Monotonicity of the maximum of inner product norms

Boris Lavrič

Abstract. Let $\mathbb{K}$ be the field of real or complex numbers. In this note we characterize all inner product norms $p_1, \dots, p_m$ on $\mathbb{K}^n$ for which the norm $x \longmapsto \max\{p_1(x), \dots, p_m(x)\}$ on $\mathbb{K}^n$ is monotonic.

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1. Introduction

Let $\mathbb{K}^n$ be the n-dimensional real or complex vector space of column vectors $x = (x_1, \dots, x_n)^{\mathrm{T}}$, and let $\mathbb{K}^{n,n}$ be the space of all $n \times n$ matrices with entries in $\mathbb{K}$. The space $\mathbb{K}^n$ is endowed with the standard inner product $(x, y) \mapsto y^*x$, where y^* is the conjugate transpose of y, and with the standard vector space topology. If C is a positive definite matrix, the functional $p_C : x \longmapsto (x^*Cx)^{1/2}$ is an inner product norm on $\mathbb{K}^n$. As is well known, each norm on $\mathbb{K}^n$ generated by an inner product is of the form p_C for some positive definite matrix $C \in \mathbb{K}^{n,n}$.

A norm p on $\mathbb{K}^n$ is called *monotonic* if $|x| \leq |y|$ (componentwise) implies $p(x) \leq p(y)$ for all $x, y \in \mathbb{K}^n$, and *absolute* if $p(x) = p(|x|)$ for all $x \in \mathbb{K}^n$. Monotonic norms were introduced in [1] and have been extensively studied. It is well known that monotonicity and absoluteness are equivalent, and easy to see that a norm p is absolute if and only if $p(Dx) \leq p(x)$ for all $x \in \mathbb{K}^n$ and all $D \in \Delta_n(\mathbb{K})$, where $\Delta_n(\mathbb{K})$ denotes the set of all diagonal matrices $D = \mathrm{diag}(d_1, \dots, d_n) \in \mathbb{K}^{n,n}$ such that $|d_i| = 1$ for all i. A list of characterizations of monotonic norms is contained in [2] and [3].

Let $p_1, \dots, p_m$ be norms on $\mathbb{K}^n$. If all p_i are monotonic, then the norm $\max\{p_1, \dots, p_m\}$ is monotonic as well. The converse fails even in case when all p_i are inner product norms. In this paper we characterize all inner product norms $p_1, \dots, p_m$ for which the norm $p = \max\{p_1, \dots, p_m\}$ is monotonic. More precisely, if $p_i = p_{A_i}$ with $A_i \in \mathbb{K}^{n,n}$ positive definite, then we describe all A_i for which p is monotonic. The special case $m = 2$ is considered in [4, Theorem 7], where a similar characterization is obtained with a completely different method that is not applicable to the case $m > 2$.

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2. Results

From now on let $p_i = p_{A_i} : x \longmapsto (x^* A_i x)^{1/2}$, $i = 1, \dots, m$, be given inner product norms on $\mathbb{K}^n$ defined by positive definite matrices $A_i \in \mathbb{K}^{n,n}$, and let p be the norm $p = \max\{p_1, \dots, p_m\}$. For every nonempty $X \subseteq \mathbb{K}^n$ let

$$I(X) = \{i \in \{1, \dots, m\} : p_i(x) = p(x) \ \text{ for all } \ x \in X\},$$

and for each $x \in \mathbb{K}^n$ denote $I(x) = I(\{x\})$. It is clear that the sets $I(x)$ are nonempty. The following auxiliary result gives a useful information about the sets $I(X)$.

Lemma 1. *Let $p = \max\{p_1, \dots, p_m\}$, and let $\mathcal{V}$ be the collection of all nonempty open subsets $V \subseteq \mathbb{K}^n$.*

(a) *For every $U \in \mathcal{V}$ there exists a $V \in \mathcal{V}$ such that $V \subseteq U$ and $I(V)$ is nonempty.*

(b) *If $J = \bigcup_{V \in \mathcal{V}} I(V)$, then $p = \max\{p_j : j \in J\}$.*

PROOF: (a) First, let us show that for every $x_0 \in \mathbb{K}^n$ there exists a neighborhood U_0 of x_0 such that

$$I(x) \subseteq I(x_0) \ \text{ for all } \ x \in U_0. \tag{1}$$

If $i \in \{1, \dots, m\} \setminus I(x_0)$, then $p_i(x_0) < p(x_0)$. The continuity of norms implies that there is a neighborhood U_0 of x_0 such that $p_i(x) < p(x)$ for all $x \in U_0$. Therefore $i \notin I(x)$ for every $x \in U_0$, and hence (1) follows.

