A tree π-base for $\mathbb{R}^*$ without cofinal branches

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Abstract. We prove an analogue to Dordal's result in P.L. Dordal, *A model in which the base-matrix tree cannot have cofinal branches*, J. Symbolic Logic **52** (1980), 651–664. He obtained a model of ZFC in which there is a tree π-base for $\mathbb{N}^*$ with no ω_2 branches yet of height ω_2. We establish that this is also possible for $\mathbb{R}^*$ using a natural modification of Mathias forcing.

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1. Introduction

Definition 1.1. Let X be a topological space. The *Novák number*, $\mathfrak{n}(X)$, is the minimum number of dense open subsets whose intersection is empty. $w\mathfrak{n}(X)$ denotes the minimum number of dense open subsets of X whose intersection has empty interior and it is called the *weak Novák number*.

$\mathfrak{n}(X)$ is also known as the Baire number of X. J. Novák was among the first topologists who studied this characteristic for general topological spaces. J. van Mill and S. Williams in [vMW83] introduced the definition of the weak Novák number. B. Balcar, J. Pelant and P. Simon in [BPS80] studied the Novák number of $\mathbb{N}^*$ using that $\mathbb{N}^*$ always has a tree π-base. In general, a *tree π-base* for a topological space is a π-base which forms a tree when ordered by reverse inclusion. They also introduced the cardinal, now denoted by $\mathfrak{h}$ or $\mathfrak{h}(\mathbb{N}^*)$, as the minimum height of a tree π-base. In general, $\mathfrak{h}(X)$ is the distributivity number of the Boolean algebra $RO(X)$; equivalently, it is the minimum cardinal κ such that forcing with $RO(X)$ does not add a new subset of κ. By results in [Wil82], for every locally compact noncompact metric space X, its Stone-Čech remainder, X^*, always has a tree π-base, $\mathfrak{h}(X^*) = w\mathfrak{n}(X^*)$ and this cardinal coincides with the minimum height of a tree π-base.

Trivially, $\mathfrak{h}(X^\lambda) \geq \mathfrak{h}(X^\gamma)$ holds whenever the numbers are defined and $\lambda \leq \gamma$. In fact, if $\left\{D_\alpha : \alpha < \mathfrak{h}(X^\lambda)\right\}$ is a family of dense open sets of X^λ whose intersection is not dense then letting $D'_\alpha = \{f \in X^\gamma : f \restriction \lambda \in D_\alpha\}$ we get a family of

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dense open subsets of X^γ and their intersection has empty interior. We have been interested in the following questions which appeared in [Dow98]:

Is $\mathfrak{h}\left((\mathbb{R}^*)^\omega\right) = \mathfrak{h}\left(\mathbb{R}^*\right)$?

Is it true that $\mathfrak{h}(\mathbb{R}^*) \leq \mathfrak{h}(\mathbb{N}^* \times \mathbb{N}^*)$?

Is it true $\mathfrak{n}(\mathbb{R}^*) = \mathfrak{n}(\mathbb{N}^*)$?

Alan Dow has conjectured positive answers for the first two questions; the third one is due to E.K. van Douwen.

Following Dow, we denote by $\overline{\mathrm{cpt}}$ the ideal of regular open sets which are bounded, or equivalently, have compact closure and we also adopt the convention that a * adorning a word or symbol will mean that it should be interpreted modulo the ideal $\overline{\mathrm{cpt}}$ in $RO(\mathbb{R})$; e.g. $A \subseteq^* B \subseteq \mathbb{R}$ will mean that $A \setminus B$ is bounded (it has compact closure). Then as he does, we represent $RO(\beta\mathbb{R} \setminus \mathbb{R})$ as the completion of the quotient $RO(\mathbb{R})/\overline{\mathrm{cpt}}$, which from now on we will denote by $\mathfrak{R}^*$. Although the elements of $\mathfrak{R}^*$ are equivalence classes of regular open sets, for convenience, we treat them as actual regular open sets.

In *the Mathias model* (i.e. the model obtained by a countable support iteration of Mathias forcing) $\mathcal{P}(\omega)/\mathrm{fin}$ has no dense tree of height ω_1, so $\mathfrak{h}(\mathbb{N}^*) = \aleph_2$ in the Mathias model. However, while there is no dense tree of height ω_1, P.L. Dordal showed [Dor87] that there must be a dense tree in which there are no branches of length ω_2. This means that a tree π-base for $\mathbb{N}^*$ in this model has barely height ω_2. Dordal's model was actually constructed using a modified support for Mathias forcing but was easily adapted once Shelah showed [She84] that the "not filling towers" property — the key of Dordal's proof — is preserved by countable support proper iterations.

Shelah and Spinas [SS00] showed that in the Mathias model, $\mathfrak{h}(\mathbb{N}^* \times \mathbb{N}^*) = \aleph_1$ (but $\mathfrak{h}(\mathbb{N}^*) = \aleph_2$ holds). It was known that $\mathfrak{h}(\mathbb{R}^*) \leq \mathfrak{h}(\mathbb{N}^*)$, and Alan Dow proved that in the Mathias model $\mathfrak{h}(\mathbb{R}^*) = \aleph_1$. Later, armed with a better understanding of the cardinal $\mathfrak{h}(\mathbb{R}^*)$, B. Balcar and M. Hrušák proved the following result from which Dow's result easily follows:

Theorem 1.2 ([BH04]). $\mathfrak{h}(\mathbb{R}^*) \leq \{\mathfrak{h}, \mathrm{add}(\mathcal{M})\}$*, where* $\mathcal{M}$ *is the ideal of meager sets in* $\mathbb{R}$*, and* $\mathrm{add}(\mathcal{M})$ *its additivity number.*

Dordal's result implies that $\mathfrak{h}(\mathbb{N}^*) = \mathfrak{n}(\mathbb{N}^*) = \aleph_2$ in the Mathias model. However it is shown in [BPS80] that $\mathfrak{n}(\mathbb{R}^*)$ is always at least $\aleph_2$, hence, in the Mathias model, $\mathfrak{h}(\mathbb{R}^*) < \mathfrak{n}(\mathbb{R}^*)$. Nevertheless,

Theorem 1.3 ([Dow89]). *There is a model for* ZFC *in which* $\mathfrak{h}(\mathbb{R}^*) = \mathfrak{n}(\mathbb{R}^*)$.

2. Modification of Mathias forcing

Our terminology is mostly standard. The bar over a set denotes the closure of it with respect to the ambient space understood from the context. Remember that u is a *regular open set* if $\mathrm{int}\,(\overline{u}) = u$. We often use the first lowercase letters to denote open intervals in $\mathbb{R}$, while letters like s, t, u, v, etc. for regular open sets in $\mathbb{R}$. Letters like k, m, n, etc. denote positive integers. Unless otherwise explicitly stated, the first capital letters denote finite unions of open intervals with rational endpoints. For convenience we will sometimes confuse the elements of the algebra $\mathfrak{R}^*$ with unbounded regular open subsets of the reals.

