A characterization of holomorphic germs on compact perfect sets

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Abstract. Let $K \subseteq \mathbb{C}$ be a perfect compact set, E a quasi-complete locally convex space over $\mathbb{C}$ and $f : K \to E$ a map. In this note we give a necessary and sufficient condition — in terms of differential quotients — for f to have a holomorphic extension on a neighborhood of K.

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Introduction

Assume that $K \subseteq \mathbb{R}^n$ is a compact set, E is a locally convex space (briefly: LC-space) over $\mathbb{R}$ or $\mathbb{C}$, and $f : K \to E$ is a map. It is well known that f has a C^∞ extension if and only if there exists a (non uniquely determined) sequence of maps $f_n : K \to E$, with $f_0 = f$, such that it satisfies appropriate conditions (see [5] and [6]).

In this note we propose a similar criterion for the analytic case, that is, we characterize in terms of an adequate boundedness condition on well specified differential quotients, those maps $f : K \to E$ (where $K \subseteq \mathbb{C}$ is a compact perfect set) which admit a holomorphic extension $\overline{f} : U \to E$ to some neighborhood U of K (Proposition 2.9).

In order to formulate it, we need to fix some notations. Let $U \subseteq \mathbb{C}$ be an open set and E a quasi-complete LC-space. We let $H(U, E)$ denote the space of all holomorphic maps $u : U \to E$, with the topology of uniform convergence on compact subsets of U (for the definitions and basic properties see [1], [3]). If $V \subseteq U$ is another open set, then there is an obvious restriction map $H(U, E) \to H(V, E)$. For a non-void compact set $K \subseteq \mathbb{C}$, we let $\mathcal{U}(K)$ denote the directed set of all open neighborhoods of K. Clearly we obtain a basis (a cofinal subset) of $\mathcal{U}(K)$ by taking the sets $W_r(K) = \{z \in \mathbb{C} : d(z, K) < r\}$ for $r > 0$ (or else $r = \frac{1}{n}, n \geq 1$). Denote by $\mathcal{O}(K, E)$ the space of holomorphic E-valued germs on K, that is the LC-space $\varinjlim_r H(W_r(K), E)$.

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1. Assume that K is a compact perfect set. Let $C(K,E)$ denote the space of all continuous maps $f : K \to E$, endowed with the topology of uniform convergence. It is clear that the natural continuous map $\mathcal{O}(K,E) \to C(K,E)$ is injective. The main aim of this note is to characterize the image of this map.

In the sequel we use the notations $B(a,\epsilon) = \{z \in \mathbb{C} : |z-a| < \epsilon\}$ and $\overline{B(a,\epsilon)} = \{z \in \mathbb{C} : |z-a| \leq \epsilon\}$.

Technical Lemma 1.1. *Let $K \subset \mathbb{C}$ be a compact perfect set. Let $\epsilon_0 > 0$ and let $f : K \to E$ be a map. Then the following assertions are equivalent:*

- (i) *there exists $\epsilon_0 > 0$, such that for all $a \in K$, there exists $f_a \in H(B(a,\epsilon_0),E)$ such that $f_a(z) = f(z)$ for all $z \in K \cap B(a,\epsilon_0)$;*
- (ii) *there exist $\epsilon_1 > 0$ and $g \in H(W_{\epsilon_1}(K),E)$ such that $g(z) = f(z)$ for all $z \in K$.*

Proof: It is clear that (ii)⇒(i). Let us see that (i)⇒(ii). It suffices to show that if $a,b \in K$ and $z \in B(a,\epsilon_0/4) \cap B(b,\epsilon_0/4)$, then $f_a(z) = f_b(z)$. In fact, in this case, $K \cap B(a,\epsilon_0/2) \cap B(b,\epsilon_0/2) \neq \emptyset$, which implies that $K \cap \overline{B(a,\epsilon_0/2)} \cap \overline{B(b,\epsilon_0/2)}$ is an infinite set, since K is a perfect set. From this, it follows that $f_a(z) = f_b(z)$. □

If K satisfies suitable conditions, then we can replace differential quotients by ordinary derivatives in the statement of the criterion. For instance this is the case if K is uniformly C^1-regular. Recall the definition: we say that a perfect set X is C^1-connected when for every $a,b \in X$ there exists a piecewise C^1-curve $\Gamma \subseteq X$, such that $a,b \in \Gamma$. We can define then the geodesic distance $D(a,b)$ in the obvious way. We recall that for an open set X this distance is equivalent to the usual distance $d(a,b) = |a-b|$. Since this fails for general X we say that a compact perfect set is uniformly C^1-regular if D is equivalent to d (see [2] for a similar and more general definition).