Suppose $U \in \mathcal{V}$ does not satisfy (a). Take any $x_1 \in U$ and choose an open neighborhood U_1 of x_1 such that $U_1 \subseteq U$ and

$$I(x) \subseteq I(x_1) \ \text{ for all } \ x \in U_1.$$

If $I(x) = I(x_1)$ for all $x \in U_1$, then $I(U_1) = I(x_1)$, and hence $V = U_1$ satisfies (a). Since by assumption this is not the case, there exists an $x_2 \in U_1$ such that $I(x_2) \subsetneqq I(x_1)$. Choose an open neighborhood U_2 of x_2 such that $U_2 \subseteq U_1$ and

$$I(x) \subseteq I(x_2) \ \text{ for all } \ x \in U_2.$$

Proceeding like before we get an infinite sequence $I(x_1) \supsetneqq I(x_2) \supsetneqq \dots$ Since $I(x_1)$ is finite, this is impossible, hence (a) follows.

(b) Suppose $p(x_0) > \max\{p_j(x_0) : j \in J\}$ for some $x_0 \in \mathbb{K}^n$. Then there exists a $U \in \mathcal{V}$ such that $p(x) > \max\{p_j(x) : j \in J\}$ for all $x \in U$. It follows that $I(V) = \emptyset$ for every $V \in \mathcal{V}$ such that $V \subseteq U$. This contradicts (a), thus $p = \max\{p_j : j \in J\}$. □

The set J in Lemma 1 can be replaced by any minimal subset $M \subseteq \{1, \dots, m\}$ for which $p = \max\{p_i : i \in M\}$. For the proof it suffices to apply Lemma 1 with M instead of $\{1, \dots, m\}$.

If $A \in \mathbb{K}^{n,n}$ is positive definite, let from now on

$$\mathcal{F}_A = \{D^* A D : D \in \Delta_n(\mathbb{K})\}.$$

Lemma 2. *Let $p = \max\{p_1, \dots, p_m\}$, and let J be as in Lemma 1. Then the following statements are equivalent:*

(a) *p is monotonic;*
(b) *$\mathcal{F}_{A_j} \subseteq \{A_1, \dots, A_m\}$ for each $j \in J$.*

Proof: (a) $\Rightarrow$ (b). Suppose (a), and let $j \in J$, $D \in \Delta_n(\mathbb{K})$. Lemma 1 ensures the existence of a nonempty open subset $U_0 \subseteq \mathbb{K}^n$ such that $p_j(x) = p(x)$ for all $x \in U_0$. Since p is monotonic, $p_j(Dx) = p(Dx) = p(x)$ for every $x \in U = D^*(U_0)$. The set U is nonempty and open, hence by Lemma 1 there exists a nonempty open subset $V \subseteq U$ and a $k \in J$ such that $p(x) = p_k(x)$ for all $x \in V$. It follows that $p_j(Dx) = p_k(x)$ and therefore

$$x^* D^* A_j D x = x^* A_k x \quad \text{for all} \quad x \in V.$$

Let us prove that this implies $A_k = D^* A_j D$. Put $A = D^* A_j D - A_k$, notice that $A^* = A$, and take any $x_0 \in V$, $y \in \mathbb{K}^n$. Then there exists a $\delta > 0$ such that for every positive $\epsilon < \delta$ we have $x_0 + \epsilon y \in V$, and therefore $(x_0 + \epsilon y)^* A (x_0 + \epsilon y) = 0$. It is clear that $x_0^* A x_0 = 0$, and hence $x_0^* A y + y^* A x_0 + \epsilon y^* A y = 0$ for every positive $\epsilon < \delta$. It follows that $y^* A y = 0$ for all $y \in \mathbb{K}^n$, thus $A = 0$ and therefore $A_k = D^* A_j D$.