We will use the following forcing notion which we will denote by $\mathbb{M}_{\mathbb{R}^*}$. Let us first remember that Mathias forcing $\mathbb{M}$ is the set of all pairs $(s,A) \in [\omega]^{<\aleph_0} \times [\omega]^{\aleph_0}$ such that $s \cap A = \emptyset$ with the ordering $(s,A) \leq (t,B)$ if and only if $t \subseteq s \subseteq t \cup B$ and $A \subseteq B$. Mathias forcing is the natural forcing to increase $\mathfrak{h}(\mathbb{N}^*)$ since it adds a branch to any given tree and at the same time it diagonalizes that branch. However, Mathias forcing does not increase $\mathfrak{h}(\mathbb{R}^*)$; actually, as we said before, after an ω_2-iteration of $\mathbb{M}$ over a model of CH, $\mathfrak{h}(\mathbb{R}^*)$ remains $\aleph_1$. To increase $\mathfrak{h}(\mathbb{R}^*)$ we consider an analogue of $\mathbb{M}$ for the regular open algebra on $\mathbb{R}$.

Definition 2.1. $\mathbb{M}_{\mathbb{R}^*}$ will be the following forcing notion:

(1) conditions are pairs (A,s) where A is a finite union of separated open intervals in $\mathbb{R}$ with rational endpoints, s is an unbounded regular open set and $\overline{A} \cap \overline{s} = \emptyset$,

(2) the ordering is $(A,s) \leq (B,t)$ if and only if $B \subseteq A \subseteq B \cup t$ and $s \subseteq^* t$.

Like conditions in the Mathias forcing, in our forcing we have that $(A,s) \perp (A,t)$ if and only if $s \cap t =^* \emptyset$. From this follows that $\mathbb{M}_{\mathbb{R}^*}$ has the κ^+-cc, where $\kappa = c(\mathbb{R}^*) \leq w(\mathbb{R}^*) \leq w(\beta\mathbb{R}) \leq 2^{d(\mathbb{R})} = \mathfrak{c}$; for example, $\mathbb{M}_{\mathbb{R}^*}$ has the $\aleph_2$-cc under CH. If (A,s) and (B,t) are two conditions in $\mathbb{M}_{\mathbb{R}^*}$, we have that $\neg((A,s) \perp (B,t))$ if and only if (1) $A \setminus B \subseteq s$ and $B \setminus A \subseteq t$, (2) $s \cap t \neq^* \emptyset$.

Like Mathias forcing, $\mathbb{M}_{\mathbb{R}^*}$ can be factored as an $\aleph_1$-closed forcing followed by a σ-centred forcing.

Lemma 2.2. *There is a two-stage iteration $\mathbb{P} * \dot{\mathbb{Q}}$ such that $\mathbb{P}$ is $\aleph_1$-closed, $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is σ-centred", and $\mathbb{M}_{\mathbb{R}^*}$ can be densely embedded in $\mathbb{P} * \dot{\mathbb{Q}}$.*

Proof: $\mathbb{P}$ is $\langle \mathfrak{R}^*, \subseteq^* \rangle$. A generic $\dot{G}$ for $\mathbb{P}$ is an ultrafilter over $\mathbb{R}$; we now define $\dot{\mathbb{Q}}$ in $V^{\mathbb{P}}$ to be the σ-centred notion of forcing that adds an unbounded regular open $x \subseteq \mathbb{R}$ almost contained in every $y \in \dot{G}$. Specifically, $\mathbb{Q}$ can be defined as the set of all pairs (A,s) with A a finite union of separated open intervals with rational endpoints and $s \in \dot{G}$. As before, we say that $(A',s') \leq (A,s)$ if $A \subseteq A' \subseteq A \cup s$ and $s' \subseteq s$. It is possible to check that the image of $\mathbb{M}_{\mathbb{R}^*}$ under the embedding $(A,s) \mapsto (s,(A,s))$ is dense. □

We will denote $\mathbb{Q}$ in the previous lemma by $\mathbb{M}_{\mathbb{R}^*}(\dot{G})$, where $\dot{G}$ is the generic filter added by $\mathfrak{R}^*$. So, we can express $\mathbb{M}_{\mathbb{R}^*} = \mathfrak{R}^* * \mathbb{M}_{\mathbb{R}^*}(\dot{G})$.

Corollary 2.3. *The modification of Mathias forcing is a proper forcing, and hence, it does not collapse cardinals over a model of* CH.

If G is $\mathfrak{R}^*$-generic, then it is a P-point on $\mathbb{R}$; that is, if $\{s_n\}_{n \in \omega} \subseteq G$ then there exists $s \in G$ such that $(\forall\, n \in \omega)\,(s \subseteq^* s_n)$. Indeed, this is a consequence of the fact that countably closed posets are $\aleph_1$-Baire. If $\{s_n : n \in \omega\} \subseteq G$, and $D_n = \{t \in \Re^* : s_n \cap t =^* \emptyset \text{ or } s_n \supseteq^* t\}$, then D_n is open dense subset of $\Re^*$. Hence $D = \bigcap_{n\in\omega} D_n$ is still dense. Thus, if $s \in G \cap D$, then $(\forall\, n \in \omega)\,(s \subseteq^* s_n)$. This diagonalization method will be used often in the sequel. Another trivial observation which will be important later is that for each $(A, s) \in \mathbb{M}_{\mathbb{R}^*}$ we can find $(A, s') \leq (A, s)$, where s' is a countable union of separated open intervals with rational endpoints.

Our next theorem shows that after an ω_2-iteration of $\mathbb{M}_{\mathbb{R}^*}$ we get $\mathfrak{h}\,(\mathbb{R}^*) = \aleph_2$. The rest of the section will show that the forcing $\mathbb{M}_{\mathbb{R}^*}$ preserves towers in $\Re^*$ and in the next section we conclude that despite $\mathfrak{h}\,(\mathbb{R}^*) = \aleph_2$ there is a tree π-base for $\mathbb{R}^*$ without cofinal branches.

In the following proof we use the concept of almost disjoint family. A family of unbounded regular open sets E is *almost disjoint* (with respect to the ideal $\overline{\text{cpt}}$) if $U \cap V \in \overline{\text{cpt}}$ whenever U and V are distinct elements of E. A *maximal almost disjoint family* of unbounded regular open sets is a maximal element in the collection of all the almost disjoint families with the containment order.

