonal.*

2. The rank of the diagonal and condensations

In this section we study to what extent the rank of the diagonal is responsible for submetrizability type properties of a space. Every regular separable space with a zero-set diagonal is submetrizable [12]. In [5] further results in this direction were obtained and it was asked whether every separable space with a regular G_δ-diagonal is submetrizable as well. We answer this question in negative below, and also show that in a special case the answer is "yes".

A space X is *star-Lindelöf* if, for each open cover γ of X, there is a countable subset A of X such that $\mathrm{St}(A,\gamma) = X$. Every separable space is star-Lindelöf, and every space of the countable extent is star-Lindelöf as well.

Lemma 2.1. *Let $\{\mathcal{U}_n\}_n$ be a diagonal sequence on X of rank r. Let x, y be any distinct elements of X.*

1. *If $r \geq 2$, then there exists n such that $y \notin \mathrm{St}(z,\mathcal{U}_n)$ whenever $x \in \mathrm{St}(z,\mathcal{U}_n)$.*
2. *If $r \geq 3$ then there exists n such that $y \notin \overline{\mathrm{St}(z,\mathcal{U}_n)}$ whenever $x \in \mathrm{St}(z,\mathcal{U}_n)$.*
3. *If $r \geq 4$ then there exists n such that $\mathrm{St}(z_x,\mathcal{U}_n) \cap \mathrm{St}(z_y,\mathcal{U}_n) = \emptyset$ whenever $x \in \mathrm{St}(z_x,\mathcal{U}_n)$ and $y \in \mathrm{St}(z_y,\mathcal{U}_n)$.*
4. *If $r \geq 5$, then there exists n such that $\overline{\mathrm{St}(z_x,\mathcal{U}_n)} \cap \overline{\mathrm{St}(z_y,\mathcal{U}_n)} = \emptyset$ whenever $x \in \mathrm{St}(z_x,\mathcal{U}_n)$ and $y \in \mathrm{St}(z_y,\mathcal{U}_n)$.*

Proof: Let us prove 1. Assume the contrary. Since $r \geq 2$, $y \notin \mathrm{St}^2(x,\mathcal{U}_n)$, for some $n \in \omega$. Then there exists $z \in X$ such that $\mathrm{St}(z,\mathcal{U}_n)$ contains both x and y. Therefore, there exist $U_x \ni x, z$ and $U_y \ni y, z$ in $\mathcal{U}_n$. Clearly, U_x and U_y form a two-link path from x to y within $\mathcal{U}_n$, a contradiction.

Proof of 2: Assume the contrary. Since $r \geq 3$, there exists n such that $y \notin \mathrm{St}^3(x,\mathcal{U}_n)$. Then $x \in \mathrm{St}(z,\mathcal{U}_n)$ and $y \in \overline{\mathrm{St}(z,\mathcal{U}_n)}$, for some $z \in X$. Pick $U_y \in \mathcal{U}_n$ that contains y. Then U_y meets $\mathrm{St}(z,\mathcal{U}_n)$. Therefore, there is $U_{z,y} \in \mathcal{U}_n$ that contains z and meets U_y. Since $x \in \mathrm{St}(z,\mathcal{U}_n)$, there exists $U_{x,z} \in \mathcal{U}_n$ that contains x and z. The sets $U_{x,z}, U_{z,y}, U_y$ provide a 3-link path from x to z within $\mathcal{U}_n$, a contradiction.

The proofs of 3 and 4 are analogous to the proofs of 1 and 2. □

A space X is said to be *weakly M-normal* (*weakly normal*) if, for every closed disjoint subsets A and B of X there is a continuous mapping f from X to a metrizable space M (respectively, to a separable metrizable space M) such that $f(A) \cap f(B) = \emptyset$. Clearly, every normal space is weakly normal. On the other hand, every submetrizable space is weakly M-normal.

