Martin boundary associated with a system of PDE

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Abstract. In this paper, we study the Martin boundary associated with a harmonic structure given by a coupled partial differential equations system. We give an integral representation for non negative harmonic functions of this structure. In particular, we obtain such results for biharmonic functions (i.e. $\Delta^2 \varphi = 0$) and for non negative solutions of the equation $\Delta^2 \varphi = \varphi$.

Keywords: Martin boundary, biharmonic functions, coupled partial differential equations

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1. Introduction

Let $D$ be a domain in $\mathbb{R}^d$, $d \geq 1$, and let $L_i$, $i = 1, 2$, be two second order elliptic differential operators on $D$ leading to harmonic spaces $(D, H_{L_i})$ with Green functions $G_i$ (see [18]). Moreover, we assume that every ball $B \subset \bar{B} \subset D$ is an $L_i$-regular set. Throughout this paper we consider two positive Radon measures $\mu_1$ and $\mu_2$ such that $K_{D}^{\mu_i} = \int_{D} G_i(\cdot,y) \mu_i(dy)$ is a bounded continuous real function on $D$, $i = 1, 2$, and

$$\|K_{D}^{\mu_1}\|_{\infty} \|K_{D}^{\mu_2}\|_{\infty} < 1.$$ 

We consider the system:

$$\begin{cases}
L_1 u = -v \mu_1, \\
L_2 v = -u \mu_2.
\end{cases}$$

(S)

Note that if $U$ is a relatively compact open subset of $D$, $\mu_1 = \lambda^d$, where $\lambda^d$ is the Lebesgue measure, $\mu_2 = 0$ and $L_1 = L_2 = \Delta$, then we obtain the classical biharmonic case on $U$. In the case when $\mu_1 = \mu_2 = \lambda^d$ and $\lambda^d(D) < \infty$, we obtain equations of type $\Delta^2 \varphi = \varphi$. In this work, we shall study the Martin boundary associated with the balayage space given by the system $(S)$ (see [7], [14] and [19]), and we shall characterize minimal points of this boundary in order to give an integral representation for non negative solutions of the system $(S)$.

Let us note that the notion of a balayage space defined by J. Bliedtner and W. Hansen in [7] is more general than that of a P-harmonic space. It covers harmonic structures given by elliptic or parabolic partial differential equations, Riesz potentials, and biharmonic equations (which are a particular case of this
work). In the biharmonic case, a similar study can be done using couples of functions as presented in [3], [5], [8], [9], [21] and [22].

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2. Notations and preliminaries

For \( j = 1, 2 \), let \( X_j = D \times \{ j \} \), and let \( X = X_1 \cup X_2 \). Moreover, let \( i_j \) and \( \pi_j \) be the mappings defined by

\[
i_j : \begin{cases} 
D \to X_j \\
x \mapsto (x, j)
\end{cases}
\quad \text{and} \quad
\pi_j : \begin{cases} 
X_j \to D \\
(x, j) \mapsto x.
\end{cases}
\]

Let \( \mathcal{U}_0 \) be the set of all balls \( B \) such that \( B \subset \bar{B} \subset D \), \( \mathcal{U}_j \) be the image of \( \mathcal{U}_0 \) by \( i_j \), \( j = 1, 2 \), and \( \mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \).

**Definition 2.1.** Let \( v \) be a measurable function on \( X \). For \( U \in \mathcal{U}_1 \), we define the kernel \( S_U \) by

\[
S_U v = (H^1_{\pi_1(U)}(v \circ i_1) \circ \pi_1) + (K^{\mu_1}_{\pi_1(U)}(v \circ i_2) \circ \pi_1).
\]

For \( U \in \mathcal{U}_2 \), we define the kernel \( S_U \) by

\[
S_U v = (H^2_{\pi_2(U)}(v \circ i_2) \circ \pi_2) + (K^{\mu_2}_{\pi_2(U)}(v \circ i_1) \circ \pi_2),
\]

where \( H^j_{\pi_j(U)} \), \( j = 1, 2 \), denote the harmonic kernels associated with \( (D, H_{L_j}) \) and

\[
K^{\mu_i}_{\pi_i(U)}(w) = \int G^{\pi_i(U)}_i(\cdot, y) w(y) \mu_i(dy) \quad i = 1, 2,
\]

where \( w \) is a measurable function on \( D \) and \( G^{\pi_i(U)}_i \) is the Green function associated with the operator \( L_i \) on \( \pi_i(U) \). Let \( G_j \), \( j = 1, 2 \), be the Green kernel associated with \( L_j \) on \( D \). The family of kernels \( (S_U)_{U \in \mathcal{U}} \) yields a balayage space on \( X \) as defined in [7] and [14].

Let \( \ast \mathcal{H}(X) \) denote the set of all hyperharmonic functions on \( X \), i.e.

\[
\ast \mathcal{H}(X) := \{ v \in \mathcal{B}(X) : v \text{ is l.s.c. and } S_U v \leq v \text{ } \forall U \in \mathcal{U} \},
\]

where \( \mathcal{B}(X) \) denotes the set of all Borel functions on \( X \). Let \( \mathcal{S}(X) \) be the set of all superharmonic functions on \( X \), i.e.

\[
\mathcal{S}(X) := \{ v \in \ast \mathcal{H}(X) : (S_U v)|_U \in C(U) \text{ } \forall U \in \mathcal{U} \},
\]

and let \( \mathcal{H}(X) \) be the set of all harmonic functions on \( X \):

\[
\mathcal{H}(X) := \{ h \in \mathcal{S}(X) : S_U h = h \text{ } \forall U \in \mathcal{U} \}.
\]
Denoting $\mathcal{W} := \ast \mathcal{H}^+(X)$, the space $(X, \mathcal{W})$ is a balayage space (see [7] and [14]).

For every positive numerical function $\varphi$ on $X$ and for every $U \in \mathcal{U}$, the reduit $R_{\varphi}^U$ is defined by

$$R_{\varphi}^U := \inf\{v \in \ast \mathcal{H}(X) : v \geq \varphi \text{ on } U\}.$$ 

Let $\hat{R}_{\varphi}^U$ be the lower semi-continuous regularization of $R_{\varphi}^U$, i.e.

$$\hat{R}_{\varphi}^U(x) := \liminf_{y \to x} R_{\varphi}^U(y), \ x \in X.$$

**Theorem 2.1.** Let $s$ be a function on $X$ such that

$$K_D^{\mu_j}(s \circ i_k) < \infty, \ j \neq k, \ j, k = 1, 2.$$ 

The following statements are equivalent.

1. $s$ is a superharmonic function on $X$.
2. $s_j := s \circ i_j - K_D^{\mu_j}(s \circ i_k), \ j \neq k, j, k \in \{1, 2\}$, are $L_j$-superharmonic on $D$.

**Proof:** Let $s$ be a superharmonic function on $X$ and let $U \in \mathcal{U}_0$. We have

$$i_1(U) \in \mathcal{U}_1 \text{ and } \pi_1(i_1(U)) = U.$$ 

Since $S_{i_1(U)}s \leq s$, we have

$$H_U^1(s \circ i_1) + K_D^{\mu_1}(s \circ i_2) \leq s \circ i_1.$$ 

Knowing that

$$K_D^{\mu_1}(s \circ i_2) = K_D^{\mu_1}(s \circ i_2) - H_U^1(K_D^{\mu_1}(s \circ i_2)),$$

we obtain

$$H_U^1(s \circ i_1) + K_D^{\mu_1}(s \circ i_2) - H_U^1(K_D^{\mu_1}(s \circ i_2)) \leq s \circ i_1.$$ 

Therefore

$$H_U^1(s \circ i_1 - K_D^{\mu_1}(s \circ i_2)) \leq s \circ i_1 - K_D^{\mu_1}(s \circ i_2).$$

So, $s_1 := s \circ i_1 - K_D^{\mu_1}(s \circ i_2)$ is an $L_1$-superharmonic function on $D$. Similarly, we prove that $s_2 := s \circ i_2 - K_D^{\mu_2}(s \circ i_1)$ is $L_2$-superharmonic on $D$. Conversely, we assume that $s_i, i = 1, 2$, are $L_i$-superharmonic functions. Let $U \in \mathcal{U}_j, j = 1, 2$ and $k \neq j$. Since $s_j$ is an $L_j$-superharmonic function,

$$H_{\pi_j(U)}^j s_j \leq s_j.$$ 

Hence

$$H_{\pi_j(U)}^j(s \circ i_j - K_D^{\mu_j}(s \circ i_k)) \leq s \circ i_j - K_D^{\mu_j}(s \circ i_k).$$

Therefore

$$H_{\pi_j(U)}^j(s \circ i_j) + K_D^{\mu_j}(s \circ i_k) \leq s \circ i_j.$$ 

So,

$$S_U s \leq s, \ \forall U \in \mathcal{U}.$$ 

Thus $s$ is superharmonic on $X$. 

\[\square\]
Corollary 2.1. Let $v$ be a function on $X$ such that $K_D^j(v \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$, is a finite function. Then the following properties are equivalent.

1. $v$ is harmonic on $X$.
2. $v \circ i_1 - K_D^1(v \circ i_2)$ and $v \circ i_2 - K_D^2(v \circ i_1)$ are $L_1$-harmonic and $L_2$-harmonic function on $D$, respectively.

Remarks 2.1. (1) Note that if $v$ is a positive harmonic function on $X$, then $K_D^j(v \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$, is a finite function.

(2) If $v \in \mathcal{H}(X)$, then the couple $(v \circ i_1, v \circ i_2)$ is a solution of $(S)$.

Corollary 2.2. Let $v$ be a positive function defined on $X$. Then the following properties are equivalent.

1. $v$ is hyperharmonic on $X$.
2. The function $v_j := \begin{cases} v \circ i_j - K_D^j(v \circ i_k) & \text{if } K_D^j(v \circ i_k) < \infty, \\ +\infty & \text{otherwise} \end{cases}$ for $j \neq k$, $j, k \in \{1, 2\}$.