Theorem 2.4. *Let $V \vDash \mathsf{CH}$, and let $\mathbb{P}_{\omega_2}$ be an iteration of modified Mathias forcing. Then $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{h}\,(\mathbb{R}^*) = \aleph_2$.*

PROOF: It suffices to show that every collection $\{E_\gamma : \gamma < \omega_1\}$ of maximal almost disjoint families of unbounded regular open sets in $V^{\mathbb{P}_{\omega_2}}$, there is an unbounded regular open set x almost contained in a member of each E_γ. Let $\{\dot{E}_\gamma : \gamma < \omega_1\}$ be a sequence of names for such almost disjoint families and fix a $p \in \mathbb{P}_{\omega_2}$. Let $G = G_{\omega_2}$ be $\mathbb{P}_{\omega_2}$-generic, with $G_\alpha = G \restriction \alpha$ $\mathbb{P}_\alpha$-generic for all $\alpha < \omega_2$. Let B_α denote the α-th generic subset. We will find $\alpha < \omega_2$ such that $p \Vdash$ "B_α is almost contained in a member of each E_γ".

Given a $\mathbb{P}_{\omega_2}$-name $\dot{x}$ and $p \in \mathbb{P}_{\omega_2}$ such that $p \Vdash \dot{x} \in \Re^*$, there is some $\alpha = \alpha\,(x) < \omega_2$ such that $\alpha < \omega_2$ and $p \Vdash \dot{x} \in V\,[G_\alpha]$. Let each $\dot{E}_\gamma$ be enumerated as a sequence of $\mathbb{P}_{\omega_2}$-names $\{\dot{x}_{\xi\gamma} : \xi < \omega_2\}$, where we have assumed for convenience that

$$p \Vdash (\forall\, \gamma < \omega_1)\left(\left|\dot{E}_\gamma\right| = \aleph_2\right).$$

Define $f : \omega_2 \to \omega_2$ by $f(\beta) = \sup\{\alpha(\dot{x}_{\xi\gamma}) : \xi < \beta,\ \gamma < \omega_1\}$. Thus for every $\beta < \omega_2$ and $\gamma < \omega_1$, we have $p \Vdash \{\dot{x}_{\xi\gamma} : \xi < \beta\} \in V\left[G_{f(\beta)}\right]$. Since $V\,[G_\beta] \vDash \mathsf{CH}$ for $\beta < \omega_2$, define $g : \omega_2 \to \omega_2$ such that

$$p \Vdash (\exists\, \xi < g(\beta))\left(\overline{\dot{y} \cap \dot{x}_{\xi\gamma}} \text{ is not compact}\right),$$

for every $\beta < \omega_2$, every $\gamma < \omega_1$, and every $\mathbb{P}_\beta$-name $\dot{y}$ such that $p \Vdash \dot{y} \in \Re^*$.

Finally, choose $\alpha < \omega_2$ so that $\operatorname{cf}(\alpha) = \omega_1$ and for every $\beta < \alpha$, $f(\beta) < \alpha$ and $g(\beta) < \alpha$. We claim that α is as desired. First, the definition of f implies that $p \Vdash \{\dot{x}_{\xi\gamma} : \xi < \alpha\} \in V\,[G_\alpha]$ for all $\gamma < \omega_1$. Secondly, the preservation of

ω_1 implies that every regular open subset in $V[G_\alpha]$ in fact lies in some $V\left[G_\beta\right]$ with $\beta<\alpha$. By definition of g, each $\left\{\dot{x}_{\xi\gamma}:\xi<\alpha\right\}$, $\gamma<\omega_1$, is a maximal almost disjoint family in $V[G_\alpha]$. Now, for each $\gamma<\omega_1$, $\left\{\dot{x}_{\xi\gamma}:\xi<\alpha\right\}$ is a maximal almost disjoint family in $\Re^*$, therefore one of its elements must be a member of $\dot{G}$, the $\Re^*$-generic filter. Therefore B_α must be contained in a member of each $\left\{\dot{x}_{\xi\gamma}:\xi<\alpha\right\}$, as we wanted to show. □

Theorem 2.5. *Let $\{x_\alpha:\alpha<\kappa\}$ be a tower in $\Re^*$. Then it remains a tower in $V^{\mathbb{M}_{\mathbb{R}^*}}$.*

Proof: Suppose not, and let $\dot{x}$ be an $\mathbb{M}_{\mathbb{R}^*}$-name so that $(A_0,s_0)\Vdash$ "$\dot{x}\in\Re^*$ and $(\forall\,\alpha<\kappa)\,(\dot{x}\subseteq^* x_\alpha)$" for some $(A_0,s_0)\in\mathbb{M}_{\mathbb{R}^*}$. Without loss of generality assume $(A_0,s_0)\Vdash$ "$\dot{x}\subseteq\mathbb{R}^+\setminus\mathbb{N}=\emptyset$ and $\dot{x}\cap[n,n+1)$ is a finite union of separate open intervals with rational endpoints, for each $n\in\mathbb{N}$". We will need some lemmas.

Lemma 2.6. *Suppose $D\subseteq\mathbb{M}_{\mathbb{R}^*}$ is a dense open subset and that $(A,s)\in\mathbb{M}_{\mathbb{R}^*}$. Then there exists some regular open $t\subseteq s$ such that if $(C,u)\le(A,t)$ and $(C,u)\in D$, then there is $m\in\mathbb{N}$ such that $(C,t\setminus[0,m])\in D$.*

Proof: By induction construct sequences $\{b_n:n\in\omega\}$ and $\{t_n:n\in\omega\}$ of separate open intervals with rational endpoints and elements of $\Re^*$. Suppose that $\{b_n:n\le k\}$ and t_k are already constructed. Let $\mathcal{B}_n=\{B_i:i\in\omega\}$ be the family of all C extending A and such that $C\setminus A\subseteq\bigcup_{i=0}^k b_i$. Construct regular open subsets $u_{i+1}\subseteq u_i\subseteq t_k$ $(i\in\omega)$ as follows: Suppose the set u_i is given; if there exists $v\subseteq u_i$ such that $(B_{i+1},v)\in D$, then $u_{i+1}=v$; otherwise $u_{i+1}=u_i$.

Finally let $t_{k+1}\in\Re^*$ be such that $t_{k+1}\subseteq^* u_i$, for all $i\in\omega$, and let $b_{k+1}\subseteq\bigcap_{i=0}^{k+1}t_i$ be an interval with rational endpoints. Then $t=\bigcup_{n\in\omega}b_n$ has the required properties.