Theorem 2.2. *Let X be a star-Lindelöf space with a rank r-diagonal.*

1. *If $r \geq 2$ then X condenses onto a second-countable T_1-space.*
2. *If $r \geq 3$ then X condenses onto a second-countable T_2-space.*
3. *If $r \geq 5$ then X condenses onto a second-countable Urysohn space. If, in addition, X is weakly M-normal, then X is submetrizable.*

Proof: Let $\{\mathcal{U}_n\}_n$ be a diagonal sequence on X of rank r. By virtue of star-Lindelöfness, for every n we can fix a countable $X_n \subset X$ such that $X = \bigcup\{\mathrm{St}(x,\mathcal{U}_n) : x \in X_n\}$. Let $\mathcal{B}$ be the family of all $\mathrm{St}(x,\mathcal{U}_n)$'s and $X \setminus \overline{\mathrm{St}(x,\mathcal{U}_n)}$'s, where $x \in X_n$ and $n \in \omega$. Clearly, $\mathcal{B}$ is countable. Fix distinct $x, y \in X$.

To prove part 1, apply 1 of Lemma 2.1. For part 2, apply 2 of Lemma 2.1. For part 3, apply 4 of Lemma 2.1. If X is weakly normal, we can fix a countable family $\xi = \{f_n : n \in \omega\}$ of continuous mappings of X to metrizable spaces M_n so that any two elements of $\mathcal{B}$ with disjoint closures are separated by some f_n. Then the diagonal product of the mappings f_n is a continuous one-to-one mapping of X to a metrizable space $\Pi\{M_n : n \in \omega\}$. Hence, X is submetrizable. □

Corollary 2.3. *A star-Lindelöf space X is submetrizable if and only if X is weakly normal and has a rank 5-diagonal.*

Corollary 2.4. *Every separable Moore space with a regular G_δ-diagonal condenses onto a Hausdorff space with a countable base.*

Proof: Indeed, a Moore space has a rank 3-diagonal if and only if it has a regular G_δ-diagonal (Ph. Zenor, [16]). It remains to apply Theorem 2.2. □

Proposition 2.5. *Every pseudocompact subspace Y of a Hausdorff first countable space X is closed in X.*

Proof: Assume the contrary, and fix a point $a \in \overline{Y} \setminus Y$. Fix also a countable decreasing base $\{U_n : n \in \omega\}$ of X at a. Put $V_n = U_n \cap Y$ for $n \in \omega$. Then $\xi = \{V_n : n \in \omega\}$ is an infinite family of non-empty open subsets of Y such that no point of Y is an accumulation point for ξ, since X is Hausdorff and ξ converges to the point a which is not in Y. This contradicts pseudocompactness of Y. □

Theorem 2.6. *Every condensation f from a regular pseudocompact space X onto a Hausdorff first countable space Z is a homeomorphism.*

Proof: Since f is continuous, one-to-one, and onto, we only have to show that f is closed. Take any closed subset F of X. Since X is regular, $F = \bigcap\{\overline{U} : U \in \gamma_F\}$, where γ_F is the family of all open neighbourhoods of F in X. We put $\eta = \{\overline{U} : U \in \gamma_F\}$. Take any $P \in \eta$. Clearly, P is pseudocompact. Therefore, $f(P)$ is a pseudocompact subspace of Z. It follows from Proposition 2.5 that $f(P)$ is closed in Z, for every $P \in \eta$. We have $f(F) = \bigcap\{f(P) : P \in \eta\}$, since f is one-to-one. Hence, $f(F)$ is closed in Z, and the mapping f is closed. □

Corollary 2.7. *If a regular pseudocompact space X can be condensed onto a Hausdorff space with a countable base, then X is metrizable and compact.*

Proof: Indeed, it follows from Theorem 2.6 that X itself has a countable base. Therefore, X is compact and metrizable. □

Corollary 2.8. *Mrowka space Ψ does not condense onto a second-countable Hausdorff space.*

Mrowka space is a Moore space and has a rank 2-diagonal. Thus, conditions 1 and 2 in Theorem 2.2 cannot be improved in the obvious way.

Example 2.9. There exists a Tychonoff Moore space Z that is separable, non-submetrizable, and has a diagonal of the rank exactly 3. Hence, Z has a regular G_δ-diagonal.