(b) $\Rightarrow$ (a). Suppose (b) and let $x \in \mathbb{K}^n$, $D \in \Delta_n(\mathbb{K})$. Lemma 1(b) ensures that there is some $j \in J$ such that $p(Dx) = p_j(Dx)$. It follows from (b) that there exists a $k \in J$ such that $A_k = D^* A_j D$, hence

$$p_j(Dx) = ((Dx)^* A_j Dx)^{1/2} = (x^* A_k x)^{1/2} = p_k(x) \le p(x).$$

Therefore, $p(Dx) \le p(x)$ for all $x \in \mathbb{K}^n$ and all $D \in \Delta_n(\mathbb{K})$, and hence p is monotonic. □

Lemma 3. *Let $A \in \mathbb{K}^{n,n}$ be positive definite.*

(a) *If $\mathbb{K} = \mathbb{C}$, then $\mathcal{F}_A$ is finite if and only if A is diagonal. Both conditions are equivalent to $\mathcal{F}_A = \{A\}$.*
(b) *If $\mathbb{K} = \mathbb{R}$, then $\mathcal{F}_A$ has $2^{n-\kappa(A)}$ elements, where $\kappa(A)$ is the number of connected components of the directed graph $\Gamma(A)$.*

Proof: (a) If A is diagonal, then $D^* A D = A$ for all $D \in \Delta_n(\mathbb{C})$, and hence $\mathcal{F}_A = \{A\}$.

Suppose that A is not diagonal, and take a nonzero entry a_{ij} of A such that $i \neq j$. Let $(\delta_k)_{k=1}^{\infty}$ be a sequence of different complex numbers of absolute value 1, and let

$$D_k = I_n + (\delta_k - 1) E_{jj} \in \mathbb{C}^{n,n}, \quad k = 1, 2, \dots,$$

where I_n is the identity and E_{jj} is an elementary matrix. Then $D_k \in \Delta_n(\mathbb{C})$ and

$$(D_k^* A D_k)_{ij} = \delta_k a_{ij}, \quad k = 1, 2, \dots,$$

hence $\mathcal{F}_A$ contains an infinite number of different matrices $D_k^* A D_k$.

(b) We shall prove first that the subset

$$\Delta_A = \{D \in \Delta_n(\mathbb{R}) : D^* A D = A\}$$

of $\Delta_n(\mathbb{R})$ has $2^{\kappa(A)}$ elements.

It is clear that a $D = \operatorname{diag}(d_1, \dots, d_n) \in \Delta_n(\mathbb{R})$ satisfies $D^* A D = A$ if and only if $d_i d_j a_{ij} = a_{ij}$ for all $i, j \in \{1, \dots, n\}$. This implies that $D \in \Delta_n(\mathbb{R})$ belongs to Δ_A if and only if

$$d_i = d_j \quad \text{for all} \quad i, j \quad \text{such that} \quad a_{ij} \neq 0.$$

It follows that $d_i \in \{1, -1\}$ depends only on the connected component of $\Gamma(A)$, and that therefore Δ_A has $2^{\kappa(A)}$ elements.

Observe now that Δ_A is a subgroup of the multiplicative group $\Delta_n(\mathbb{R})$. Since for each $D_1, D_2 \in \Delta_n(\mathbb{R})$ we have the equivalence

$$D_1^* A D_1 = D_2^* A D_2 \iff D_1 D_2^{-1} \in \Delta_A,$$

the map $\phi : D \longmapsto D^* A D$ is constant on equivalence classes from the quotient group $\Delta_n(\mathbb{R})/\Delta_A$. It may be easily verified that ϕ generates a bijection $\Delta_n(\mathbb{R})/\Delta_A \longrightarrow \mathcal{F}_A$, hence $\mathcal{F}_A$ has $2^{n-\kappa(A)}$ elements. □

Theorem 4. *The norm $p = \max\{p_1, \dots, p_m\}$ is monotonic if and only if there exists a subset $J \subseteq \{1, \dots, m\}$ such that $p = \max\{p_j : j \in J\}$ and one of the following conditions is satisfied.*

(a) *If $\mathbb{K} = \mathbb{C}$, then A_j is diagonal for every $j \in J$;*

(b) *If $\mathbb{K} = \mathbb{R}$, then $\{A_j : j \in J\}$ is a union of a pairwise disjoint sets of the form $\mathcal{F}_A = \{D^* A D : D \in \Delta_n(\mathbb{R})\}$ each consisting of $2^{n-\kappa(A)}$ elements.*

Proof: Suppose that p is monotonic and put $J = \bigcup_{V \in \mathcal{V}} I(V)$. Then Lemma 2 ensures that $\{A_j : j \in J\}$ is a union of sets of the form $\mathcal{F}_A$, $A \in \{A_1, \dots, A_m\}$. If $\mathbb{K} = \mathbb{C}$, then by Lemma 3(a) each A_j, $j \in J$, is diagonal. If $\mathbb{K} = \mathbb{R}$, then by Lemma 3(b) each $\mathcal{F}_A$ has $2^{n-\kappa(A)}$ elements. It can be easily verified that the sets $\mathcal{F}_{A_i}$ and $\mathcal{F}_{A_j}$ are either equal or disjoint (they are the equivalence classes of $\{A_j : j \in J\}$ for the equivalence relation $B \sim A$ if $B \in \mathcal{F}_A$).