Indeed, suppose $(C,u)\le(A,t)$ and $(C,u)\in D$. Let $n\in\omega$ be such that $C\setminus A\subseteq\bigcup_{i=0}^n b_i$. Then $C=B_{j+1}$, for some $j\in\omega$ and $B_{j+1}\in\mathcal{B}_n$. Note that $u\subseteq^* t\subseteq^* t_{j+1}\subseteq^* u_i$. Thus we can choose $m\in\mathbb{N}$ such that $v=u\setminus[0,m]\subseteq u_i$ and so that $(C,t\setminus[0,m])\in\mathbb{M}_{\mathbb{R}^*}$. As $(C,v)\in D$ we have chosen some $u_{i+1}\subseteq u_i$ such that $(C,u_{i+1})\in D$. It follows that $(C,t\setminus[0,m])\in D$. □

The previous lemma says that $\mathbb{M}_{\mathbb{R}^*}$ has the *pure decision property*; that is to say: whenever φ is a sentence of the forcing language for $\mathbb{M}_{\mathbb{R}^*}$ and $(A,s)\in\mathbb{M}_{\mathbb{R}^*}$, there is a regular open set $t\subseteq s$ such that, if $(C,u)\le(A,t)$ and (C,u) decides φ, then there is $m\in\mathbb{N}$ such that $(C,t\setminus[0,m])$ also decides φ. Another consequence of Lemma 2.6 is the following.

Lemma 2.7. *Let $(A,s)\Vdash\dot{x}\in\Re^*$ be given. Then there is $(A,t)\le(A,s)$ so that if $(B,u)\le(A,t)$ and (B,u) decides that an open interval with rational endpoints $b\subseteq\dot{x}$, then there exists $m\in\mathbb{N}$ so that $(B,t\setminus[0,m])$ decides $b\subseteq\dot{x}$.*

We may now suppose that our original (A_0,s_0) has the property of the (A,t) in the conclusion of Lemma 2.7. We then make the following definition. Denote by

$\mathcal{I}$ the family of all intervals with rational endpoints, and for all $(A, s) \le (A_0, s_0)$, let

$$F_A = \{b \in \mathcal{I} : (\exists\, m \in \mathbb{N})\, ((A, s_0 \setminus [0, m]) \Vdash b \subseteq \dot{x})\}$$
$$= \{b \in \mathcal{I} : (\exists\, (A, s) \le (A_0, s_0))\, ((A, s) \Vdash b \subseteq \dot{x})\}\,.$$

Our previous lemma implies that F_A only depends on A, it does not change as s_0 becomes smaller.

Lemma 2.8. *There is $(A, s) \le (A_0, s_0)$ such that $\bigcup F_B$ is bounded for every $(B, t) \le (A, s)$.*

Proof: Otherwise we would have that for every $(A, s) \le (A_0, s_0)$ there is some $(B, t) \le (A, s)$ such that $\bigcup F_B$ is unbounded. There are only countably many F_B's so we may find x_α such that $\bigcup F_B \not\subseteq^* x_\alpha$ for all possible B for which $\bigcup F_B$ is unbounded. Now choose $(A, s) \le (A_0, s_0)$ such that $(A, s) \Vdash$ "$\dot{x} \setminus [0, n] \subseteq x_\alpha$ for some $n \in \mathbb{N}$". Also choose $(B, t) \le (A, s)$ so that $\bigcup F_B$ is unbounded, and select an interval $b \in \mathcal{I}$ with

$$b \subseteq \bigcup F_B \setminus ([0, n] \cup x_\alpha)\,.$$

Then $(A, s \setminus [0, m]) \Vdash$ "$b \subseteq \dot{x} \setminus [0, n]$ and $b \not\subseteq x_\alpha$", a contradiction. □

Simplify again by assuming that our original (A_0, s_0) has the property of (A, s) in the statement of Lemma 2.8. Moreover, we have assumed that (A_0, s_0) forces that $\dot{x} \cap [0, n]$ is a finite union of separate open intervals. To simplify even more, without loss of generality, let us assume further that (A_0, s_0) forces that $\dot{x} \cap [n, n+1)$ is at most one interval for every $n \in \mathbb{N}$. Then, under these assumptions, we have that $\bigcup F_B$ is a finite union of separate open intervals, for every $(B, t) \le (A_0, s_0)$. Indeed, let $m \in \mathbb{N}$ be the least natural such that $\bigcup F_B \subseteq [0, m]$ and fix $1 \le k \le m$. Put $q_0^k = \inf \bigcup F_B \cap [k-1, k]$ and $q_1^k = \bigcup F_B \cap [k-1, k]$. We claim that the interval $\left(q_0^k, q_1^k\right)$ is equal to $\bigcup F_B \cap [k-1, k]$; in other words that $\left(q_0^k, q_1^k\right) \in F_B$. If not, there is no unbounded regular open set $s \subseteq s_0$ such that $(B, s) \Vdash \left(q_0^k, q_1^k\right) \subseteq \dot{x}$. Thus given $s \subseteq s_0$, there is an extension (C, t) of (B, s) and some open interval $b' \subseteq \left(q_0^k, q_1^k\right)$ such that $(C, t) \Vdash b' \cap \dot{x} = \emptyset$.

On the other hand, by the choice of q_0^k and q_1^k there are intervals $b_0, b_1 \in F_B$ such that q_0^k is less than every member of b_0, every member of b_0 is less than every member of b', every member of b' is less than every member of b_1 and every member of b_1 is less than q_1^k. For b_0 and b_1, there are $n_0, n_1 \in \mathbb{N}$ such that $(B, s_0 \setminus [0, n_0]) \Vdash b_0 \subseteq \dot{x}$ and $(B, s_0 \setminus [0, n_1]) \Vdash b_1 \subseteq \dot{x}$. Letting $n = \max\{n_0, n_1\}$ we get that $(C, t \setminus [0, n])$ forces that $\dot{x} \cap [k-1, k]$ is the union of at least two disjoint intervals, contradicting the fact that (A_0, s_0) forces that $\dot{x} \cap [k-1, k]$ is at most one interval.

Lemma 2.9. *Suppose* (A_0, s_0) *is as assumed. Then there is* $(A_0, t) \leq (A_0, s)$ *such that for all* $(B, u) \leq (A_0, t)$*, there is some* $m \in \mathbb{N}$ *such that* $(B, t \setminus [0, m]) \Vdash \bigcup F_B \subseteq \dot{x}$*. Thus,*

$$F_B = \{b \in \mathcal{I} : (\exists\, m \in \mathbb{N})\, ((B, t \setminus [0, m]) \Vdash b \subseteq \dot{x})\}\,.$$

Proof: We will construct a sequence of open intervals with rational endpoints $a_0, a_1, \dots, a_n, \dots$ and a $\subseteq^*$-decreasing sequence of unbounded regular open sets s_n in such a way that if t is the union of the a_i's then t satisfies the conclusion of the lemma.