Construction. Let S be the subset of the Euclidean plane that consists of all points on the line $y = 1$ and all points with rational coordinates that are above this line. Let S' be the subset of the Euclidean plane that consists of all points on the line $y = -1$ and all points with rational coordinates that are below this line. In short,

$$S = \{(x, y) \in R^2 : y = 1\} \cup \{(x, y) \in R^2 : x, y \text{ are rational and } y > 1\},$$
$$S' = \{(x, y) \in R^2 : y = -1\} \cup \{(x, y) \in R^2 : x, y \text{ are rational and } y < -1\}.$$

Let Q be the set of rationals in R. The underlying set for our space Z is the set of all elements p that fall in one of the following categories:

1. $p = \{(x, 1), (x, -1)\}$, where $x \in Q$;
2. $p = (x, y) \in S \cup S'$, where either $x \notin Q$ or $y \notin \{1, -1\}$.

In words, Z is obtained from $S \cup S'$ by identifying each point on the line $y = 1$ that has rational x-coordinate with the corresponding point on the line $y = -1$. Now let us topologize Z. Fix $p \in Z$. If $p = (x, y)$ and $y \notin \{1, -1\}$, then we declare p isolated. Otherwise, one of the following three cases takes place. Before we discuss each case let us agree on terminology. In all cases below a "basic triangle at q" will mean a triangle which has the sides adjacent to the vertex q of equal length and an angle at q of measure 30°. The height (or bisector) at q will be used to orient the triangle vertically or with slope -1.

Case: [$p = (x, 1)$ and $x \notin Q$]. In the half-plane above the point p draw a basic triangle at p with the height slope equal to -1.

The trace of the triangle (the boundary and interior included) on Z is a basic neighborhood at p. The length of the height at p will be called the height of the neighborhood.

Case: [$p = (x, -1)$ and $x \notin Q$]. In the half-plane below the point p draw a basic triangle with the height slope equal to -1.

As in Case 1 the trace of the triangle on Z will determine a basic neighborhood at p.

Case: [$p = \{(x, 1), (x, -1)\}$ and $x \in Q$]. Construct two basic triangles, with vertical heights (of the same length) — one above the vertex $q = (x, 1)$ and one below the vertex $q' = (x, -1)$.

The point p plus the traces of the boundary and interior of the two triangles on Z is a basic neighborhood at p. The length of the height of the upper triangle will be the height of the neighborhood. The construction of Z is complete.

The space Z is Tychonoff, since each basic neighborhood is a clopen set. The rest will be proved in the two lemmas below. Notice that Lemma 2.11 implies that Z is not submetrizable. □

Lemma 2.10. *The diagonal rank of Z is at least* 3.

PROOF: If $p \in Z$ is isolated, put $U_n(p) = \{p\}$. If p is not isolated, let $U_n(p)$ be a basic neighborhood at p such that each participating triangle has Euclidean diameter less than $1/n$. Let $\mathcal{U}_n = \{U_n(p) : p \in Z\}$. Notice that if p is not isolated then it belongs to only one element of $\mathcal{U}_n$, namely, to $U_n(p)$. Let us show that $\{\mathcal{U}_n\}_n$ has rank at least 3. Fix any two distinct points $p_1, p_2 \in Z$.

Assume $p_1 = (x,1)$ and $p_2 = (x,-1)$. Let us show that $p_1 \notin \mathrm{St}^3(p_2, \mathcal{U}_1)$. Take any $U \in \mathcal{U}_1$. We need to show that U misses $U_1(p_1)$ or $U_1(p_2)$. Recall that $U_1(p_1)$ is the point p_1 plus a triangle facing north-west above the line $y = 1$, while $U_1(p_2)$ is p_2 plus a triangle facing south-east below the line $y = -1$. The only chance for U to meet both sets is if U is a base neighborhood at $\{(q,1),(q,-1)\}$ for some $q \in Q$. Since triangles we used to define neighborhoods have small angle measures, the upper triangle of U can meet $U_1(p_1)$ only if $q < x$. For the lower triangle of U to meet $U_1(p_2)$ we need $q > x$. Consequently, U misses $U_1(p_1)$ or $U_1(p_2)$.