If we identify a function $s$ on $X$ with the couple $(s \circ i_1, s \circ i_2)$ defined on $D$, then we get the following N. Bouleau’s decomposition [9]:

Theorem 2.2. Any superharmonic function $s$ on $X$ can be written as $s = t + Vs$, where

$$V = \begin{pmatrix} 0 & K_D^1 \\ K_D^2 & 0 \end{pmatrix}$$

and $t$ is a function on $X$ defined by

$$t := \begin{cases} s_1 \circ \pi_1 & \text{on } X_1, \\ s_2 \circ \pi_2 & \text{on } X_2, \end{cases}$$

where $s_j := s \circ i_j - K_D^j(s \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$.

Proof: It follows from Theorem 2.1 that $s_j$, $j = 1, 2$, is $L_j$-superharmonic on $D$. Then, if we identify the function $s$ with the couple $(s \circ i_1, s \circ i_2)$ defined on $D$ and the function $t$ with the couple $(t \circ i_1, t \circ i_2) = (s_1, s_2)$ defined on $D$, we have

$$\begin{pmatrix} 0 & K_D^1 \\ K_D^2 & 0 \end{pmatrix} \begin{pmatrix} s \circ i_1 \\ s \circ i_2 \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} s \circ i_1 \\ s \circ i_2 \end{pmatrix}.$$

Remark 2.1. In the classical biharmonic case, we obtain the N. Bouleau’s decomposition [9]. Indeed, if we identify a function $s$ on $X$ with the couple $(s \circ i_1, s \circ i_2)$ on $D$, then

$$s \circ i_1 = s_1 + K_D^1(s \circ i_2),$$

with $s_1$ $L_1$-superharmonic on $D$ and the N. Bouleau’s kernel $V$ is given by $V = K_D^1$. 

3. Martin boundary associated with \((S)\)

Let us fix \(x_0 \in D\) and set for all \(x, y \in D\)

\[
g^1(x, y) := \begin{cases} \frac{G_1(x, y)}{G_1(x_0, y)} & \text{if } x \neq x_0 \text{ or } y \neq x_0, \\ 1 & \text{if } x = y = x_0, \end{cases}
\]

and

\[
g^2(x, y) := \begin{cases} \frac{G_2(x, y)}{G_2(x_0, y)} & \text{if } x \neq x_0 \text{ or } y \neq x_0, \\ 1 & \text{if } x = y = x_0. \end{cases}
\]

Let \(A_1 = \{g^1(x, \cdot), x \in D\}, A_2 = \{g^2(x, \cdot), x \in D\}\) and \(A = A_1 \cup A_2\).

As in [10] and [12], we consider the Martin compactification \(\hat{D}\) of \(D\) associated with \(A\). The boundary \(\Delta = \hat{D} \setminus D\) of \(D\) is called the Martin boundary of \(D\) associated with the system \((S)\).

The function \(g^k(x, \cdot), k = 1, 2, x \in D\) can be extended, on \(\hat{D}\), to a continuous function denoted \(g^k(x, \cdot), k = 1, 2, x \in D\) as well.

In the following, we denote \(Q := \sum_{n=0}^{+\infty} (K^\mu_1 K^\mu_2 D)^n\) (resp. \(T := \sum_{n=0}^{+\infty} (K^\mu_2 K^\mu_1 D)^n\)) which coincides with \((I - K^\mu_1 K^\mu_2 D)^{-1}\) (resp. \((I - K^\mu_2 K^\mu_1 D)^{-1}\)) on \(B_b(D)\), where \((I - K^\mu_1 K^\mu_2 D)^{-1}\) (resp. \((I - K^\mu_2 K^\mu_1 D)^{-1}\)) is the inverse of the operator \((I - K^\mu_1 K^\mu_2 D)\) (resp. \((I - K^\mu_2 K^\mu_1 D)\)) on \(B_b(D)\), and \(B_b(D)\) denotes the set of all bounded Borel measurable functions on \(D\). We recall the following equalities

\[
(K^\mu_1 K^\mu_2 D)Q = Q(K^\mu_1 K^\mu_2 D),
\]

\[
(K^\mu_1 K^\mu_2 D)Q + I = Q.
\]

Similarly we have

\[
(K^\mu_2 K^\mu_1 D)T = T(K^\mu_2 K^\mu_1 D),
\]

\[
(K^\mu_2 K^\mu_1 D)T + I = T,
\]

\[
K^\mu_2 D Q = T K^\mu_2 D
\]

and

\[
K^\mu_1 D T = Q K^\mu_1 D.
\]

**Remark 3.1.** Note that if \(\varphi\) is a finite positive Borel measurable function on \(D\) such that \(K^\mu_1 K^\mu_2 D \varphi\) is bounded, then \(Q \varphi < +\infty\).
Theorem 3.1. Let \( t_i, i = 1, 2 \), be two \( L_i \)-harmonic functions on \( D \) such that \( K_D^{i_j} t_k \) is finite and \( K_D^{i_k} K_D^{i_j} t_k \) is bounded, \( j \neq k, j, k \in \{1, 2\} \), on \( D \). Then the functions \( v \) and \( w \) defined on \( X \) by

\[
v := \begin{cases} (Qt_1) \circ \pi_1 & \text{on } X_1, \\ (K_D^{i_2} Qt_1) \circ \pi_2 & \text{on } X_2, \end{cases}
\]

and

\[
w := \begin{cases} (QK_D^{i_1} t_2) \circ \pi_1 & \text{on } X_1, \\ (T t_2) \circ \pi_2 & \text{on } X_2 \end{cases}
\]

are harmonic on \( X \).

Remark 3.2. In the biharmonic case, if we assume that \( K_D^d t_2 < \infty \), then \((t_1, 0)\) and \((K_D^d t_2, t_2)\) are biharmonic.

Proof: Let us prove first that \( v \) and \( w \) are finite.

(i) We have

\[(Qt_1) \circ \pi_1 = (QK_D^{\mu_1} K_D^{\mu_2} t_1) \circ \pi_1 + t_1 \circ \pi_1.\]

Since \( K_D^{\mu_1} K_D^{\mu_2} t_1 \) is bounded and \( t_1 \) is finite,

\[(Qt_1) \circ \pi_1 < \infty.\]

(ii) We have also

\[(K_D^{\mu_2} Qt_1) \circ \pi_2 = (TK_D^{\mu_2} t_1) \circ \pi_2,

hence

\[(K_D^{\mu_2} Qt_1) \circ \pi_2 = (TK_D^{\mu_2} K_D^{\mu_1} K_D^{\mu_2} t_1) \circ \pi_2 + (K_D^{\mu_2} t_1) \circ \pi_2.\]

Since \( K_D^{\mu_1} K_D^{\mu_2} t_1 \) is bounded and \( K_D^{\mu_2} t_1 \) is finite,

\[(K_D^{\mu_2} Qt_1) \circ \pi_2 < \infty.\]

(iii) We have

\[(QK_D^{\mu_1} t_2) \circ \pi_1 = (QK_D^{\mu_1} K_D^{\mu_2} K_D^{\mu_1} t_2) \circ \pi_1 + (K_D^{\mu_1} t_2) \circ \pi_1.\]

Knowing that \( K_D^{\mu_2} K_D^{\mu_1} t_2 \) is bounded and \( K_D^{\mu_1} t_2 \) is finite, we have

\[(QK_D^{\mu_1} t_2) \circ \pi_1 < \infty.\]

(iv) We have

\[(T t_2) \circ \pi_2 = (TK_D^{\mu_2} K_D^{\mu_1} t_2) \circ \pi_2 + t_2 \circ \pi_2.\]
Since $K_D^{\mu_2}K_D^{\mu_1}t_2$ is bounded and $t_2$ is finite,

$$(Tt_2) \circ \pi_2 < \infty.$$ 

Let us show now that $v$ and $w$ are harmonic. From Corollary 2.1, it suffices to show that $v \circ i_j - K_D^{\mu_j}(v \circ i_k)$ and $w \circ i_j - K_D^{\mu_j}(w \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$, are $L_j$-harmonic functions on $D$.

(v) On the one hand,

$$v \circ i_1 - K_D^{\mu_1}(v \circ i_2) = Qt_1 - (K_D^{\mu_1}K_D^{\mu_2})Qt_1.$$ 

As

$$Qt_1 = (K_D^{\mu_1}K_D^{\mu_2})Qt_1 + t_1,$$

we get

$$v \circ i_1 - K_D^{\mu_1}(v \circ i_2) = t_1.$$ 

Since $t_1$ is an $L_1$-harmonic function on $D$, $v \circ i_1 - K_D^{\mu_1}(v \circ i_2)$ is $L_1$-harmonic on $D$.

On the other hand,

$$v \circ i_2 - K_D^{\mu_2}(v \circ i_1) = K_D^{\mu_2}Qt_1 - K_D^{\mu_2}Qt_1 = 0,$$

i.e. $v \circ i_2 - K_D^{\mu_2}(v \circ i_1)$ is $L_2$-harmonic on $D$. Then we conclude that $v$ is harmonic on $X$.

(vi) Since

(*)

$$T = K_D^{\mu_2}QK_D^{\mu_1} + I,$$

we have

$$w \circ i_1 - K_D^{\mu_1}(w \circ i_2) = (QK_D^{\mu_1} - K_D^{\mu_1}K_D^{\mu_2}QK_D^{\mu_1} - K_D^{\mu_1})t_2.$$ 

As

$$Q = (K_D^{\mu_1}K_D^{\mu_2})Q + I,$$

we obtain

$$w \circ i_1 - K_D^{\mu_1}(w \circ i_2) = 0.$$ 

Using (*), we have

$$w \circ i_2 - K_D^{\mu_2}(w \circ i_1) = (K_D^{\mu_2}QK_D^{\mu_1} + I - K_D^{\mu_2}QK_D^{\mu_1})t_2 = t_2.$$ 

Then $w \circ i_j - K_D^{\mu_j}(w \circ i_k)$ is $L_j$-harmonic on $D$ and therefore, $w$ is a harmonic function on $X$. \qed
Corollary 3.1. Let $t_i$, $i = 1, 2$, be two positive $L_i$-hyperharmonic functions on $D$. Then the functions $v$ and $w$ defined on $D$ by

$$v := \begin{cases} (Q t_1) \circ \pi_1 & \text{on } X_1, \\ (K_D^\mu_1 Q t_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$w := \begin{cases} (Q K_D^\mu_1 t_2) \circ \pi_1 & \text{on } X_1, \\ (T t_2) \circ \pi_2 & \text{on } X_2 \end{cases}$$

are hyperharmonic on $X$.