The converse is clear. □

Theorem 4 shows how to form all monotonic norms that are maximum of inner product norms. In the case $\mathbb{K} = \mathbb{C}$ such norms are exactly the norms $p = \max\{p_1, \dots, p_m\}$ with diagonal positive definite $A_1, \dots, A_m$, while in the case $\mathbb{K} = \mathbb{R}$ such norm are the norms $q = \max\{q_1, \dots, q_m\}$ with each q_i of the form $q_i = \max\{p_A : A \in \mathcal{F}_{A_i}\}$ for some positive definite $A_i \in \mathbb{R}^{n,n}$. To prove this observation it suffices to apply Theorem 4 and use the fact that all norms p_i and q_i are monotonic.

The following characterization facilitates to check the monotonicity of the maximum of inner product norms.

Theorem 5. *Let $p = \max\{p_1, \dots, p_m\}$, and let K be the set of all indices $k \in \{1, \dots, m\}$ for which $\mathcal{F}_{A_k} \subseteq \{A_1, \dots, A_m\}$ (if $\mathbb{K} = \mathbb{C}$, then K consists of all indices k for which A_k is diagonal). Then p is monotonic if and only if K is nonempty and*

$$p_i \leq \max\{p_k : k \in K\} \quad \text{for each} \quad i \in \{1, \dots, m\} \setminus K. \tag{2}$$

Proof: First, notice that if $K \neq \emptyset$, then (2) is equivalent to $p = \max\{p_k : k \in K\}$.

Now, suppose that p is monotonic. Then by Lemma 2 $J \subseteq K$, thus K is nonempty. If (2) is not satisfied, take an $x_0 \in \mathbb{K}^n$ such that $p(x_0) > \max\{p_k(x_0) : k \in K\}$. A continuity argument gives an open neighborhood U of x_0 such that

$$p(x) > \max\{p_k(x) : k \in K\} \quad \text{for all} \quad x \in U.$$

Lemma 1 ensures that there exists a nonempty open $V \subseteq U$ such that $I(V) \neq \emptyset$. It follows that each $j \in I(V)$ satisfies

$$p_j(x) = p(x) > \max\{p_k(x) : k \in K\} \quad \text{for all} \quad x \in V.$$

Therefore $j \notin K$, and hence $\mathcal{F}_{A_j} \nsubseteq \{A_1, \dots, A_m\}$. By Lemma 2 this contradicts the monotonicity of p, hence (2) follows.

To show the converse suppose K is nonempty. Then (2) gives $p = \max\{p_k : k \in K\}$, hence Lemma 2 ensures that p is monotonic. □

It follows from Theorem 5 that if $\mathbb{K} = \mathbb{C}$, then A_k is diagonal for each $k \in K$, and that if $\mathbb{K} = \mathbb{R}$, then $2^{n-\kappa(A_k)} \leq m$ for each $k \in K$. If $m \leq 3$ and $k \in K$, then $\kappa(A_k)$ equals n or $n-1$. In the first case A_k is diagonal, while in the second case A_k is of the form $D + E$, where D is diagonal, and

$$E = \lambda(E_{rs} + E_{sr}), \quad \lambda \in \mathbb{R} \setminus \{0\}, \quad r \neq s. \tag{3}$$

For $m = 2$ this implies [4, Theorem 7], while for $m = 3$ we get the following result.

Corollary 6. *The norm $p = \max\{p_1, p_2, p_3\}$ is monotonic if and only if one of the following conditions in which $\{i, j, k\} = \{1, 2, 3\}$ is satisfied:*

(a) *A_1, A_2, A_3 are diagonal;*
(b) *A_i, A_j are diagonal, and $p_k \leq \max\{p_i, p_j\}$;*
(c) *A_i is diagonal, $A_i - A_j$ and $A_i - A_k$ are positive semidefinite;*
(d) *$\mathbb{K} = \mathbb{R}$, $A_i = D + E$, $A_j = D - E$ with D diagonal, E of the form (3), and A_k is diagonal or $p_k \leq \max\{p_i, p_j\}$.*

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Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail: Boris.Lavric@fmf.uni-lj.si

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