Now let $p_1 = (a,1)$ and $p_2 = (b,1)$. Let d be the Euclidean distance between $(a,1)$ and $(b,1)$. Pick n such that $3/n < d$. Let us show that $p_1 \notin \mathrm{St}^3(p_2, \mathcal{U}_n)$. By the definition of $\mathcal{U}_n$, $U_n(p_1)$ and $U_n(p_2)$ are triangles of diameters less than $1/n$ in the upper half-plane bounded by the line $y = 1$. Take any $U \in \mathcal{U}_n$. The portion of U that lies in the upper half-plane has diameter less than $1/n$. Since $1/n + 1/n + 1/n$ is less than the Euclidean distance between p_1 and p_2, by triangle inequality, U misses $U_n(p_1)$ or $U_n(p_2)$.

Other cases are similar to the latter case. □

Lemma 2.11. *The diagonal rank of Z is at most* 3.

PROOF: Assume the contrary, and let $\{\mathcal{U}_n\}_n$ be a diagonal sequence of rank at least 4. We may assume that each $\mathcal{U}_n$ consists of basic neighborhoods. Put $A_n = \{x \in R \setminus Q : (x,1) \notin \mathrm{St}^4((x,-1), \mathcal{U}_n)\}$. For each A_n define $A_{n,m}$ as follows: $x \in A_n$ is in $A_{n,m}$ iff there are basic neighborhoods $U(x,1), U(x,-1) \in \mathcal{U}_n$ of heights at least $1/m$ at $(x,1)$ and $(x,-1)$, respectively. Since the diagonal sequence has rank at least 4, every $x \in R \setminus Q$ is in at least one $A_{n,m}$. Therefore, there exist N and M such that $\mathrm{cl}_R(A_{N,M})$ has a non-empty interior in R.

Pick any rational q in the interior of $\mathrm{cl}_R(A_{N,M})$. Let $U(q) \in \mathcal{U}_N$ be a basic neighborhood at $\{(q,1),(q,-1)\}$. It is clear that if a big triangle is moved just a little along a straight line, then the new triangle meets the old one. Recall that all basic neighborhoods of the same height at points of the form $(x,1)$ are obtained from each other by sliding along the line $y = 1$. Therefore, we can pick distinct $a, b \in A_{N,M}$ very close to each other so that a basic neighborhood at $(a,1)$ of height at least $1/M$ meets a basic neighborhood at $(b,1)$ of height at least $1/M$. Let $U(a,1), U(b,1), U(a,-1), U(b,-1) \in \mathcal{U}_N$ be basic neighborhoods of heights at least $1/M$ at $(a,1), (b,1), (a,-1)$, and $(b,-1)$, respectively. Thus we have:

(1) $U(a,1) \cap U(b,1) \neq \emptyset$ and $U(a,-1) \cap U(b,-1) \neq \emptyset$.

Since q is in the interior of $\mathrm{cl}_R(A_{N,M})$, we can require that $a < q$ and $b > q$. We can also pick these a, b so close that

(2) $U(b, 1)$ meets the upper triangle of $U(q)$, and

(3) $U(a, -1)$ meet the lower triangle of $U(q)$.

From (1)–(3) we see that $U(a, 1), U(b, 1), U(q), U(a, -1)$ form a 4-link path from $(a, 1)$ to $(a, -1)$ within $\mathcal{U}_N$, contradicting the inclusion $a \in A_N$. □

Corollary 2.12. *There is a Tychonoff space with a regular G_δ-diagonal such that the diagonal is not a zero-set.*

Proof: By Zenor's theorem [16], any space with a rank 3-diagonal has a regular G_δ-diagonal. By H. Martin's theorem [12], any separable space with a zero-set diagonal is submetrizable. Therefore, Z is a Tychonoff space with a regular G_δ-diagonal which is not a zero-set. □

Note, that the space Z is not weakly normal.