On this line of work we use later the following estimate:

Lemma 1.2. *Let $K \subseteq \mathbb{C}$ be a compact set, and $\Gamma \subseteq K$ a piecewise C^1-curve which connects the points $a,z \in K$. Then*

$$\int_\Gamma |\xi - z|^r \, |d\xi| \leq \frac{\operatorname{length}(\Gamma)^{r+1}}{r+1}.$$

Proof: We can assume $z = 0$. If we parametrize Γ by arc length $\varphi : [0,L] \to K$, then $|\varphi'(s)| = 1$. So,

$$|\varphi(t)| = |\varphi(t) - \varphi(0)| \leq \int_0^t |\varphi'(s)| \, ds \leq t,$$

and then

$$\int_\Gamma |\xi|^r \, |d\xi| \leq \int_0^L |\varphi(t)|^r \, dt \leq \int_0^L t^r \, dt = \frac{L^{r+1}}{r+1}.$$

□

2. In the sequel we consider an LC-space E and a infinite compact set $K \subseteq \mathbb{C}$. For each $r \in \mathbb{N}$ we define $K^{(r)} = \{(x_0, \dots, x_{r-1}) \in K^r : \; x_i \neq x_j \text{ for every } i \neq j\}$.

Definition 2.1. We define "differential quotients" $\Delta^r \colon E^K \to E^{K^{(r+1)}}$ inductively for $r \geq 0$ by

(i) Δ^0 is the identity map,

(ii) $\Delta^{r+1}(f)(x_0, \dots, x_{r+1}) = \frac{r+1}{x_0 - x_{r+1}}(\Delta^r(f)(x_0, \dots, x_r) - \Delta^r(f)(x_1, \dots, x_{r+1}))$.

For the real case, some of the following properties can be found in [4].

Lemma 2.2. *The operator Δ^r has the following properties:*

(a) *Δ^r is $\mathbb{C}$-linear;*

(b) *$\Delta^r(f)(x_0, \dots, x_r) = r! \sum_{k=0}^r f(x_k) \prod_{j \neq k} \frac{1}{x_k - x_j}$;*

(c) *$\Delta^r(f)$ is symmetric in $(x_0, \dots, x_r)$;*

(d) *Let F be a LC-space. If $T : E \to F$ is a linear map, then $T(\Delta^r(f)) = \Delta^r(T \circ f)$ for each $r \geq 0$;*

(e) *Let E_i $(i = 1, 2)$ and F be LC-spaces, and $f \in E_1^K$, $g \in E_2^K$. If $B : E_1 \times E_2 \to F$ is a bilinear map, and $\widetilde{B}(f, g) : K \to F$ is the map $z \to B(f(z), g(z))$, then $\sum_{k=0}^r \binom{r}{k} B(\Delta^k(f)(x_0, \dots, x_k), \Delta^{r-k}(g)(x_k, \dots, x_r)) = \Delta^r(\widetilde{B}(f, g))(x_0, \dots, x_r)$;*

(f) *Assume that E is an algebra. Then,*

$$\Delta^r(fg)(x_0, \dots, x_r) = \sum_{k=0}^r \binom{r}{k} \Delta^k(f)(x_0, \dots, x_k) \Delta^{r-k}(g)(x_k, \dots, x_r).$$

Proof: The statements (a)–(e) can be proved by induction on r and (f) is an immediate consequence of (e). □

Definition 2.3. We define

$$\begin{aligned} \mathcal{B}(K, E) &= \{f \in E^K : \text{ for each } r \geq 0 \text{ the set } \frac{\Delta^r(f)(K^{(r+1)})}{r!} \text{ is bounded in } E\} \\ &= \{f \in E^K : \text{ for each } r \geq 0 \text{ the set } \Delta^r(f)(K^{(r+1)}) \text{ is bounded in } E\}. \end{aligned}$$