Theorem 3.2. Let $\nu_1$ and $\nu_2$ be two positive Radon measures on $\triangle$ such that

$$\int_\triangle K_D^\mu_j g^k(\cdot, y) \, d\nu_k(y) < \infty$$

and

$$\int_\triangle K_D^\mu_j K_D^\mu_k g^j(\cdot, y) \, d\nu_j(y)$$

is bounded on $D$, $j \neq k$, $j, k \in \{1, 2\}$. Then the function $v$ defined on $X_1$ by

$$v := \int_\triangle (Q g^1(\cdot, y)) \circ \pi_1 \, d\nu_1(y) + \int_\triangle (Q K_D^\mu_1 g^2(\cdot, y)) \circ \pi_1 \, d\nu_2(y)$$

and on $X_2$ by

$$v := \int_\triangle (K_D^\mu_2 Q g^1(\cdot, y)) \circ \pi_2 \, d\nu_1(y) + \int_\triangle (T g^2(\cdot, y)) \circ \pi_2 \, d\nu_2(y)$$

is harmonic on $X$.

Proof: It suffices to replace the functions $t_j$ from Theorem 3.1 with the $L_j$-harmonic functions $\int_\triangle g^j(\cdot, y) \, d\nu_j(y)$. \hfill \square

Corollary 3.2. Let $\nu_1$ and $\nu_2$ be two positive Radon measures on $\triangle$ such that

$$\int_\triangle K_D^\mu_1 g^2(\cdot, y) \, d\nu_2(y) < \infty.$$ Then

$$(v, w) = \left( \int_\triangle g^1(\cdot, y) \, d\nu_1(y) + \int_\triangle K_D^\mu_1 g^2(\cdot, y) \, d\nu_2(y), \int_\triangle g^2(\cdot, y) \, d\nu_2(y) \right)$$

is a biharmonic couple in the classical sense.
Theorem 3.3. Let \( v \) be a positive harmonic function on \( X \) such that \( K_D^{\mu_j} K_D^{\mu_k}(v \circ i_j) \) is bounded on \( D \), \( j, k \in \{1, 2\}, j \neq k \). Then there exist two positive Radon measures \( \nu_1 \) and \( \nu_2 \) supported by \( \triangle \) such that \( v \) can be represented on \( X_1 \) by

\[
v = \int_\triangle (Qg^1(\cdot, y)) \circ \pi_1 \, d\nu_1(y) + \int_\triangle (QK_D^{\mu_1} g^2(\cdot, y)) \circ \pi_1 \, d\nu_2(y)
\]

and on \( X_2 \) by

\[
v = \int_\triangle (K_D^{\mu_2} Qg^1(\cdot, y)) \circ \pi_2 \, d\nu_1(y) + \int_\triangle (Tg^2(\cdot, y)) \circ \pi_2 \, d\nu_2(y).
\]

**Proof:** Let \((D_n)_n\) be an increasing sequence of relatively compact open subsets of \( D \) such that \( D = \bigcup D_n \), and let \( v \) be a positive harmonic function on \( X \). From Corollary 2.1, the positive functions \( v \circ i_1 - K_D^{\mu_1}(v \circ i_1) \) and \( v \circ i_2 - K_D^{\mu_2}(v \circ i_1) \) are \( L_1 \)-harmonic and \( L_2 \)-harmonic on \( D \), respectively. Then for all \( n \in \mathbb{N} \), both \( \widehat{R}_{v \circ i_1 - K_D^{\mu_1}(v \circ i_2)}^{D_n} \) and \( \widehat{R}_{v \circ i_2 - K_D^{\mu_2}(v \circ i_1)}^{D_n} \) are \( L_1 \)-potential and \( L_2 \)-potential on \( D \), respectively. Therefore, there exist two positive Radon measures \( \mu_1^n \) and \( \mu_2^n \) on \( D \) such that

\[
\widehat{R}_{v \circ i_1 - K_D^{\mu_1}(v \circ i_2)}^{D_n} = \int_D G_1(\cdot, y) \, d\mu_1^n(y)
\]

and

\[
\widehat{R}_{v \circ i_2 - K_D^{\mu_2}(v \circ i_1)}^{D_n} = \int_D G_2(\cdot, y) \, d\mu_2^n(y).
\]

Then we have

\[
\widehat{R}_{v \circ i_1 - K_D^{\mu_1}(v \circ i_2)}^{D_n} = \int_D g^1(\cdot, y) \, d\nu_1^n(y)
\]

and

\[
\widehat{R}_{v \circ i_2 - K_D^{\mu_2}(v \circ i_1)}^{D_n} = \int_D g^2(\cdot, y) \, d\nu_2^n(y)
\]

with

\[
d\nu_1(y) = G_1(x_0, \cdot) d\mu_1^n(y)
\]

and

\[
d\nu_2(y) = G_2(x_0, \cdot) d\mu_2^n(y).
\]

Since \( \widehat{R}_{v \circ i_j - K_D^{\nu_j}(v \circ i_k)}^{D_n} \) is \( L_j \)-harmonic on \( D \setminus D_n \), \( j \neq k \), \( j, k \in \{1, 2\} \), \( \nu_1^n \) and \( \nu_2^n \) are necessarily supported by \( D \setminus D_n \).

Because of \( \|\nu^n_1\| \leq (v \circ i_j)(x_0) - K_D^{\mu_j}(v \circ i_k)(x_0), j = 1, 2 \), we may extract two subsequences \( (\nu_{p(n)}^1) \) and \( (\nu_{p(n)}^2) \) converging vaguely to two positive Radon measures \( \nu^1 \) and \( \nu^2 \) on \( \widehat{D} = \widehat{D} \). So, \( \nu^1 \) and \( \nu^2 \) are supported by \( \triangle \). Therefore

\[
\begin{cases}
\nu \circ i_1 - K_D^{\mu_1}(v \circ i_2) = \int_\triangle g^1(\cdot, y) \, d\nu^1(y), \\
\nu \circ i_2 - K_D^{\mu_2}(v \circ i_1) = \int_\triangle g^2(\cdot, y) \, d\nu^2(y).
\end{cases}
\]
Since we obtain
\[
\begin{align*}
\{ v \circ i_1 & = \int_\triangle g^1(\cdot, y) \, d\nu^1(y) + K_D^{\mu_1} \left( \int_\triangle g^2(\cdot, y) \, d\nu^2(y) + K_D^{\mu_2}(v \circ i_1) \right), \\
v \circ i_2 & = \int_\triangle g^2(\cdot, y) \, d\nu^2(y) + K_D^{\mu_2}(v \circ i_1),
\end{align*}
\]
and
\[
\begin{align*}
\{ v \circ i_1 & = \int_\triangle g^1(\cdot, y) \, d\nu^1(y) + \int_\triangle K_D^{\mu_1} g^2(\cdot, y) \, d\nu^2(y) + K_D^{\mu_1} K_D^{\mu_2}(v \circ i_1), \\
v \circ i_2 & = \int_\triangle g^2(\cdot, y) \, d\nu^2(y) + K_D^{\mu_2}(v \circ i_1).
\end{align*}
\]

Thus,
\[
\begin{align*}
\{ Q(v \circ i_1) & = \int_\triangle Qg^1(\cdot, y) \, d\nu^1(y) + \int_\triangle QK_D^{\mu_1} g^2(\cdot, y) \, d\nu^2(y) \\
& \quad + QK_D^{\mu_1} K_D^{\mu_2}(v \circ i_1), \\
v \circ i_2 & = \int_\triangle g^2(\cdot, y) \, d\nu^2(y) + K_D^{\mu_2}(v \circ i_1).
\end{align*}
\]

Since
\[
QK_D^{\mu_1} K_D^{\mu_2} + I = Q,
\]
we obtain
\[
\begin{align*}
\{ K_D^{\mu_1} K_D^{\mu_2} Q(v \circ i_1) + v \circ i_1 & = \int_\triangle Qg^1(\cdot, y) \, d\nu^1(y) + \int_\triangle QK_D^{\mu_1} g^2(\cdot, y) \, d\nu^2(y) \\
& \quad + QK_D^{\mu_1} K_D^{\mu_2}(v \circ i_1), \\
v \circ i_2 & = \int_\triangle g^2(\cdot, y) \, d\nu^2(y) + K_D^{\mu_2}(v \circ i_1).
\end{align*}
\]

Since \( K_D^{\mu_1} K_D^{\mu_2}(v \circ i_1) \) is bounded,
\[
\begin{align*}
\{ v \circ i_1 & = \int_\triangle Qg^1(\cdot, y) \, d\nu_1(y) + \int_\triangle QK_D^{\mu_1} g^2(\cdot, y) \, d\nu_2(y), \\
v \circ i_2 & = \int_\triangle K_D^{\mu_2} Qg^1(\cdot, y) \, d\nu_1(y) + \int_\triangle Tg^2(\cdot, y) \, d\nu_2(y).
\end{align*}
\]

So the function \( v \) can be written on \( X_1 \) as
\[
v = \int_\triangle (Qg^1(\cdot, y)) \circ \pi_1 \, d\nu_1(y) + \int_\triangle (QK_D^{\mu_1} g^2(\cdot, y)) \circ \pi_1 \, d\nu_2(y)
\]
and on \( X_2 \) as
\[
v = \int_\triangle (K_D^{\mu_2} Qg^1(\cdot, y)) \circ \pi_2 \, d\nu_1(y) + \int_\triangle (Tg^2(\cdot, y)) \circ \pi_2 \, d\nu_2(y).
\]

\[\square\]

**Corollary 3.3** ([5]). Let \((v, w)\) be a positive biharmonic couple in the classical sense. Then there exist two positive Radon measures \(\mu\) and \(\nu\) supported by \(\triangle\) such that
\[
\begin{align*}
\{ v & = \int_\triangle g^1(\cdot, y) \, d\mu(y) + \int_\triangle K_D^{\mu_1} g^2(\cdot, y) \, d\nu(y), \\
w & = \int_\triangle g^2(\cdot, y) \, d\nu(y).
\end{align*}
\]
4. Minimal points and uniqueness of the integral representation

**Definition 4.1.** (1) A positive $L_1$-harmonic (resp. $L_2$-harmonic) function $h$ on $D$ is called $L_1$-minimal (resp. $L_2$-minimal) if for any positive $L_1$-harmonic (resp. $L_2$-harmonic) function $u$ on $D$, $u \leq h$ implies $u = \alpha h$ with a factor $\alpha > 0$.