Problem 2.13. *Is there a Tychonoff space with a rank 4-diagonal such that the diagonal is not a zero-set? Which is not a rank 5-diagonal?*

Problem 2.14 (A. Bella). *Is every regular G_δ-diagonal a rank 2-diagonal?*

Conjecture. *For every natural number n there is a Tychonoff space X_n with a rank n-diagonal that is not a rank $n+1$-diagonal.*

Observe that, for $n \geq 5$, the space X_n in the above conjecture cannot be normal. Hence, it cannot be paracompact. Can it be metacompact? Can it be subparacompact?

Recall that a space X is said to be perfect if every closed subset of X is a G_δ-set in X.

Theorem 2.15. *Let X be a normal star-Lindelöf perfect space with a rank 2-diagonal. Then X condenses onto a separable metrizable space.*

Proof: Let $\{\mathcal{U}_n\}_n$ be a diagonal sequence on X of rank at least 2. By virtue of star-Lindelöfness, for every n we can fix a countable $X_n \subset X$ such that $X = \bigcup\{\mathrm{St}(x, \mathcal{U}_n) : x \in X_n\}$. Let $\mathcal{B} = \{\mathrm{St}(x, \mathcal{U}_n) : x \in X_n \text{ and } n \in \omega\}$. Clearly, $\mathcal{B}$ is countable. Fix distinct $x, y \in X$. By 1 of Lemma 2.1, there is $W \in \mathcal{B}$ such that $x \in W$ and $y \notin W$. For each $W \in \mathcal{B}$ fix a continuous real-valued function f_W on X such that $X \setminus W = f^{-1}(0)$. We can do this, since X is normal and perfect. Clearly, the countable family $\mathcal{F} = \{f_W : W \in \mathcal{B}\}$ of continuous functions separates points of X. Hence, the diagonal product of functions in $\mathcal{F}$ is a condensation from X onto a separable metrizable space. □

Corollary 2.16. *Every star-Lindelöf normal Moore space condenses onto a separable metrizable space.*

G.M. Reed [14] proved that every separable normal Moore space is submetrizable. He has also constructed a Moore space with a regular G_δ-diagonal that is not submetrizable [14]. The two crucial properties of Reed's space were verified in [2]. A description and some further interesting properties of Reed's space are given below.

Example 2.17. Let $X = X_0 \cup X_1 \cup U$, where $X_0 = \mathbb{R} \times \{0\}$, $X_1 = \mathbb{R} \times \{-1\}$, and $U = \mathbb{R} \times (0, \infty)$. If $x = (a, 0) \in X_0$, then x' denotes the twin element $(a, -1) \in X_1$. For $n \in \omega$ and $x = (a, 0) \in X_0$ let $V_n(x) = \{x\} \cup \{(s, t) \in U : (t = s - a) \wedge (0 < t < \frac{1}{n})\}$, and $V_n(x') = \{x'\} \cup \{(s, t) \in U : (t = a - s) \wedge (0 < t < \frac{1}{n})\}$.

The topology $\mathcal{T}$ on X is such that all elements of U are isolated, and the collections $\{V_n(x) : n \in \omega, n \geq 1\}$ and $\{V_n(x') : n \in \omega, n \geq 1\}$ are bases of the topology at x and x', respectively.

Let γ be an open cover of the space X. We associate with it a subset $J(\gamma)$ of the usual space $\mathbb{R}$ of real numbers as follows. First, we define sets $J_0(\gamma)$ and $J_1(\gamma)$. Let $y \in \mathbb{R}$. Then $y \in J_0(\gamma)$ if, for some $n \in \omega$ and for some $c, d \in \mathbb{R}$, the following two conditions are satisfied:

(1) $c < y < d$, and

(2) The set of all $z \in \mathbb{R}$ such that $c < z < d$ and $V_n(z, 0)$ is contained in some element of γ is dense in the interval $[c, d]$.

Similarly, we define the set $J_1(\gamma)$ replacing in the above definition the set $V_n(z, 0)$ with the set $V_n(z, -1)$.

From the Baire property of $\mathbb{R}$ and from the definition of the topology of X it follows that $J_0(\gamma)$ and $J_1(\gamma)$ are open and dense in $\mathbb{R}$.