Suppose we have succeeded constructing $a_0, a_1, \dots, a_{n-1}$ and an unbounded regular open set s_n such that for every (B, u) with $A_0 \subseteq B \subseteq A_0 \cup \bigcup_{i<n} a_i$ and $u \subseteq^* s_n$ we have that $(B, s_n \setminus [0, m]) \Vdash \bigcup F_B \subseteq \dot{x}$, for some $m \in \mathbb{N}$. Let a_n be an open interval with rational endpoints contained in $\bigcap_{i \leq n} s_i \setminus [0, \sup a_{n-1} + 1]$. Now let $\{B_i\}_{i\in\omega}$ be an enumeration of all possible finite unions of open intervals with rational endpoints B such that $A_0 \subseteq B \subseteq A_0 \cup \bigcup_{i\leq n} a_i$. We are going to construct $s_n^{B_i} \subseteq s_n \setminus [0, \sup a_n + 1]$ such that $(B_i, s_n^{B_i}) \Vdash \bigcup F_{B_i} \subseteq \dot{x}$.

Suppose we have $s_n^{B_i}$ and we want to get $s_n^{B_{i+1}}$. Firstly, we know that $\bigcup F_{B_{i+1}}$ is bounded; let $m_{B_{i+1}} \in \mathbb{N}$ such that $\bigcup F_{B_{i+1}} \subseteq \left[0, m_{B_{i+1}}\right]$. By the remarks before the lemma, there is an extension (B_{i+1}, v) of $(B_{i+1}, s_n^{B_i})$ such that $(B_{i+1}, v) \Vdash \dot{x} \cap \left[0, m_{B_{i+1}}\right] = d_1 \cup \dots \cup d_{m_{B_{i+1}}}$. Since $d_j \in F_{B_{i+1}}$, for each $j \leq m_{B_{i+1}}$, there is some $n_j \in \mathbb{N}$ such that $(B_{i+1}, s_n^{B_i} \setminus [0, n_i]) \Vdash d_j \subseteq \dot{x}$. Then let $n_{B_{i+1}} = \sup a_n + \max\{n_j : 1 \leq j \leq m_{B_{i+1}}\} + 1$. Then

$$(B_{i+1}, s_n^{Bi} \setminus \left[0, n_{B_{i+1}}\right]) \Vdash \bigcup F_{B_{i+1}} \subseteq \dot{x}.$$

Letting $s_n^{B_{i+1}} = v \cap s_n^{Bi} \setminus \left[0, n_{B_{i+1}}\right]$ we finish with the inductive step in the construction of the $s_n^{B_i}$'s. Now let $s_{n+1} \subseteq^* s_n^{B_i}$, for every $i \in \omega$. This completes the inductive construction of the s_n's.

Finally let $t = \bigcup_{n\in\omega} a_n$. Then t is clearly an unbounded regular open set. To verify that t satisfies the conclusion of the lemma, let $(B, u) \leq (A_0, t)$ and let n be the least natural number such that $A_0 \subseteq B \subseteq A \cup \bigcup_{i\leq n} a_i$. Then considering the enumeration of all possible B's at stage n of our construction we have that $B = B_i$ for some $i \in \omega$. Then $(B_i, s_{n+1}) \Vdash \bigcup F_{B_i} \subseteq \dot{x}$ and since $t \subseteq^* s_{n+1}$ there is some $m \in \mathbb{N}$ such that $(B_i, t \setminus [0, m]) \Vdash F_B \subseteq \dot{x}$. □

Now we will be able to finish the proof of Theorem 2.5. Assume (A_0, s_0) satisfies the conclusion of Lemmas 2.8 and 2.9, so that if $(B, s) \leq (A_0, s_0)$ then $\bigcup F_B$ is bounded and $(B, s) \Vdash \bigcup F_B \subseteq \dot{x}$.

First build a condition $(A_0, s) \leq (A_0, s_0)$ such that, for all $(B, t) \leq (A_0, s)$, one of the following two cases holds:

(i) $\left(\overline{B} \cap \overline{a} = \emptyset \Rightarrow F_{B\cup a} = F_B\right)$, for all intervals with rational endpoints $a \subseteq s$, or

(ii) there exists $\alpha < \kappa$ such that for all $j \in \mathbb{N}$ there is some $m \in \mathbb{N}$ such that $\bigcup F_{B\cup a} \setminus (x_\alpha \cup [0, j]) \neq \emptyset$, for all intervals with rational endpoints $a \subseteq s \setminus [0, m]$.

For this construction use again recursion as before. Suppose that, at stage n, we have defined $a_0, \ldots, a_{n-1}$ and $s_n \in \Re^*$ such that either (i) or (ii) holds for every $(B, t) \leq (A_0, s_n)$ with $B \setminus A_0 \subseteq \bigcup\{a_0, \ldots, a_{n-1}\}$. Take one interval from s_n and call it a_n, and consider sequentially all possible B with $B \setminus A_0 \subseteq \bigcup\{a_0, \ldots, a_{n-1}, a_n\}$. Let $\{B_i\}_{i\in\omega}$ be an enumeration of all the possible B's. We will construct a regular open sets $s_n^{B_i} \supseteq s_n^{B_{i+1}}$ for every $i \in \omega$. Suppose we have taken care of B_i. For B_{i+1}, one of the following holds:

Case 1. $\bigcup\left\{\bigcup F_{B_{i+1}\cup a} : a \subseteq s_n^{B_i}\right\}$ is bounded. In this case we select an unbounded $s_n^{B_{i+1}} \subseteq s_n^{B_i}$ such that

$$\bigcup\left\{F_{B_{i+1}\cup a} : a \subseteq s_n^{B_{i+1}}\right\} = F_{B_{i+1}},$$

to make (i) hold for B_{i+1}. First, it follows from 2.9 that $F_{B_{i+1}} \subseteq F_{B_{i+1}\cup a}$ for $a \subseteq s_n^{B_i}$. Now suppose $\bigcup\bigcup\left\{F_{B_{i+1}\cup a} : a \subseteq s_n^{B_i}\right\} = d_0 \cup \cdots \cup d_k$. For each $j \leq k$ we have that either there exists $s' \subseteq s_n^{B_i}$ such that $d_j \in F_{B_{i+1}\cup a}$ if and only if $a \subseteq s_n^{B_i} \setminus s'$ and $s_n^{B_i} \setminus s'$ is bounded in which case just replace $s_n^{B_i}$ by s', or else $v = \bigcup\left\{a \subseteq s_n^{B_i} : b \in F_{B_{i+1}\cup a}\right\}$ is unbounded. In this latter case

$$(B_{i+1} \cup a, v \setminus [0, \sup a]) \Vdash d_j \subseteq \dot{x}$$

for every $a \subseteq v$, and so $(B_{i+1}, v) \Vdash d_j \subseteq \dot{x}$; that is, $d_j \in F_{B_{i+1}}$. Thus we replace $s_n^{B_i}$ by v to take care of d_j. In summary, after $k+1$ replacements we can get $s_n^{B_{i+1}}$ with the desired property.