(2) A positive harmonic function $h$ on $X$ is called minimal if for any positive harmonic function $u$ on $X$, $u \leq h$ implies $u = \alpha h$ with a factor $\alpha > 0$.

Denote

$$\triangle_1 = \{y \in \triangle : g^1(\cdot, y) \text{ is } L_1\text{-minimal}\},$$

$$\triangle_2 = \{y \in \triangle : g^2(\cdot, y) \text{ is } L_2\text{-minimal}\}.$$ 

Note that for all $y \in \triangle$, the function $g^1(\cdot, y)$ (resp. $g^2(\cdot, y)$) is $L_1$-harmonic (resp. $L_2$-harmonic) on $D$.

**Proposition 4.1.** Any positive harmonic function $v$ on $X$ such that $K_{D}^\mu_k K_{D}^\mu_j (v \circ i_k)$ is bounded for all $j \neq k, j, k \in \{1, 2\}$, can be written as $v = w + s$, where $w$ and $s$ are defined by

$$w := \begin{cases} (Qv_1) \circ \pi_1 & \text{on } X_1, \\ (K_{D}^\mu_2 Qv_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$s := \begin{cases} (QK_{D}^\mu_1 v_2) \circ \pi_1 & \text{on } X_1, \\ (Tv_2) \circ \pi_2 & \text{on } X_2, \end{cases}$$

with $v_j := v \circ i_j - K_{D}^\mu_j (v \circ i_k)$, $j \neq k, j, k \in \{1, 2\}$.

**Remark 4.1.** (1) Note that if $v = w' + s'$ is another decomposition of $v$ with

$$w' := \begin{cases} (Qt_1) \circ \pi_1 & \text{on } X_1, \\ (K_{D}^\mu_2 Qt_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$s' := \begin{cases} (QK_{D}^\mu_1 t_2) \circ \pi_1 & \text{on } X_1, \\ (Tt_2) \circ \pi_2 & \text{on } X_2, \end{cases}$$

where $t_j, j = 1, 2$, are $L_j$-harmonic on $D$, then $t_1 = v_1$ and $t_2 = v_2$.

(2) In the classical case, for any biharmonic couple $(h_1, h_2)$ the following holds:

$$(h_1, h_2) = (t, 0) + (K_{D}^\mu_1 h_2, h_2),$$

where $t$ is a harmonic function on $D$. Note that $(K_{D}^\mu_1 h_2, h_2)$ is a pure biharmonic couple (see [3] and [21], [22]).
Corollary 4.1. Let $v$ be a positive minimal harmonic function on $X$ such that $K_D^{\mu_k}K_D^{\mu_j}(v \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$, is bounded. Then $v = \alpha w$ or $v = \beta s$, where $\alpha$ and $\beta$ are positive constants; $w$ and $s$ are defined as in Proposition 4.1.

Proposition 4.2. Let $v$ be a positive function on $X$ such that $K_D^{\mu_j}(v \circ i_k)$ is finite and $K_D^{\mu_k}K_D^{\mu_j}(v \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$, is bounded. The following statements are equivalent.

1. $v$ is a minimal harmonic function on $X$.
2. $v_1$ is a positive minimal $L_1$-harmonic function on $D$, or $v_2$ is a positive minimal $L_2$-harmonic function on $D$, where $v_j := v \circ i_j - K_D^{\mu_j}(v \circ i_k)$.

Proof: Let $v$ be a positive minimal harmonic function on $X$. Then we have $v = \alpha w$ or $v = \beta s$ by Corollary 4.1.

We shall show that if $v = \alpha w$, then $v_1$ is $L_1$-minimal and if $v = \beta s$, then $v_2$ is $L_2$-minimal.

(i) Case $v = \alpha w$:
Suppose that $v_1$ is not $L_1$-minimal. Then there exist two $L_1$-harmonic functions $u_1$ and $u_2$ such that $v_1 = u_1 + u_2$. So $v = \alpha f_1 + \alpha f_2$, with

$$f_1 = \begin{cases} (Qu_1) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2}Qu_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$f_2 = \begin{cases} (Qu_2) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2}Qu_2) \circ \pi_2 & \text{on } X_2. \end{cases}$$

It follows from Theorem 3.1 that $f_1$ and $f_2$ are harmonic on $X$. This contradicts that $v$ is minimal.

(ii) Case $v = \beta s$:
Suppose that $v_2$ is not $L_2$-minimal. Then there exist two $L_2$-harmonic functions $u_1$ and $u_2$ such that $v_2 = u_1 + u_2$. Therefore $v = \beta s_1 + \beta s_2$, with

$$s_1 = \begin{cases} (QK_D^{\mu_1}u_1) \circ \pi_1 & \text{on } X_1, \\ (Tu_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$s_2 = \begin{cases} (QK_D^{\mu_1}u_2) \circ \pi_1 & \text{on } X_1, \\ (Tu_2) \circ \pi_2 & \text{on } X_2. \end{cases}$$

It follows from Theorem 3.1 that $s_1$ and $s_2$ are harmonic on $X$. This contradicts that $v$ is minimal.

Conversely, suppose that $v_1$ is $L_1$-minimal and let us show that $v$ is minimal. Assume the contrary and put $v = g_1 + g_2$, where $g_1$ and $g_2$ are harmonic functions
on $X$. Then, from Proposition 4.1, there exist two $L_1$-harmonic functions $s_1$ and $s_2$, and two $L_2$-harmonic functions $w_1$ and $w_2$ such that

$$g_1 = \begin{cases} (Qs_1) \circ \pi_1 + (QK_{\mu_1}D w_1) \circ \pi_1 & \text{on } X_1, \\ (K_{\mu_2}^D Qs_1) \circ \pi_2 + (Tw_1) \circ \pi_2 & \text{on } X_2, \end{cases}$$

and

$$g_2 = \begin{cases} (Qs_2) \circ \pi_1 + (QK_{\mu_1}D w_2) \circ \pi_1 & \text{on } X_1, \\ (K_{\mu_2}^D Qs_2) \circ \pi_2 + (Tw_2) \circ \pi_2 & \text{on } X_2. \end{cases}$$

Therefore the function $g_1 + g_2$ is defined on $X_1$ by

$$g_1 + g_2 := (Q(s_1 + s_2)) \circ \pi_1 + (QK_{\mu_1}^D (w_1 + w_2)) \circ \pi_1$$

and on $X_2$ by

$$g_1 + g_2 := (K_{\mu_2}^D Q(s_1 + s_2)) \circ \pi_2 + (T(w_1 + w_2)) \circ \pi_2.$$

We deduce, from Proposition 4.1 and Remark 4.1.1, that $v_1 = s_1 + s_2$, which leads to a contradiction because $v_1$ is $L_1$-minimal.

In the same way, we suppose that $v_2$ is an $L_2$-minimal function and we show that $v$ is a minimal function. □

By using the fact that any positive minimal $L_j$-harmonic function on $D$ is proportional to $g^j(\cdot, y)$, $y \in \Delta_j$ (see [10]), $w$ and $s$ from Corollary 4.1 can be given more precisely.

**Corollary 4.2.** Let $v$ be a positive minimal harmonic function defined on $X$ such that the function $K_{\mu_k}^D K_{\mu_j}^D (v \circ i_k)$, $j \neq k$, $j, k \in \{1, 2\}$, is bounded. Then

$$v = \alpha w \quad \text{or} \quad v = \beta s,$$

with

$$w := \begin{cases} (Qg^1(\cdot, y)) \circ \pi_1 & \text{on } X_1, y \in \Delta_1, \\ (K_{\mu_2}^D Qg^1(\cdot, y)) \circ \pi_2 & \text{on } X_2, y \in \Delta_1, \end{cases}$$

and

$$s := \begin{cases} (QK_{\mu_1}^D g^2(\cdot, y)) \circ \pi_1 & \text{on } X_1, y \in \Delta_2, \\ (Tg^2(\cdot, y)) \circ \pi_2, & \text{on } X_2, y \in \Delta_2. \end{cases}$$

**Proof:** This result follows immediately from Proposition 4.2 and Corollary 4.1. □
Remark 4.2. Note that $K_D^{ij}(v \circ i_k) < \infty$, $j \neq k$, $j, k \in \{1, 2\}$, because $v$ is a positive harmonic function on $X$.

Consider the family of mappings on the real vector space $\mathcal{H}(X)$ defined by

$$\varphi_K : \{ \mathcal{H}(X) \rightarrow \mathbb{R}^+, h \mapsto \varphi_K(h) \},$$

where

$$\varphi_K(h) = \sup_{x \in K} (|h \circ i_1(x)| + |h \circ i_2(x)|),$$

and $K$ is a compact subset of $D$. $(\varphi_K)$ is a family of semi-norms on $\mathcal{H}(X)$ and these semi-norms define a topology that makes $\mathcal{H}(X)$ a metrizable topological space. It follows that this space is locally convex.

The cone $\mathcal{H}^+(X) = \{ h \in \mathcal{H}(X) : h \geq 0 \}$ defines on $\mathcal{H}(X)$ an order relation called specific order:

$$h_1 \prec h_2 \iff h_2 = h_1 + g, \quad g \in \mathcal{H}^+(X).$$

Equipped with this order, $\mathcal{H}^+(X)$ is a lattice. The minimal harmonic functions are the points of the extreme generatrices of $\mathcal{H}^+(X)$. We recall that a base of $\mathcal{H}^+(X)$ is the intersection of $\mathcal{H}^+(X)$ with a closed hyperplane.