Now take any diagonal sequence $\xi = \{\gamma_n : n \in \omega\}$ of open covers on X. By the Baire property of the space $\mathbb{R}$, the set $K = \bigcap\{J_0(\gamma_n) \cap J_1(\gamma_n) : n \in \omega\}$ is not empty. Fix any $a \in K$, and put $x_1 = (a, 0)$ and $x_1' = (a, -1)$. Take any $k \in \omega$ and consider the sets $A = \mathrm{St}_{\gamma_k}(a)$, $B = \mathrm{St}_{\gamma_k}(A)$, and $C = \mathrm{St}_{\gamma_k}(B)$. Clearly, $V_n(x) \subset A$, for some $n \in \omega$. From $a \in J_1(\gamma_k)$ it follows that there is $c \in \mathbb{R}$ such that $c < a$ and, for some $m \in \omega$ and for some dense subset P of $[c, a]$ (in the usual topology of $\mathbb{R}$) we have $V_m(s, -1) \subset B$ for each $s \in P$.

Since $a \in J_0(\gamma_k)$, it follows from that there is $d \in \mathbb{R}$ such that $a < d$ and, for some $l \in \omega$ and for some dense subset H of $[a, d]$ (in the usual topology of $\mathbb{R}$) we have $V_l(s, 0) \subset C$ for each $s \in H$. However, the last fact immediately implies that $(a, -1) \in \overline{C}$, that is, the closure of the triple star of the point $(a, 0)$ with respect to γ_k, for each $k \in \omega$, always contains the point $(a, -1)$. Hence, the space X does not have a strong rank 3-diagonal. In fact, it is clear from the above argument that the rank of the diagonal of X is precisely 3, which implies that X is not submetrizable.

It was observed by G.M. Reed that X is a Moore space and that X is continuously symmetrizable (see the details in [2]), and therefore, X has a zero-set diagonal and a regular G_δ-diagonal. Thus, we see that *neither zero-set diagonal,* *nor the regular G_δ-diagonal imply that X has a rank 4-diagonal.* However, we do not know the answer to the following question:

Problem 2.18. *Is every rank 4-diagonal a zero-set?*

Problem 2.19. *Suppose that X is a normal space with a zero-set-diagonal. Is X submetrizable? Is the rank of the diagonal of X at least 2?*

Note, that the Reed's space X is not weakly normal.

3. Diagonal properties, bounded sets, and extent

An important ingredient of submetrizability is Dieudonné completeness (i.e. completeness with respect to the largest uniformity on X generating the topology of X). Mrowka space Ψ witnesses that a Tychonoff space may have a rank 2-diagonal without being Dieudonné complete (recall that every pseudocompact Dieudonné complete space is compact [7]). However, we do not know the answers to the following questions:

Problem 3.1. *Is every Tychonoff space with a rank 3-diagonal (with a rank 5-diagonal) Dieudonné complete? What if the rank of the diagonal is infinite?*

Problem 3.2. *Is every Tychonoff space with a rank 4-diagonal (with a zero-set-diagonal) Dieudonné complete?*

Problem 3.3. *Is every normal space with a G_δ-diagonal Dieudonné complete?*

Observe that *the spaces X and Z constructed in Section 2 are hereditarily Dieudonné complete*, since each of them obviously admits a continuous finite-to-one mapping onto a hereditarily realcompact space (see [7, 3.11.B]).

The diagonal of a space X will be called a *strong rank k-diagonal*, where $k \in \omega$, if X has a diagonal sequence $\{\gamma_n : n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\overline{\mathrm{St}^k(x, \gamma_n)} : n \in \omega\}$ for every $x \in X$. The next statement is obvious:

Proposition 3.4. *Every rank 2-diagonal is a strong rank 1-diagonal.*

On the other hand, every space with a regular G_δ-diagonal also has a strong rank 1-diagonal. This was noticed by R. Hodel [10], who introduced the concept of the strong rank 1-diagonal and was the first to show how much stronger this property is than the G_δ-diagonal property.

We study below properties of bounded subsets of regular spaces with the strong rank 1-diagonal (at least).