Case 2. $\bigcup\left\{\bigcup F_{B_{i+1}\cup a} : a \subseteq s_n^{B_i}\right\}$ is unbounded. Arrange for (ii) to hold. Choose $\alpha < \kappa$ so as that set is not almost contained in x_α. Now choose $s_n^{B_{i+1}} \subseteq s_n^{B_i}$ so that for every $j \in \mathbb{N}$, if $a \subseteq s_n^{B_{i+1}}$ then there is an interval disjoint from $[0, j]$ which is contained in $\bigcup F_{B_{i+1}\cup a} \setminus x_\alpha$. Note that if $(B, t) \leq \left(B, s_n^{B_{i+1}}\right)$ and $j \in \mathbb{N}$, there is an extension of (B, t) which forces some interval b disjoint from $[0, j]$ into $\dot{x} \setminus x_\alpha$.

Let s_{n+1} be almost contained in these $s_n^{B_i}$, with $i \in \omega$, and then let $s = \bigcup_{n\in\omega} a_n$. This completes the construction. It can be checked that either (i) or (ii) holds.

Now choose $\alpha < \kappa$ larger than any required for (ii). Fix $(A,t) \leq (A_0, s)$ and $j \in \mathbb{N}$ such that $(A,t) \Vdash \dot{x} \setminus [0,j] \subseteq x_\alpha$. We again have two cases.

Case (a). For all $(B,u) \leq (A,t)$, (i) holds. Then $F_{B\cup b} = F_B$ for all $b \subseteq t$. By induction on the length of the union forming B, we have $F_B = F_A$ for all $(B,u) \leq (A,t)$. Hence $(A,t) \Vdash \dot{x} \subseteq \bigcup F_A$ contradicting the assumption that $\dot{x}$ was forced to be unbounded.

Case (b). There is $(B,u) \leq (A,t)$ so that (ii) holds. Then we can find $b \subseteq t$ so that there is some $b \in F_{B\cup b}$ which is disjoint from $x_\alpha \cup [0,j]$ and hence (B,t) has an extension which forces $b \subseteq \dot{x} \setminus (x_\alpha \cup [0,j])$ contradicting the choice of α, j and (A,t). □

3. Countable support preservation

Now we prove that a countable support iteration of length ω_2 of $\mathbb{M}_{\mathbb{R}^*}$ does not fill towers. When Baumgartner and Dordal show this for Mathias forcing they modified the support since they did not have at hand the very useful Lemma 1.13 of Shelah [She84]. Shelah's lemma is very general and applies to multiple situations.

Definition 3.1. Let R be a binary relation on ω^ω. Then $\mathcal{F} \subseteq \omega^\omega$ will be said to be *R-bounding* if and only if for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ such that $g\, R\, f$.

Typical examples of R and $\mathcal{F}$ are: (1) $f\, R\, g$ holds if and only if

$$|\{n \in \omega : f(n) \geq g(n)\}| < \aleph_0.$$

In this case a family $\mathcal{F} \subseteq \omega^\omega$ is R-bounding if and only if it is dominating.
(2) $f\, R\, g$ holds if and only if $|\{n \in \omega : f(n) < g(n)\}| < \aleph_0$. In this case a family $\mathcal{F} \subseteq \omega^\omega$ is R-bounding if and only if it is unbounded.

Definition 3.2. Let R be a binary relation on ω^ω, $\mathcal{F} \subseteq \omega^\omega$, $M \subseteq \omega^\omega$, and $f \in \mathcal{F}$. Then $\mathsf{G}(M, R, f)$ is the game of length ω with the following rules:

(1) at stage n of the game Player I chooses $g_n \in M$ such that $g_n \restriction k_{n-1} = g_{n-1} \restriction k_{n-1}$ (where $k_{-1} = 0$);
(2) at stage n of the game Player II then chooses $k_n \in \omega$.

Player II is declared to be the winner of the game if

$$\bigcup \{g_n \restriction k_n : n \in \omega\}\; R\, f.$$

The idea is best understood if R is the usual $\not<^*$ relation and the g_n's are thought of as approximations to a generic g. Player II is demanding a decision from the forcing out to k_{n+1} where this is chosen large enough so that f is bigger at least one more time. There are difficulties in general with the meaning of R and of Player II's strategy when passing from one model of set theory to a larger one.

Definition 3.3. If R is a binary relation on ω^ω, let $R(\cdot, f) = \{g : g\, R\, f\}$. The pair $(\mathcal{F}, R)$ will be said to be *nice* if and only if

(1) $\mathcal{F}$ is R-bounding;
(2) for every $M \in [\mathcal{F}]^{\aleph_0}$ there is some $f \in \mathcal{F}$ such that Player II has a winning strategy in the game $\mathsf{G}(\bigcup \{R(\cdot, g) : g \in M\}, R, f)$;
(3) Player II's strategy remains a winning strategy in any extension of the universe in which $\mathcal{F}$ remains R-bounding.

The following lemma of Shelah (see Lemma 1.13 in [She84, p. 189] for the original proof or Lemma 3.13 of [She98, p. 313]) shows that if $(\mathcal{F}, R)$ is nice then in order to show that a countable support iteration of proper forcing notions preserves that $\mathcal{F}$ is R-bounding, one need only be concerned with the successor stages of the construction. The case of limit stages of uncountable cofinality are easily handed by Claim III 4.1B(2) of [She98, p. 120].

Lemma 3.4. *Let $\mathbb{P}$ be the countable support iteration of proper forcing notions $\mathbb{P}_n$ and suppose that $(\mathcal{F}, R)$ is nice and that $\mathbb{P}_n \Vdash$ "$\mathcal{F}$ is R-bounding" for each $n \in \omega$. Then $\mathbb{P} \Vdash$ "$\mathcal{F}$ is R-bounding".*

Now we are going to apply this lemma for our purposes. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{(p, q) \subseteq \mathbb{R} : p, q \in \mathbb{Q},\ n < p < q\} \cup \{\emptyset\}\,.$$

We agree that $\{I^n_m : m \in \omega\}$ enumerates $\mathcal{U}_n$. $\prod_{n \in \mathbb{N}} \mathcal{U}_n$ generates a subalgebra $\mathcal{A}$ of $\Re^*$. Basically $\mathcal{A}$ is built up from countable unions of finite unions of separate intervals from each $\mathcal{U}_n$. To each $f \in \omega^{\leq \omega}$ we can assign an element $O(f)$ of $\mathcal{A}$ given by

$$O(f) = \operatorname{int} \overline{\bigcup \left\{ I^n_{f(n)} : n \in \operatorname{dom} f \right\}}.$$