Let us consider the set

$$B := \{ h \in \mathcal{H}^+(X) : (h \circ i_1)(x_0) + (h \circ i_2)(x_0) = 1 \}, \quad x_0 \in D.$$ 

$B$ is a compact base of the cone $\mathcal{H}^+(X)$. Indeed, the mapping

$$\varphi_{x_0} : \{ \mathcal{H}^+(X) \rightarrow \mathbb{R}, h \mapsto (h \circ i_1)(x_0) + (h \circ i_2)(x_0) = 1 \}$$

is a continuous linear form. Then it defines a closed hyperplane $B$ such that the origin $0 \notin B$. Then, $B$ is equicontinuous at any point $x \in X$. So, we conclude, by Ascoli’s theorem, that $B$ is compact. Note that $\mathcal{H}^+(X) = \mathbb{R}^+ B$. Let $\mathcal{E}(B)$ denote the set of all extreme points of $\mathcal{H}^+(X)$ belonging to $B$ (see [11]). Moreover, using Corollary 4.2, we have

$$\mathcal{E}(B) = \mathcal{E}_1(B) \cup \mathcal{E}_2(B),$$

where

$$\mathcal{E}_1(B) = \left\{ h \in \mathcal{E}(B) : \exists \alpha \in \mathbb{R}^+, \exists y \in \triangle_1 : h = \begin{cases} (\alpha Q g^1(\cdot, y)) \circ \pi_1 & \text{on } X_1 \\ (\alpha K_D^{h_2} g^1(\cdot, y)) \circ \pi_2 & \text{on } X_2 \end{cases} \right\}$$

and

$$\mathcal{E}_2(B) = \left\{ h \in \mathcal{E}(B) : \exists \beta \in \mathbb{R}^+, \exists y \in \triangle_2 : h = \begin{cases} (\beta Q K_D^{h_1} g^2(\cdot, y)) \circ \pi_1 & \text{on } X_1 \\ (\beta T g^2(\cdot, y)) \circ \pi_2 & \text{on } X_2 \end{cases} \right\}.$$ 

We recall the following results which are useful for showing the uniqueness of an integral representation (see [16]).
Definition 4.2 ([16]). Let Γ a closed convex cone. A mapping \( \ell : \lambda \mapsto e_\lambda \) of a separated topological space \( \Omega \) in \( \mathcal{E}(\Gamma) \) is called a parametrization of \( \mathcal{E}(\Gamma) \), if any element \( \gamma \in \mathcal{E}(\Gamma) \) is proportional to a unique element \( e_\lambda \). It is called admissible if it is continuous and the inverse mapping \( \mathcal{E}(\Gamma) \longrightarrow \Omega \) is universally measurable.

Theorem A ([16]). Let a closed cone convex \( \Gamma \) and an admissible parametrization \( \ell \) of \( \mathcal{E}(\Gamma) \) be given. For any \( \gamma \in \Gamma \), there exist a positive Radon measure \( \mu \) on \( \Omega \) such that
\[
\gamma = \int_\Omega e_\lambda d\mu(\lambda).
\]

Theorem B ([16]). The measure \( \mu \) given by Theorem A is unique for any \( \gamma \in \Gamma \), if and only if the cone \( \Gamma \) is a lattice.

Theorem 4.1. If \( g_1(x, \cdot) \), \( x \in D \), separates \( \Delta_1 \) and \( g_2(x, \cdot) \), \( x \in D \), separates \( \Delta_2 \), then for any positive harmonic function \( v \) on \( X \) such that the function \( K_{\mu_k}^{\mu_j}(v \circ i_k) \), \( j \neq k, j, k \in \{1, 2\} \), is bounded, there exist two unique measures \( \nu_1 \) and \( \nu_2 \) supported respectively by \( \Delta_1 \) and \( \Delta_2 \) such that \( v \) can be represented on \( X_1 \) by
\[
v = \int_{\Delta_1} (Qg_1(\cdot, y)) \circ \pi_1 d\nu_1(y) + \int_{\Delta_2} (QK_{\mu_1}^{\mu_2}g_2(\cdot, y)) \circ \pi_1 d\nu_2(y)
\]
and on \( X_2 \) by
\[
v = \int_{\Delta_1} (K_{\mu_2}^{\mu_1}Qg_1(\cdot, y)) \circ \pi_2 d\nu_1(y) + \int_{\Delta_2} (Tg_2(\cdot, y)) \circ \pi_2 d\nu_2(y).
\]

Proof: If \( v = 0 \), we have \( \nu_1 = \nu_2 = 0 \).

If \( v \neq 0 \), we may assume without loss of generality that \( v \in B \). Consider the mapping
\[
\Psi : \begin{cases} 
\Delta_1 \cup \Delta_2 \longrightarrow \mathcal{E}(B) \\
y \mapsto \Psi(y)
\end{cases}
\]
where \( \Psi(y) \) is defined by
\[
\Psi(y) := \begin{cases} 
(Qg_1(\cdot, y)) \circ \pi_1 & \text{on } X_1, \\
(K_{\mu_2}^{\mu_1}Qg_1(\cdot, y)) \circ \pi_2 & \text{on } X_2,
\end{cases} \quad y \in \Delta_1,
\]
\[
\Psi(y) := \begin{cases} 
(QK_{\mu_1}^{\mu_2}g_2(\cdot, y)) \circ \pi_1 & \text{on } X_1, \\
(Tg_2(\cdot, y)) \circ \pi_2 & \text{on } X_2,
\end{cases} \quad y \in \Delta_2.
\]
The mapping \( \Psi \) is bijective because \( g_1(x, \cdot) \) and \( g_2(x, \cdot) \) separate \( \Delta_1 \) and \( \Delta_2 \), respectively. \( \Psi \) and its inverse \( \Psi^{-1} \) are continuous because \( g_1 \) and \( g_2 \) are continuous on \( \Delta \times D \). Then there exists, by Theorem B, a unique measure \( \nu \) supported by \( \Delta_1 \cup \Delta_2 \) such that
\[
v = \int_{\Delta_1 \cup \Delta_2} \Psi(y) d\nu(y).
\]
Let $\nu_j, j = 1, 2,$ be the restriction of the measure $\nu$ to $\triangle_j$. Then $v$ may be written on $X_1$ as

$$v = \int_{\triangle_1} (Qg^1(\cdot, y)) \circ \pi_1 \, d\nu_1(y) + \int_{\triangle_2} (QK_D^{\mu_1}g^2(\cdot, y)) \circ \pi_1 \, d\nu_2(y)$$

and on $X_2$ as

$$v = \int_{\triangle_1} (K_D^{\mu_2}Qg^1(\cdot, y)) \circ \pi_2 \, d\nu_1(y) + \int_{\triangle_2} (Tg^2(\cdot, y)) \circ \pi_2 \, d\nu_2(y).$$

□

Let $t_i, i = 1, 2,$ be two positive $L_i$-harmonic functions on $D$ such that the function $K_D^{\mu_j}t_k$ is finite and the function $K_D^{\mu_k}K_D^{\mu_j}t_k, j \neq k, j, k \in \{1, 2\}$, is bounded on $D$. By [10] and [12], there exists a unique measure $\nu_{t_j}$, supported by $\triangle_j$, such that $t_j = \int_{\triangle_j} g^j(\cdot, y) \, d\nu_{t_j}(y), j = 1, 2$. We consider the harmonic function $w$ from Theorem 3.1 defined on $X$ by

$$w := \begin{cases} (Qt_1 + QK_D^{\mu_1}t_2) \circ \pi_1 & \text{on } X_1, \\ (K_D^{\mu_2}Qt_1 + Tt_2) \circ \pi_2 & \text{on } X_2. \end{cases}$$

Corollary 4.3. If the functions $g^j(x, \cdot), x \in D$, separate $\triangle_j, j = 1, 2$, then $w$ is written on $X_1$ by

$$w = \int_{\triangle_1} (Qg^1(\cdot, y)) \circ \pi_1 \, d\nu_{t_1}(y) + \int_{\triangle_2} (QK_D^{\mu_1}g^2(\cdot, y)) \circ \pi_1 \, d\nu_{t_2}(y),$$

and on $X_2$ by

$$w = \int_{\triangle_1} (K_D^{\mu_2}Qg^1(\cdot, y)) \circ \pi_2 \, d\nu_{t_1}(y) + \int_{\triangle_2} (Tg^2(\cdot, y)) \circ \pi_2 \, d\nu_{t_2}(y).$$

Proof: It suffices to replace $t_j, j = 1, 2,$ with their Martin representations in the expression of $w$, and the result follows from the uniqueness of the measures $\nu_{t_j}$ in Theorem 4.1. □

Remark 4.3. By Corollary 4.3, we have $\nu_{t_j}(\triangle \setminus \triangle_j) = 0$, thus $\nu_{t_j}(\triangle \setminus (\triangle_1 \cup \triangle_2)) = 0, j = 1, 2$. 
5. Dirichlet problem on the Martin boundary associated with \((S)\)

Given a couple of functions \((u_1, u_2)\) defined on \(\triangle\), the Dirichlet problem on \(\triangle\) consists to find a couple of functions \((h_1, h_2)\) solving the system \((S)\) such that

\[
\lim_{x \to y} h_i(x) = u_i(y) \quad \forall y \in \triangle.
\]

The couple \((u_1, u_2)\) can be identified with a function \(f\) on \(\bar{\triangle} := \bigcup_{j=1}^{2} \triangle \times \{j\}\) such that \(f \circ i_j = u_j\), where \(i_j, j = 1, 2\), denote always the mappings of \(\triangle\) in \(\triangle \times \{j\}\) defined by \(i_j(z) := (z, j), z \in \triangle\). The Dirichlet problem may be stated as follows: for a given function \(f\) defined on \(\bar{\triangle}\), determine, if possible, a harmonic function \(H_f\) on \(X\) such that \(H_f(x) \to f(y)\) as \(x \to y\) for each \(y \in \bar{\triangle}\). As in harmonic and biharmonic cases, there are some examples where there is no solution of this problem. In this section, we will discuss the Perron-Wiener-Brelot (PWB) approach to the Dirichlet problem. To this end, we give the following definition.