A subset A of a space X is said to be *bounded* in X, if every infinite collection $\{U_n : n \in \omega\}$ of open subsets of X such that $U_n \cap A \neq \emptyset$ has a point of accumulation in X. A subset A of a Tychonoff space X is bounded in X if and only if every continuous real-valued function on X is bounded on A. In any Dieudonné complete space every closed bounded subset is compact. So our interest in bounded sets is motivated by the above problems.

The next fact was established in [2]:

Proposition 3.5. *Suppose that X is a regular space with a G_δ-diagonal, and that Y is a bounded subset of X. Then the space Y is first countable.*

Theorem 3.6. *Suppose that X is a Tychonoff space with a G_δ-diagonal, and that Y is a closed bounded subset of X. Then the space Y is Čech-complete.*

PROOF: Fix a Hausdorff compactification B of X. Since X has a G_δ-diagonal, we can also fix a sequence $\{\gamma_n : n \in \omega\}$ of families γ_n of open subsets of B such that $\{x\} = \bigcap\{\mathrm{St}(x, \gamma_n) : n \in \omega\} \cap X$ for each $x \in X$.

Put $G_n = \mathrm{St}(Y, \gamma_n)$, for $n \in \omega$. Clearly, G_n is an open subset of B and $Y \subset G_n$, for any $n \in \omega$.

We claim that $\bigcap\{G_n : n \in \omega\} \cap \overline{Y} = Y$. Clearly, $Y \subset Z = \bigcap\{G_n : n \in \omega\} \cap \overline{Y}$. It remains to show that $Z \setminus Y = \emptyset$.

Assume the contrary, and fix $z \in Z \setminus Y$. Clearly, $z \in \overline{Y}$. Since $z \in G_n$, we can fix $V_n \in \gamma_n$ such that $z \in V_n$. Put $P = \bigcap\{V_n : n \in \omega\}$. If $x \in P \cap X$, then $P \cap X \subset \bigcap\{\mathrm{St}(x, \gamma_n) : n \in \omega\} \cap X$, which implies that $P \cap X$ is either empty or contains at most one point. Since $z \notin X$, it follows that we can find a zero-set F in B such that $z \in B$ and $F \cap X = \emptyset$. Fix a continuous real-valued function g on B such that $g^{-1}(0) = F$. Define a real-valued function h on X by: $h(x) = \frac{1}{g(x)}$, for each $x \in X$. Clearly, h is continuous. Notice, that h is unbounded on Y, since $z \in \overline{Y}$ and $g(z) = 0$. This contradiction shows that Y is a G_δ-set in its Hausdorff compactification $\overline{Y}$. Hence, Y is Čech-complete. □

Theorem 3.7. *Suppose that X is a regular space with a strong rank 1-diagonal. Then any bounded subset Y of X is a Moore space.*

PROOF: Take a diagonal sequence $\{\gamma_n : n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\overline{\mathrm{St}(x, \gamma_n)} : n \in \omega\}$, for every $x \in X$. Clearly, we may assume that γ_{n+1} refines γ_n for each $n \in \omega$. We are going to show that the traces of the families γ_n on Y form a development of Y. Fix $y \in Y$, and let $O(y)$ be an open neighbourhood of y in X. Since X is regular, there is an open $V \subset X$ such that $y \in V \subset \overline{V} \subset O(y)$. Consider $W_n = \mathrm{St}(y, \gamma_n) \setminus \overline{V}$. To achieve the goal, we have to show that $W_n \cap Y = \emptyset$, for some $n \in \omega$.

Assume the contrary. Then the family $\eta = \{W_n : n \in \omega\}$ accumulates to some point $a \in X$, since Y is bounded in X. Note that the family η is decreasing. It follows that a must belong to the closure of each W_n. Therefore, $a \notin V$ and hence, $a \neq y$. On the other hand, we have

$$a \in \bigcap\{\overline{W_n} : n \in \omega\} \subset \bigcap\{\overline{\mathrm{St}(y, \gamma_n)} : n \in \omega\} = \{y\},$$

which implies that $a = y$. This contradiction completes the proof. □

Theorem 3.7 should be compared to a result from [2]: *any bounded subspace of a regular space with a regular G_δ-diagonal is metrizable* which implies that every pseudocompact regular space with a regular G_δ-diagonal is metrizable and compact [13]. The result in [2] can be now strengthened as follows:

Theorem 3.8. *Any closed bounded subspace Y of a regular space X with a regular G_δ-diagonal is metrizable by a complete metric and therefore, any such Y has the Baire property.*

PROOF: By the above mentioned result from [2], Y is metrizable. By Theorem 3.6, Y is Čech-complete. It follows that Y is metrizable by a complete metric (P.S. Alexandroff, F. Hausdorff, see [7]) and that Y has the Baire property. □

Theorem 3.9. *Suppose that X is a Tychonoff space of countable extent and with a strong rank 1-diagonal. Then any bounded subspace Y of X is separable and metrizable.*

PROOF: The closure of Y in X is also bounded, therefore, we may assume that Y is closed in X. Then the extent of Y is also countable. By Theorem 3.7, Y is a Moore space. It follows that Y has a σ-discrete network. Since the extent of Y is countable, this network is, in fact, countable. By Theorem 3.6, Y is Čech-complete. It remains to refer to a theorem in [1] that every Čech-complete space with a countable network has a countable base and is, therefore, separable and metrizable. □

If we drop the assumption that the extent of X is countable, then the above conclusion is no longer true, even for separable spaces. Indeed, Mrowka space Ψ is a Tychonoff space with a strong rank 1-diagonal, Ψ is bounded in itself and is not metrizable. However, we have the following related to Theorem 3.9 result:

Theorem 3.10. *Suppose that X is a Tychonoff space with a G_δ-diagonal, and that Y is a bounded subspace of X such that the Souslin number of Y is countable. Then Y is separable.*

PROOF: By Theorem 3.6, Y is Čech-complete. By a well known result of Šapirovskij [15], Y contains a dense paracompact Čech-complete subspace Z. Clearly, Z has a G_δ-diagonal. Hence (see [7]), Z is metrizable. Since Z is dense in Y, the Souslin number of Z is also countable. Therefore, Z and Y are separable. □

Problem 3.11. *Is every bounded subset of a regular (Tychonoff) space with a regular G_δ-diagonal compact? Separable?*

Theorem 3.8 suggests that the answer to the last question might well be "yes". The above statements imply several corollaries for pseudocompact spaces.

Theorem 3.12. *Suppose that X is a Tychonoff pseudocompact space. Then the following three conditions are equivalent:*

(1) *X has a strong rank 1-diagonal;*
(2) *X is a Moore space;*
(3) *X is a separable Moore space.*

Proof: Clearly, (3) implies (2), and (2) implies (1). Now, let us assume that (1) holds. Then, by Theorem 3.7, X is a Moore space. Hence, X is perfect. Therefore, the Souslin number of X is countable (an obvious standard argument shows that the Souslin number of every regular perfect pseudocompact space is countable). Hence, by Theorem 3.10, the space X is separable. □

Corollary 3.13. *Suppose that X is a Tychonoff pseudocompact space of the countable extent and that X also has a strong rank 1-diagonal. Then X is metrizable and compact.*

On the other hand, R. Buzyakova has shown [5] that, consistently, there exists a pseudocompact Tychonoff space X of the countable extent and with a G_δ-diagonal such that X is not metrizable [5]. Hence, the condition "strong rank 1-diagonal" cannot be replaced above by the condition "G_δ-diagonal".

Corollary 3.14. *Suppose that X is a regular pseudocompact space. Then the rank $r(X)$ of the diagonal of X can take only four values: $0, 1, 2$, and ∞. More precisely, we have:*

(1) *$r(X) = 0$ if and only if X does not have a G_δ-diagonal;*
(2) *$r(X) = 1$ if and only if X has a G_δ-diagonal but is not a Moore space;*
(3) *$r(X) = 2$ if and only if X is a non-metrizable Moore space;*
(4) *$r(X) = \infty$ if and only if X is metrizable.*

It follows from Corollary 3.14 that the rank of the diagonal of any Mrowka space Ψ is precisely 2.

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