Proposition 2.4. *The following properties hold:*

(a) *$\mathcal{B}(K, E)$ is a vector subspace of E^K. Moreover, if E is an algebra, then $\mathcal{B}(K, E)$ is a subalgebra of E^K;*

(b) *If K is a perfect set and $f \in \mathcal{B}(K, E)$, then each $\Delta^r(f) \colon K^{(r+1)} \to E$ is uniformly continuous.*

Proof: (a) is obvious. For (b), let p be a continuous seminorm in E and let $M_{r,p} = \sup\{p(\Delta^r(f)(x_0,\dots,x_r)) : (x_0,\dots,x_r) \in K^{(r+1)}\}$. It suffices to prove that

$$(1) \qquad p(\Delta^r(f)(x_0,\dots,x_r) - \Delta^r(f)(y_0,\dots,y_r)) \le \frac{M_{r+1,p}}{r+1}\sum_{j=0}^{r}|x_j - y_j|.$$

This is clear when $x_i \neq y_j$ for every pair (i,j). In fact, in this case,

$$\begin{aligned}
&p(\Delta^r(f)(x_0,\dots,x_r) - \Delta^r(f)(y_0,\dots,y_r))\\
&\le \sum_{j=0}^{r} p(\Delta^r(f)(x_j,\dots,x_r,y_0,\dots,y_{j-1}) - \Delta^r(f)(x_{j+1},\dots,x_r,y_0,\dots,y_j))\\
&= \sum_{j=0}^{r}\frac{x_j - y_j}{r+1} p(\Delta^{r+1}(f)(x_j,\dots,x_r,y_0,\dots,y_j)) \le \frac{M_{r+1,p}}{r+1}\sum_{j=0}^{r}|x_j - y_j|.
\end{aligned}$$

Assume now that $x_{i_0} = y_{j_0}$ for some (i_0,j_0). For $\epsilon > 0$ we can select $(z_0,\dots,z_r) \in K^{(r+1)}$, such that
(a) $\sum_{j=0}^{r}|x_j - z_j| < \epsilon$,
(b) $z_i \neq x_j$ y $z_i \neq y_j$ for all (i,j).
Then, by the previous case we have

$$\begin{aligned}
&p(\Delta^r(f)(x_0,\dots,x_r) - \Delta^r(f)(y_0,\dots,y_r))\\
&\le p(\Delta^r(f)(x_0,\dots,x_r) - \Delta^r(f)(z_0,\dots,z_r))\\
&\quad + p(\Delta^r(f)(z_0,\dots,z_r) - \Delta^r(f)(y_0,\dots,y_r))\\
&\le \frac{M_{r+1,p}}{r+1}\sum_{j=0}^{r}|x_j - z_j| + \frac{M_{r+1,p}}{r+1}\sum_{j=0}^{r}|z_j - y_j|\\
&\le \frac{M_{r+1,p}}{r+1}\Big(2\epsilon + \sum_{j=0}^{r}|x_j - y_j|\Big).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, (1) follows. □

In the sequel we assume that K is a perfect compact subset of the complex plane $\mathbb{C}$, and that E is a quasi-complete LC-space.

Corollary 2.5. *If $f \in \mathcal{B}(K,E)$, then all differential quotients $\Delta^r(f)$ can be extended to uniformly continuous maps $\Delta^r(f)\colon K^{r+1} \to E$.*

We remark that all properties of Lemma 2.2 remain valid for these extensions. Now, we introduce the following notation:

$$f^{(r)}(a) = \Delta^r(f)(a,\dots,a) = \lim_{x_i \to a}\Delta^r(f)(x_0,\dots,x_r) \quad \text{for each } \ a \in K.$$

Remark 2.6. A map $f : K \to E$ is said to be differentiable in K if for every $a \in K$, there exists $f'(a) = \lim_{z \to a} \dfrac{f(z) - f(a)}{z - a}$ $(z \in K)$ (see [2] for instance). From the previous discussion it follows that f has derivatives of any order when $f \in \mathcal{B}(K, E)$.