**Definition 5.1.** Let \(h_1\) (resp. \(h_2\)) be a strictly positive \(L_1\)-harmonic (resp. \(L_2\)-harmonic) function on \(D\), and let \(h\) be the function defined on \(X\) by

\[
h := \begin{cases} h_1 \circ \pi_1 & \text{on } X_1, \\ h_2 \circ \pi_2 & \text{on } X_2. \end{cases}
\]

A function \(v\) on \(X\) is called \(h\)-harmonic (resp. \(h\)-hyperharmonic, \(h\)-superharmonic) on \(X\) if and only if the function \(u\) defined on \(X\) by

\[
u := \begin{cases} (h_1(v \circ i_1)) \circ \pi_1 & \text{on } X_1, \\ (h_2(v \circ i_2)) \circ \pi_2 & \text{on } X_2. \end{cases}
\]

is harmonic (resp. hyperharmonic, superharmonic) on \(X\).

We also define the upper and lower class associated with a function defined on \(\bar{\triangle}\). Let \(f\) be a function defined on \(\bar{\triangle}\) and let \(h\) be a function defined on \(X\) as in Definition 5.1. We define:

\[
\bar{U}_f := \{v : v \text{ is } h\text{-hyperharmonic and bounded from below on } X \text{ and } \liminf_{x \to y} v(x) \geq f(y), \forall y \in \bar{\triangle}\} \]

and

\[
\bar{U}_f := \{s : s \text{ is } h\text{-hypoharmonic and bounded from above on } X \text{ and } \limsup_{x \to y} v(x) \leq f(y), \forall y \in \bar{\triangle}\}.
\]
We note that $\bar{U}_f$ and $U_f$ are never empty since they contain the constant functions $+\infty$ and $-\infty$ respectively, and that $\bar{U}_f = -U_{-f}$. Put

$$\bar{H}_f := \inf \bar{U}_f \quad \text{and} \quad H_f := \sup U_f.$$  

$f$ is called $h$-resolutive if $\bar{H}_f$ and $H_f$ are equal and $h$-harmonic on $X$. If $f$ is $h$-resolutive, then we define $H_f^h := \bar{H}_f = H_f$ and call $H_f^h$ the PWB-solution of the Dirichlet problem on $X$ with boundary function $f$. If $f \circ i_j$ is $h_j$-resolutive on $\triangle$, we call $H_{f \circ i_j}^{h_j}$ the PWB-solution of Dirichlet problem on $D$ associated with $f \circ i_j$, $j = 1, 2$.

**Further properties of PWB solutions.**

Let $f$ and $g$ be two functions defined on $\bar{\triangle}$. Then we have

(i) $H_f^h = -\bar{H}_h^h f$.

(ii) $H_f^h \leq \bar{H}_h^h$.

(iii) $H_f^h \leq H_g^h$ and $\bar{H}_f^h \leq H_g^h$ if $f \leq g$.

(iv) Let $f, g$ be two $h$-resolutive functions and $\alpha \in \mathbb{R}$. Then $f + g$ and $\alpha f$ are $h$-resolutive and

$$H_{f+g}^h = H_f^h + H_g^h, \quad H_{\alpha f}^h = \alpha H_f^h.$$  

(v) If $U_f \cap (-S(X)) \neq \emptyset$ (resp. $\bar{U}_f \cap S(X) \neq \emptyset$), then the function $\bar{H}_f^h$ (resp. $H_f^h$) is identically $\infty$, or $h$-harmonic on $X$.

Let $f$ be a positive function on $\bar{\triangle}$ such that $f \circ i_2 = 0$ and $w$ the function defined on $X$ by

$$w := \left\{ \begin{array}{ll}
\left( \frac{1}{n_1} Q(h_1, \bar{H}_f^h_{i_1}) \right) \circ \pi_1 & \text{on } X_1, \\
\left( \frac{1}{n_2} H_{f \circ i_2}^h \right) \circ \pi_2 & \text{on } X_2.
\end{array} \right.$$  

We have $\bar{H}_f^h \leq w$. Indeed, it follows from Corollary 3.1 that $w$ is a positive $h$-hyperharmonic function on $X$ and moreover, we have

$$\liminf_{x \to y} (w \circ i_1)(x) \geq (f \circ i_1)(y), \quad \text{for all } y \in \triangle$$

and

$$\liminf_{x \to y} (w \circ i_2)(x) \geq 0, \quad \text{for all } y \in \triangle.$$  

Hence, $w \in \bar{U}_f$. Thus $\bar{H}_f^h \leq w$ and therefore if $\bar{H}_f^h = +\infty$ then $w = +\infty$. If $\bar{H}_f^h < \infty$, we have
Lemma 5.1. Let $f$ be a positive function on $\bar{\triangle}$ such that $f \circ i_2 = 0$ and $K_D^{\mu_1} K_D^{\mu_2} (h_1(\bar{H}_f^h \circ i_1))$ is bounded on $D$. Then we have

$$\bar{H}_f^h = \begin{cases} 
\frac{1}{h_1} Q(h_1 \bar{H}_f^{h_1}) \circ \pi_1 & \text{on } X_1, \\
\frac{1}{h_2} K_D^{\mu_2} Q(h_1 \bar{H}_f^{h_1}) \circ \pi_2 & \text{on } X_2.
\end{cases}$$

Proof: It suffices to show that $w \leq \bar{H}_f^h$.

(a) Let us show that $w \circ i_1 \leq \bar{H}_f^h \circ i_1$.

It follows from property (v) of PWB solutions that the function $\bar{H}_f^h$ is $h$-harmonic on $X$. Then the function

$$\bar{u} := \begin{cases} 
(h_1(\bar{H}_f^h \circ i_1)) \circ \pi_1 & \text{on } X_1, \\
(h_2(\bar{H}_f^h \circ i_2)) \circ \pi_2 & \text{on } X_2
\end{cases}$$

is a positive harmonic function on $X$, and by Corollary 2.1, the functions $\bar{u}_j = h_j(\bar{H}_f^h \circ i_j) - K_D^{\mu_j} (h_k(\bar{H}_f^h \circ i_k))$, $j, k \in \{1, 2\}$, $j \neq k$ are positive and $L_j$-harmonic on $D$. Put $v_j := \frac{1}{h_j} \bar{u}_j$. On the one hand, we have

$$K_D^{\mu_2} (h_1(\bar{H}_f^h \circ i_1)) \leq (h_2(\bar{H}_f^h \circ i_2)),$$

hence

$$K_D^{\mu_1} K_D^{\mu_2} (h_1(\bar{H}_f^h \circ i_1)) \leq K_D^{\mu_1} (h_2(\bar{H}_f^h \circ i_2)),$$

i.e.

$$K_D^{\mu_1} K_D^{\mu_2} (h_1(\bar{H}_f^h \circ i_1)) \leq (h_1(\bar{H}_f^h \circ i_1) - h_1 v_1).$$

So,

$$Q(h_1 v_1) + Q K_D^{\mu_1} K_D^{\mu_2} (h_1(\bar{H}_f^h \circ i_1)) \leq Q(h_1(\bar{H}_f^h \circ i_1)).$$

Since

$$Q K_D^{\mu_1} K_D^{\mu_2} + I = Q,$$

we get

$$Q(h_1 v_1) + Q K_D^{\mu_1} K_D^{\mu_2} (h_1(\bar{H}_f^h \circ i_1)) \leq Q K_D^{\mu_1} K_D^{\mu_2} (h_1(\bar{H}_f^h \circ i_1)) + h_1(\bar{H}_f^h \circ i_1).$$

Therefore,

$$(5.1.1) \quad Q(h_1 v_1) \leq h_1(\bar{H}_f^h \circ i_1).$$
On the other hand,
\[
\liminf_{x \to y} v_1(x) = \liminf_{x \to y} (\tilde{H}_f^h \circ i_1 - \frac{1}{h_1} K^{\mu_1}_D (h_2 (\tilde{H}_f^h \circ i_2)))(x)
\]
\[
\geq (f \circ i_1)(y) - \limsup_{x \to y} \left( \frac{1}{h_1} K^{\mu_1}_D (h_2 (\tilde{H}_f^h \circ i_2)) \right)(x)
\]
for all \( y \in \Delta \). Since
\[
\limsup_{x \to y} \left( \frac{1}{h_1} K^{\mu_1}_D (h_2 (\tilde{H}_f^h \circ i_2)) \right)(x)
\]
\[
\leq \int_D \limsup_{x \to y} \frac{1}{h_1(x)} G_1(x, z) h_2(z) (\tilde{H}_f^h \circ i_2)(z) \, d\mu_1(z),
\]
and \( \limsup_{x \to y} \frac{1}{h_1(x)} G_1(x, z) = 0 \) \( \nu_{h_1} \)-a.e. on \( \Delta_1 \), where \( \nu_{h_1} \) is the measure associated with \( h_1 \) in the Martin representation ([13, p.218]), we have, by Remark 4.3, \( \nu_{h_1}(\Delta \setminus \Delta_1) = 0 \). Hence \( \limsup_{x \to y} \frac{1}{h_1(x)} G_1(x, z) = 0 \) \( \nu_{h_1} \)-a.e. on \( \Delta \).

Thus \( \liminf_{x \to y} v_1(x) \geq (f \circ i_1)(y) \) \( \nu_{h_1} \)-a.e. on \( \Delta \). Hence \( v_1 \) is a positive \( h_1 - L_1 \)-hyperharmonic function on \( D \) and \( \liminf_{x \to y} v_1(x) \geq (f \circ i_1)(y) \) \( \nu_{h_1} \)-a.e. on \( \Delta \).

So
\[
\text{(5.1.2)} \quad v_1 \geq \tilde{H}_{f \circ i_1}^{h_1}.
\]
Thus, by (5.1.1), we have
\[
Q(h_1 \tilde{H}_{f \circ i_1}^{h_1}) \leq (h_1(\tilde{H}_f^h \circ i_1)).
\]

(b) Let us show that \( w \circ i_2 \leq (\tilde{H}_f^h \circ i_2) \).

It follows from (a) that
\[
Q(h_1 \tilde{H}_{f \circ i_1}^{h_1}) \leq (h_1(\tilde{H}_f^h \circ i_1)).
\]

Then,
\[
K^{\mu_2}_D Q(h_1 \tilde{H}_{f \circ i_1}^{h_1}) \leq K^{\mu_2}_D (h_1(\tilde{H}_f^h \circ i_1)) \leq (h_2(\tilde{H}_f^h \circ i_2)).
\]
This finishes the proof. \( \square \)

Remark 5.1. The result of Lemma 5.1 is still valid if instead of the assumption
\( K^{\mu_1}_D K^{\mu_2}_D (h_1(\tilde{H}_f^h \circ i_1)) \) is bounded, we suppose only that \( Q(h_1(\tilde{H}_f^h \circ i_1)) \) is finite.