Then define $R \subseteq \omega^\omega \times \omega^\omega$ by $f\, R\, g$ if and only if $O(f) \not\subseteq^* O(g)$. Any family $\mathcal{F} \subseteq \omega^\omega$ which generates a tower in $\mathcal{A}$ (or $\Re^*$) is R-bounding: if $g \in \omega^\omega$, there must be $f \in \mathcal{F}$ such that $O(g) \not\subseteq^* O(f)$. Moreover, the pair $(\mathcal{F}, R)$ is nice. Given $M \in [\mathcal{F}]^{\aleph_0}$, we can find $f \in \mathcal{F}$ so that $O(f) \subseteq^* O(g)$ for all $g \in M$. Now, a winning strategy for Player II in the game

$$\mathsf{G}\left(\bigcup \{R(\cdot, g) : g \in M\}, R, f\right)$$

is as follows: At stage n, Player I chooses $h_n \in \omega^\omega$ so that $h_n \restriction k_{n-1} = h_{n-1} \restriction k_{n-1}$ and $h_n\, R\, g_n$ for some $g_n \in M$; then Player II only needs to choose $k_n \in \omega$ large enough so that

$$O(h_n \restriction k_n) \setminus O(f) \not\subseteq [0, n]\,.$$

That Player II's strategy remains a winning strategy in any extension of the universe in which $\mathcal{F}$ remains R-bounding is clear.

On the other hand, it is not difficult to convince that any tower in $\mathcal{A}$ can be represented by a family of functions. We already know that our forcing preserves towers from $\mathcal{A}$. Thus, we have the following conclusion.

Lemma 3.5. *If $\mathbb{P}_\delta$ is a countable support iteration (of length δ) of $\mathbb{M}_{\mathbb{R}^*}$, then $\mathbb{P}_\delta$ preserves towers; that is, if $\mathbb{M}_{\mathbb{R}^*} \Vdash$ "$\mathcal{F}$ is R-bounding" and $(\mathcal{F}, R)$ is nice, then $\mathbb{P}_\delta \Vdash$ "$\mathcal{F}$ is R-bounding".*

Theorem 3.6. *$V^{\mathbb{P}_{\omega_2}} \models$ "there is no ω_2-tower in $\mathfrak{R}^* = RO(\mathbb{R})/\overline{\mathrm{cpt}}$".*

PROOF: Suppose $p \Vdash$ "$\langle x_\xi : \xi < \omega_2 \rangle$ is a tower" for some $p \in \mathbb{P}_{\omega_2}$. We may find an $f : \omega_2 \to \omega_2$ so that no element in $\mathfrak{R}^* \cap V^{\mathbb{P}_\beta}$ lies below the sequence $\langle x_\xi : \xi < f(\beta) \rangle$, and so that the sequence $\langle x_\xi : \xi < \beta \rangle$ always lies in $V^{\mathbb{P}_{f(\beta)}}$. Choose a limit $\alpha < \omega_2$ of cofinality ω_1 so that $f(\beta) < \alpha$ whenever $\beta < \alpha$. Then $p \Vdash$ "$\langle x_\xi : \xi < \alpha \rangle$ is a tower in $V^{\mathbb{P}_\alpha}$". We may now show by induction on $\beta \geq \alpha$ using our preservation theorems for single stages and iterations that $\langle x_\xi : \xi < \alpha \rangle$ must remain a tower in $V^{\mathbb{P}_\beta}$ for all $\beta \leq \omega_2$. In particular $\langle x_\xi : \xi < \alpha \rangle$ remains a tower in $\mathbb{P}_{\omega_2}$ in contradiction to the fact that x_α must lie below all these x_ξ. □

Corollary 3.7. *There is a tree π-base for $\mathbb{R}^*$ which has no branch of length ω_2, yet it has height ω_2.*

4. Other properties of $\mathbb{M}_{\mathbb{R}^*}$

Our investigation of $\mathbb{M}_{\mathbb{R}^*}$ was motivated by the question of whether it is possible to have $\mathfrak{h}(\mathbb{R}^* \times \mathbb{R}^*) < \mathfrak{h}(\mathbb{R}^*)$. By the result in the last section $\mathbb{M}_{\mathbb{R}^*}$ seemed hopeful to use it with the methods of Shelah and Spinas [SS00]. Some time later we read their second part [SS98] where they claim that the important properties of Mathias forcing which are essential to their proof are:

(1) Mathias forcing factors into a $\aleph_1$-closed and a σ-centred forcing;
(2) Mathias forcing is Souslin-proper;
(3) every infinite subset of a Mathias real is also a Mathias real;
(4) Mathias forcing does not change the cofinality of any cardinal from above $\mathfrak{h}$ to below $\mathfrak{h}$;
(5) Mathias forcing has the pure decision property and it has the Laver property.

In this section we show that not all of those properties are shared by $\mathbb{M}_{\mathbb{R}^*}$. The first property is Lemma 2.2 and the pure decision property is shown in Lemma 2.6.

Lemma 4.1. *$\mathbb{M}_{\mathbb{R}^*}$ does not change the cofinality of any cardinal above $\mathfrak{h}(\mathbb{R}^*)$ to below $\mathfrak{h}(\mathbb{R}^*)$ over a model of GCH.*

PROOF: Let λ be a cardinal with $\mathrm{cf}(\lambda) \geq \mathfrak{h}(\mathbb{R}^*)$ and let $\kappa < \mathfrak{h}(\mathbb{R}^*)$ be a cardinal, and let us consider a $\mathbb{M}_{\mathbb{R}^*}$-name $\dot{f}$ for a function from κ to λ. Working in

V and using properness, for all $\alpha < \kappa$ we can construct a maximal antichain $\left\{p^\alpha_\beta = \left(A^\alpha_\beta, s^\alpha_\beta\right) : \beta < \mathfrak{c}\right\}$ in $\mathbb{M}_{\mathbb{R}^*}$ and $\left\{X^\alpha_\beta : \beta < \mathfrak{c}\right\}$ such that for all $\beta < \mathfrak{c}$, $A^\alpha_\beta = \emptyset$, $X^\alpha_\beta \in [V]^{\aleph_0} \cap V$ and $p^\alpha_\beta \Vdash \dot{f}(\alpha) \in X^\alpha_\beta$. Then clearly $\mathcal{S}_\alpha = \left\{s^\alpha_\beta : \beta < \mathfrak{c}\right\}$ is a maximal antichain in $\Re^*$. By $\kappa < \mathfrak{h}(\mathbb{R}^*)$, $\{\mathcal{S}_\alpha\}_{\alpha\in\kappa}$ has a refinement, say $\mathcal{S}$. Choose $s \in \mathcal{S}$. For $\alpha < \kappa$ there is $\beta(\alpha)$ such that $s \subseteq^* s^\alpha_{\beta(\alpha)}$. Then clearly $(\emptyset, s) \Vdash \operatorname{ran}(\dot{f}) \subseteq \bigcup\left\{X^\alpha_{\beta(\alpha)} : \alpha < \kappa\right\}$. Thus f cannot be cofinal in λ. □