Let \( f \) be a positive function on \( \bar{\Delta} \) such that \( f \circ i_1 = 0 \) and \( \tilde{w} \) the function defined on \( X \) by
\[
\tilde{w} := \begin{cases} 
(\frac{1}{h_1} QK^{\mu_1}_D (h_2 \tilde{H}_{f \circ i_1}^{h_2})) \circ \pi_1 & \text{on } X_1, \\
(\frac{1}{h_2} T(h_2 \tilde{H}_{f \circ i_2}^{h_2})) \circ \pi_2 & \text{on } X_2.
\end{cases}
\]

We have \( \tilde{H}_f^h \leq \tilde{w} \). Therefore if \( \tilde{H}_f^h = +\infty \), then \( \tilde{w} = +\infty \). If \( \tilde{H}_f^h < \infty \), we have:
Lemma 5.2. Let \( f \) be a positive function on \( \bar{\triangle} \) such that \( f \circ i_1 = 0 \) and \( K_D^{\mu_2}K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) \) is bounded on \( D \). Then

\[
\bar{H}_f^h = \begin{cases} 
\left( \frac{1}{h_1} QK_D^{\mu_1}(h_2\bar{H}_f^h) \right) \circ \pi_1 & \text{on } X_1, \\
\left( \frac{1}{h_2} T(h_2\bar{H}_f^h) \right) \circ \pi_2 & \text{on } X_2.
\end{cases}
\]

**Proof:** It suffices to show that \( \tilde{w} \leq \bar{H}_f^h \).

(a) Let us show that \( \tilde{w} \circ i_1 \leq \bar{H}_f^h \circ i_1 \).

By the property (v) of PWB solutions, the function \( \bar{H}_f^h \) is \( h \)-harmonic on \( X \). Then the function

\[
\tilde{u} := \begin{cases} 
(h_1(\bar{H}_f^h \circ i_1)) \circ \pi_1 & \text{on } X_1, \\
(h_2(\bar{H}_f^h \circ i_2)) \circ \pi_2 & \text{on } X_2
\end{cases}
\]

is a positive harmonic function on \( X \) and by Corollary 2.1, \( \tilde{u}_j = h_j(\bar{H}_f^h \circ i_j) - K_D^{\mu_j}(h_k(\bar{H}_f^h \circ i_k)), \) \( j, k \in \{1, 2\}, j \neq k \), are positive and \( L_j \)-harmonic functions on \( D \). Put \( v_j := \frac{1}{h_j} \tilde{u}_j \). On the one hand, we have

\[
K_D^{\mu_1}(h_2(\bar{H}_f^h \circ i_2)) \leq (h_1(\bar{H}_f^h \circ i_1)),
\]

hence

\[
K_D^{\mu_1}(h_2v_2 + K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1))) \leq h_1(\bar{H}_f^h \circ i_1)
\]

and

\[
QK_D^{\mu_1}(h_2v_2) + QK_D^{\mu_1}K_D^{\mu_2}(h_1(\bar{H}_f^h \circ i_1)) \leq Q(h_1(\bar{H}_f^h \circ i_1)).
\]

Since

\[
QK_D^{\mu_1}K_D^{\mu_2} + I = Q,
\]

we get

\[
QK_D^{\mu_1}(h_2v_2) \leq h_1(\bar{H}_f^h \circ i_1).
\]

As in the proof of Lemma 5.1, we show that \( \lim \inf_{x \to y} v_2(x) \geq (f \circ i_2)(y) \nu_{h_2}\text{-a.e. on } \triangle \). Since \( v_2 \) is a positive \( h_2-L_2 \)-hyperharmonic function and \( \lim \inf_{x \to y} v_2(x) \geq (f \circ i_2)(y), \nu_{h_2}\text{-a.e. on } \triangle, \) we obtain

\[
(5.1.2) \quad v_2 \geq \bar{H}_{f\circ i_2}^h,
\]

hence

\[
QK_D^{\mu_1}(h_2\bar{H}_{f\circ i_2}^h) \leq (h_1(\bar{H}_f^h \circ i_1)).
\]
(b) Let us show that \( \tilde{w} \circ i_2 \leq (\tilde{H}_f^{h} \circ i_2) \). We have

\[
K_D^{\mu_1} (h_2(\tilde{H}_f^{h} \circ i_2)) \leq h_1(\tilde{H}_f^{h} \circ i_1).
\]

So

\[
K_D^{\mu_2} K_D^{\mu_1} (h_2(\tilde{H}_f^{h} \circ i_2)) \leq K_D^{\mu_2} (h_1(\tilde{H}_f^{h} \circ i_1)) = h_2(\tilde{H}_f^{h} \circ i_2) - h_2 v_2.
\]

Hence

\[
T(h_2, v_2) + TK_D^{\mu_2} K_D^{\mu_1} (h_2(\tilde{H}_f^{h} \circ i_2)) \leq T(h_2(\tilde{H}_f^{h} \circ i_2)).
\]

Since

\[
TK_D^{\mu_2} K_D^{\mu_1} + I = T,
\]

we get

\[
T(h_2 \tilde{H}_{f_{\circ i_2}}^{h}) \leq (h_2(\tilde{H}_f^{h} \circ i_2)).
\]

\[\square\]

**Remark 5.2.** The result of Lemma 5.2 is still valid if instead of the assumption \( K_D^{\mu_2} K_D^{\mu_1} (h_2(\tilde{H}_f^{h} \circ i_2)) \) is bounded, we suppose only that \( T(h_2(\tilde{H}_f^{h} \circ i_2)) \) is finite.

Let \( f \) be a positive function on \( \bar{\triangle} \) and let \( w' \) be the function defined on \( X \) by

\[
w' := \left\{ \begin{array}{ll}
\frac{1}{h_1} (Q(h_1 \tilde{H}_{f_{\circ i_1}}^{h_1}) + QK_D^{\mu_1} (h_2 \tilde{H}_{f_{\circ i_2}}^{h_2})) \circ \pi_1 & \text{on } X_1, \\
\frac{1}{h_2} (K_D^{\mu_2} Q(h_1 \tilde{H}_{f_{\circ i_1}}^{h_1}) + T(h_2 \tilde{H}_{f_{\circ i_2}}^{h_2})) \circ \pi_2 & \text{on } X_2.
\end{array} \right.
\]

We have \( \tilde{H}_f^{h} \leq w' \). Therefore, if \( \tilde{H}_f^{h} = +\infty \) then \( w' = +\infty \). If \( \tilde{H}_f^{h} < \infty \), we have

**Proposition 5.1.** Let \( f \) be a positive function on \( \bar{\triangle} \) such that \( K_D^{\mu_j} K_D^{\mu_k} (h_j(\tilde{H}_f^{h} \circ i_j)) \) is bounded on \( D \), \( j, k \in \{1, 2\}, j \neq k \). Then we have

\[
\tilde{H}_f^{h} = \left\{ \begin{array}{ll}
\frac{1}{h_1} (Q(h_1 \tilde{H}_{f_{\circ i_1}}^{h_1}) + QK_D^{\mu_1} (h_2 \tilde{H}_{f_{\circ i_2}}^{h_2})) \circ \pi_1 & \text{on } X_1, \\
\frac{1}{h_2} (K_D^{\mu_2} Q(h_1 \tilde{H}_{f_{\circ i_1}}^{h_1}) + T(h_2 \tilde{H}_{f_{\circ i_2}}^{h_2})) \circ \pi_2 & \text{on } X_2.
\end{array} \right.
\]

**Proof:** It suffices to show that \( w' \leq \tilde{H}_f^{h} \).

(a) Let us show that \( w' \circ i_1 \leq \tilde{H}_f^{h} \circ i_1 \).

By the property (v) of PWB solutions, the function \( \tilde{H}_f^{h} \) is \( h \)-harmonic on \( X \). Then the function

\[
\tilde{u} := \left\{ \begin{array}{ll}
(h_1(\tilde{H}_f^{h} \circ i_1)) \circ \pi_1 & \text{on } X_1, \\
(h_2(\tilde{H}_f^{h} \circ i_2)) \circ \pi_2 & \text{on } X_2
\end{array} \right.
\]

is the solution of

\[
\left\{ \begin{array}{ll}
\tilde{u} & \text{on } X_1, \\
\tilde{u} & \text{on } X_2.
\end{array} \right.
\]
is a positive harmonic on $X$ and by Corollary 2.1, $\bar{u}_j = h_j(\bar{H}_f^h \circ i_j) - K^h_{D_j} (h_k(\bar{H}_f^h \circ i_k)), j, k \in \{1, 2\}, j \neq k$, are positive $L_j$-harmonic on $D$. Put $v_j = \frac{1}{h_j} \bar{u}_j$. On the one hand,

$$h_1v_1 + K^h_{D_1} (h_2(\bar{H}_f^h \circ i_2)) = h_1(\bar{H}_f^h \circ i_1)$$

and

$$h_2v_2 + K^h_{D_2} (h_1(\bar{H}_f^h \circ i_1)) = h_2(\bar{H}_f^h \circ i_2).$$

Hence

$$Q(h_1v_1) + QK^h_{D_1} (h_2(\bar{H}_f^h \circ i_2)) = Q(h_1(\bar{H}_f^h \circ i_1))$$

and

$$QK^h_{D_1} (h_2.v_2) + QK^h_{D_1} K^h_{D_2} (h_1(\bar{H}_f^h \circ i_1)) = QK^h_{D_1} (h_2(\bar{H}_f^h \circ i_2)).$$

Since

$$QK^h_{D_1} K^h_{D_2} + I = Q,$$

we have

$$Q(h_1.v_1) + QK^h_{D_1} (h_2.v_2) = h_1(\bar{H}_f^h \circ i_1).$$

It follows from (5.1.2) and (5.2.1) that

$$Q(h_1\bar{H}_f^{h_1}) + QK^h_{D_1} (h_2\bar{H}_f^{h_2}) \leq h_1(\bar{H}_f^h \circ i_1).$$

Similarly, we show that

$$\frac{1}{h_2} (K^h_{D_2} Q(h_1\bar{H}_f^{h_1}) + T(h_2\bar{H}_f^{h_2})) \leq h_2(\bar{H}_f^h \circ i_2).$$

$\square$

**Remark 5.3.** The result of Proposition 5.1 is still valid if instead of the assumption $K^h_{D_j} K^h_{D_k} (h_j(\bar{H}_f^h \circ i_j))$ is bounded on $D$, $j, k \in \{1, 2\}, j \neq k$, we suppose that $Q(h_1(\bar{H}_f^h \circ i_1)) < \infty$ and $T(h_2(\bar{H}_f^h \circ i_2)) < \infty$.

**$h$-negligible sets.**

**Definition 5.2.** Let $e$ be a subset of $\bar{\Delta}$. $e$ is called $h$-negligible if $\bar{H}_e^h = 0$, where $1_e$ is the indicator of the set $e$.

Let $\bar{e}$ be a subset of $\Delta$. $\bar{e}$ is called $h_j$-negligible if and only if $\bar{H}_e^{h_j} = 0, j = 1, 2$.

**Proposition 5.2.** Let $e \subset \bar{\Delta} = (\Delta \times \{1\}) \cup (\Delta \times \{2\})$ be such that $e = (e_1 \times \{1\}) \cup (e_2 \times \{2\})$, where $e_j \subset \Delta, j = 1, 2$. The following are equivalent:

1. $e$ is $h$-negligible;
2. $e_j$ is $h_j$-negligible, $j = 1, 2$. 

PROOF: Suppose that \( e \) is \( h \)-negligible; then \( \bar{H}^h_{1_e} = 0 \). By Proposition 5.1, we have

\[
\bar{H}^h_{1_e} = \begin{cases} 
\frac{1}{n_1}(Q(h_1\bar{H}^{h_1}_{1_e\circ i_1}) + QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{1_e\circ i_2})) \circ \pi_1 & \text{on } X_1, \\
\frac{1}{n_2}(K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{1_e\circ i_1}) + T(h_2\bar{H}^{h_2}_{1_e\circ i_2})) \circ \pi_2 & \text{on } X_2,
\end{cases}
\]

hence

\[
Q(h_1\bar{H}^{h_1}_{1_e\circ i_1}) = -QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{1_e\circ i_2}), \quad K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{1_e\circ i_1}) = -T(h_2\bar{H}^{h_2}_{1_e\circ i_2}).
\]

Since the functions \( h_j\bar{H}^{h_j}_{1_e\circ i_j}, \ j = 1, 2 \), are positive, \( \bar{H}^{h_j}_{1_e\circ i_j} = 0, \ j = 1, 2 \). Since \( 1_e \circ i_j = 1_{e_j}, \bar{H}^{h_j}_{1_{e_j}} = 0 \), i.e., the set \( e_j \) is \( h_j \)-negligible. The converse is obvious. \( \square \)

**Proposition 5.3.** Let \( f \) and \( \tilde{f} \) be two positive functions defined on \( \bar{\Delta} \) such that \( e = \{ f \neq \tilde{f} \} \) is a \( h \)-negligible set. Then \( \bar{H}^h_f = \bar{H}^h_{\tilde{f}} \).

PROOF: We have \( e = \{ f \neq \tilde{f} \} = (e_1 \times \{1\}) \cup (e_2 \times \{2\}) \), where \( e_j = \{ f \circ i_j \neq \tilde{f} \circ i_j \}, \ j = 1, 2 \), and \( e \) is \( h \)-negligible. Then, by Proposition 5.2, \( e_j \) is \( h_j \)-negligible. Thus \( \bar{H}^{h_j}_{f \circ i_j} = \bar{H}^{h_j}_{\tilde{f} \circ i_j}, \ j = 1, 2 \). Therefore, by Proposition 5.1, \( \bar{H}^h_f = \bar{H}^h_{\tilde{f}} \). \( \square \)

**Lemma 5.3.** Let \( f \) be a positive function on \( \bar{\Delta} \) such that \( K^{\mu_j}_D K^{\mu_k}_D (h_j(\bar{H}^h_f \circ i_j)) \) is bounded on \( D, \ j, k \in \{1, 2\}, \ j \neq k \). Then we have

\[
h_j\bar{H}^{h_j}_{f \circ i_j} = h_j(\bar{H}^h_f \circ i_j) - K^{\mu_j}_D(h_k(\bar{H}^h_f \circ i_k)).
\]

PROOF: By Proposition 5.1, we have

\[
\bar{H}^h_f \circ i_1 = \frac{1}{n_1}(Q(h_1\bar{H}^{h_1}_{f \circ i_1}) + QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{f \circ i_2})),
\]

\[
\bar{H}^h_f \circ i_2 = \frac{1}{n_2}(K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{f \circ i_1}) + T(h_2\bar{H}^{h_2}_{f \circ i_2})).
\]

Then

\[
h_1\bar{H}^h_f \circ i_1 = (Q(h_1\bar{H}^{h_1}_{f \circ i_1}) + QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{f \circ i_2})),
\]

\[
h_2\bar{H}^h_f \circ i_2 = (K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{f \circ i_1}) + T(h_2\bar{H}^{h_2}_{f \circ i_2})).
\]

Hence

\[
\begin{cases} 
K^{\mu_2}_D(h_1\bar{H}^h_f \circ i_1) = K^{\mu_2}_D(Q(h_1\bar{H}^{h_1}_{f \circ i_1}) + QK^{\mu_1}_D(h_2\bar{H}^{h_2}_{f \circ i_2})), \\
h_2\bar{H}^h_f \circ i_2 = (K^{\mu_2}_D Q(h_1\bar{H}^{h_1}_{f \circ i_1}) + T(h_2\bar{H}^{h_2}_{f \circ i_2})).
\end{cases}
\]
Since $\tilde{H}^h_f$ is $h$-harmonic on $X$, $K^\mu_{D}(h_1(\tilde{H}^h_f \circ i_1)) < \infty$. Thus,

$$h_2(\tilde{H}^h_f \circ i_2) - K^\mu_{D}(h_1(\tilde{H}^h_f \circ i_1)) = T(h_2\tilde{H}^{h_2}_{f \circ i_2}) - K^{\mu_2}(h_2\tilde{H}^{h_2}_{f \circ i_2}).$$

Since

$$T = K^{\mu_2}QK^\mu_{D} + I,$$

we get

$$h_2(\tilde{H}^h_f \circ i_2) - K^\mu_{D}(h_1(\tilde{H}^h_f \circ i_1)) = h_2\tilde{H}^{h_2}_{f \circ i_2}.$$

Similarly, we show that

$$h_1(\tilde{H}^h_f \circ i_1) - K^{\mu_1}(h_2(\tilde{H}^h_f \circ i_2)) = h_1\tilde{H}^{h_1}_{f \circ i_1}.$$

\hfill \Box

**Theorem 5.1.** Let $f$ be a positive function defined on $\bar{\Delta}$ such that $K^{\mu_j}_{D}K^{\mu_k}_{D}(h_j(\tilde{H}^h_f \circ i_j))$ is bounded, $j \neq k, j, k \in \{1, 2\}$. The following are equivalent:

(a) $f$ is $h$-resolutive;

(b) (1) $f \circ i_j$ is $j$-resolutive on $\Delta$, $j = 1, 2$, and

(2) $K^{\mu_k}_{D}(h_j \tilde{H}^{h_j}_{f \circ i_j})$ is finite, $j \neq k, j, k \in \{1, 2\}$.

**Proof:** Suppose that (b) holds. Then the function $h_j \tilde{H}^{h_j}_{f \circ i_j}$ is $L_j$-harmonic, $j = 1, 2$. Moreover, we have

$$h_j \tilde{H}^{h_j}_{f \circ i_j} \leq h_j(\tilde{H}^h_f \circ i_j).$$

Since $K^{\mu_j}_{D}K^{\mu_k}_{D}(h_j(\tilde{H}^h_f \circ i_j))$ is bounded, $j \neq k, j, k \in \{1, 2\}$, $K^{\mu_j}_{D}K^{\mu_k}_{D}(h_j \tilde{H}^{h_j}_{f \circ i_j})$ is bounded, $j \neq k, j, k \in \{1, 2\}$. Hence, by Theorem 3.1, the function

$$\tilde{H}^h_f = \left\{ \begin{array}{ll}
\frac{1}{h_1}(Q(h_1\tilde{H}^{h_1}_{f \circ i_1}) + QK^{\mu_1}_{D}(h_2\tilde{H}^{h_2}_{f \circ i_2})) \circ \pi_1 & \text{on } X_1, \\
\frac{1}{h_2}(K^{\mu_2}_{D}Q(h_1\tilde{H}^{h_1}_{f \circ i_1}) + T(h_2\tilde{H}^{h_2}_{f \circ i_2})) \circ \pi_2 & \text{on } X_2
\end{array} \right.$$

is $h$-harmonic on $X$, moreover $\tilde{H}^h_f = H^h_f = H^h_f$, therefore $f$ is $h$-resolutive.

Conversely, suppose that $f$ is $h$-resolutive. Then $\tilde{H}^h_f = H^h_f = H^h_f$ and $H^h_f$ is $h$-harmonic. On the one hand, it follows from Lemma 5.3 that

$$h_j \tilde{H}^{h_j}_{f \circ i_j} = h_j(\tilde{H}^h_f \circ i_j) - K^{\mu_j}_{D}(h_k(\tilde{H}^h_f \circ i_k)).$$
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and by Corollary 2.1, the function $H_{f^{oi_j}}^{h_j}$ is $h_j - L_j$-harmonic on $D$, i.e. $f \circ i_j$ is $h_j$-resolutive on $\Delta$. On the other hand,

$K_D^{\mu_k}(h_j H_{f^{oi_j}}^{h_j}) \leq K_D^{\mu_k}(h_j(H_f^h \circ i_j)) \leq h_k H_f^h \circ i_k,$

thus

$K_D^{\mu_k}(h_j H_{f^{oi_j}}^{h_j}) < \infty.$

□

References


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