Definition 4.2. A forcing $\mathbb{P}$ is said to have the *Laver property* if for every $\mathbb{P}$-name $\dot{f}$ for a member of ω^ω, $g \in \omega^\omega \cap V$ and $p \in \mathbb{P}$, if

$$p \Vdash (\forall\, n \in \omega)\left(\dot{f}(n) < g(n)\right),$$

then there exists $H : \omega \to [\omega]^{<\aleph_0}$ and $q \in \mathbb{P}$ such that $H \in V$,

$$(\forall\, n \in \omega)\,(|H(n)| \leq 2^n)\,,$$

$q \leq p$ and

$$q \Vdash (\forall\, n \in \omega)\left(\dot{f}(n) \in H(n)\right).$$

M. Goldstern [Gol93, 6.33] showed that the Laver property is preserved by countable support proper iterations. It is also known that if a forcing has the Laver property then it does not add any Cohen reals (see [BJ95]). In [BH04], B. Balcar and M. Hrušák show that $\mathfrak{h}\left(\mathbb{R}^*\right) \leq \min\{\mathfrak{h}, \operatorname{add}(\mathcal{M})\}$. Since forcing with $\mathbb{M}_{\mathbb{R}^*}$ we have $\mathfrak{h}\left(\mathbb{R}^*\right) = \aleph_2$ then our forcing $\mathbb{M}_{\mathbb{R}^*}$ must add Cohen reals and therefore it does not have the Laver property.

The most used property of Mathias forcing in [SS00] is that of "every infinite subset of a Mathias real is also a Mathias real"; that translated to $\mathbb{M}_{\mathbb{R}^*}$ would be: if $u \in \Re^*$ is $\mathbb{M}_{\mathbb{R}^*}$-generic and $v \in \Re^*$ is such that $v \subset u$, then v is $\mathbb{M}_{\mathbb{R}^*}$-generic as well. But in the case of $\mathbb{R}$ there is too much room to choose subsets. For example, it is easy to see that the set of conditions $(A, s) \in \mathbb{M}_{\mathbb{R}^*}$ where the last interval forming A has dyadic endpoints is dense in $\mathbb{M}_{\mathbb{R}^*}$, and therefore any $\mathbb{M}_{\mathbb{R}^*}$-generic subset of $\mathbb{R}$ must contain intervals with dyadic endpoints. Of course, it trivially contains regular open subsets that are unions of intervals of exclusively irrational endpoints.

We conclude the section showing that the forcing $\mathbb{M}_{\mathbb{R}^*}$ is not only proper but it also satisfies a more restrictive axiom which usually demands more structure over the partial order.

Definition 4.3. $\mathbb{P}$ satisfies *Axiom A* if there is a sequence of orderings $\{\leq_n\}_{n\in\omega}$ such that for all $p, q \in \mathbb{P}$:

(1) $p \leq_0 q$ iff $p \leq q$,

(2) $p \leq_{n+1} q \;\Rightarrow\; p \leq_n q$,
(3) if $\{p_n : n \in \omega\}$ is such that $p_{n+1} \leq p_n$ for all $n \in \omega$, then

$$(\exists\, q \in \mathbb{P})\,(\forall\, n \in \omega)\,(q \leq_n p_n)\,,$$

(4) if I is a pairwise incompatible subset of $\mathbb{P}$, then

$$(\forall\, p \in \mathbb{P})\,(\forall\, n \in \omega)\,(\exists\, q \in \mathbb{P})\,(q \leq_n p)$$

and $\{r \in I : \neg\,(q \perp r)\}$ is countable.

It is not hard to see that (4) is equivalent to

(4′) $(\forall\, p \in \mathbb{P})\,(\forall\, n \in \omega)\Big(p \Vdash \dot{a} \in V \;\Rightarrow\; \Big(\exists\, x \in [V]^{\aleph_0}\Big)\,(\exists\, q \leq_n p)\,(q \Vdash \dot{a} \in x)\Big)$.

Proposition 4.4. *The modification of Mathias forcing satisfies Axiom A.*

Proof: For $n \geq 1$, let $(A,s) \leq_n (B,t)$ if and only if $(A,s) \leq (B,t)$, $A = B$ and $(-n,n) \cap s = (-n,n) \cap t$. Of course, $(A,s) \leq_0 (B,t)$ if and only if $(A,s) \leq (B,t)$. If $\{(A_n, s_n) : n \in \omega\}$ is a sequence such that $(A_{n+1}, s_{n+1}) \leq_n (A_n, s_n)$ for all $n \in \omega$, then setting

$$s = \bigcap_{n\in\omega} s_n = \mathrm{int}(\overline{\bigcap_{n\in\omega} s_n}),$$

we can see that $s \in \Re^*$ and $(A_1, s) \leq_n (A_n, s_n)$ for all $n \in \omega$. $s \in \Re^*$ because s is open, so $s = \mathrm{int}(s) \subseteq \mathrm{int}\,(\overline{s})$. If $x \in \mathrm{int}\,(\overline{s})$, then $x \in (-n,n) \cap \mathrm{int}\,(\overline{s})$, for some $n \in \omega$. Thus we can find an open set $u \subseteq (-n,n) \cap \mathrm{int}\,(\overline{s})$ such that $x \in u$. For any $y \in u$ we have $y \in \overline{s} \subseteq \overline{s}_k$, for any $k \in \omega$. In particular, for $0 \leq k \leq n$, $u \subseteq \overline{s}_k$, which implies $x \in \mathrm{int}\,(\overline{s}_k) = s_k$. So $x \in \bigcap_{k=0}^{n} s_k$. And for $k > n$, since $s_k \cap (-n,n) = s_n \cap (-n,n)$, $x \in s_k$. Thus $x \in \bigcap_{k\in\omega} s_k = s$.

On the other hand, we know that if ϕ is a sentence of the forcing language for $\mathbb{M}_n$ and $(A,s) \in \mathbb{M}_{\mathbb{R}^*}$, then there is a regular open set $t \subseteq s$ so that, if $(B,u) \leq (A,t)$ and (B,u) decides ϕ, then there are rational numbers q_0, q_1 such that $(B, t \setminus [q_0, q_1])$ decides ϕ too. Also, if $p \in \mathbb{M}_{\mathbb{R}^*}$ is such that $p \Vdash \dot{a} \in V$ we can find $q \leq p$ and a countable set x in V such that $q \Vdash \dot{a} \in x$ — $\mathbb{M}_{\mathbb{R}^*}$ is proper. Thus, we can show that if $p \in \mathbb{M}_{\mathbb{R}^*}$, $n \in \omega$ and $p \Vdash \dot{a} \in V$, then there are $x \in [V]^{\aleph_0}$ and $q \leq_n p$ such that $q \Vdash \dot{a} \in x$. □

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