Lemma 2.7. *Let $a_0, a_1, \ldots, a_r, z \in K$. We have:*
(2)
$$f(z) = \sum_{i=0}^{r} \frac{\Delta^i(f)(a_0, \ldots, a_i)}{i!} \prod_{j=0}^{i-1} (z - a_j)^i + \frac{\Delta^{r+1}(f)(a_0, \ldots, a_r, z)}{(r+1)!} \prod_{j=0}^{r} (z - a_j)^{r+1}.$$

In particular,

$$f(z) = \sum_{i=0}^{r} \frac{f^{(i)}(a)}{i!}(z - a)^i + \frac{\Delta^{r+1}(f)(a, \ldots, a, z)}{(r+1)!}(z - a)^{r+1}, \tag{3}$$

for all $a, z \in K$. Moreover, if we can connect the points $a, z \in K$ by a piecewise C^1-curve $\Gamma \subseteq K$, then

$$\frac{\Delta^{r+1} f(a, \ldots, a, z)}{(r+1)!}(z - a)^{r+1} = \int_\Gamma \frac{(z - \xi)^r}{r!} f^{(r+1)}(\xi)\, d\xi. \tag{4}$$

Proof: (2) follows by induction on r and (4) follows by induction on r using integration by parts. □

Lemma 2.8. *Let U be open neighborhood of K. If $f : U \to E$ is a holomorphic map, then $f_{|K} \in \mathcal{B}(K, E)$ and there exists $R > 0$ such that for every continuous seminorm p in E we have*

$$\sup\{p(\Delta^r(f)(x_0, \ldots, x_r))\frac{R^r}{r!} : (x_0, \ldots, x_r) \in K^{r+1}\} < \infty,$$

for every $r \geq 0$.

Proof: Since for some $\delta > 0$ we have $W_\delta(K) \subseteq U$, we can take an adequate cycle $C \subseteq W_\delta(K)$ such that

$$\frac{\Delta^r(f)(x_0, \ldots, x_r)}{r!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{\prod_{j=0}^{r}(z - x_j)}\, dz,$$

for every $(x_0, \ldots, x_r) \in K^{r+1}$. If we set $R = \mathrm{dist}(K, C)$ and $L = \mathrm{length}(C)$ then for every continuous seminorm p in E we have $p(\Delta^r(f)(x_0, \ldots, x_r)) \leq \frac{r!}{2\pi} \frac{M_p L}{R^{r+1}}$ where $M_p = \sup\{p(f(z)) : z \in W_\delta(K)\}$, which finishes the proof. □

Conversely we have

Proposition 2.9. *Let $f : K \to E$ be a map. The following assertions are equivalent:*

(a) *there exist $R > 0$ and a holomorphic map $g : W_R(K) \to E$, such that $g_{|K} = f$.*

(b) *there exist $R > 0$ and a bounded set $B \subseteq E$ such that $\frac{\Delta^r(f)(x_0,\dots,x_r)}{r!}R^r \in B$ for all $r \geq 0$ and all $(x_0,\dots,x_r) \in K^{(r+1)}$.*

If K is uniformly C^1-regular, (a), (b) *are equivalent to*

(c) *$f \in \mathcal{B}(K,E)$ and there exists $R > 0$, such that the sequence $\frac{f^{(r)}}{r!}R^r$ $(r \geq 0)$ is bounded in $C(K,E)$.*

Proof: Clearly (a)⇒(b) by the previous lemma. If (b) holds, then $f \in \mathcal{B}(K,E)$ and we can define holomorphic maps $f_a : B(a,R) \to E$ by setting $f_a(z) = \sum_{j\geq 0} \frac{f(j)}{j!}(z-a)^j$ for each $a \in K$. Now (3) of Lemma 2.7 shows that $f_{a|B(a,R)\cap K} = f_{|B(a,R)\cap K}$ for each $a \in K$, hence Technical Lemma 1.1 gives (a).

On the other hand, it is obvious that always (b)⇒(c). Conversely, assume (c) and that K is uniformly C^1-regular, and let p any continuous seminorm in E. Then, for some $M_p > 0$, the inequality $p(\frac{f^{(r)}(\xi)}{r!}) \leq \frac{M_p}{R^r}$ holds for every $\xi \in K$ and $r \geq 0$. Hence, if Γ is a piecewise C^1-curve in K with origin a and final point z, then, by Lemma 1.2,

$$p\Big(\int_\Gamma \frac{(\xi-z)^r}{r!} f^{(r+1)}(\xi)\,d\xi\Big) \leq \frac{M_p}{R^{r+1}}\mathrm{length}(\Gamma)^{r+1}.$$

Since this inequality holds for every such Γ, (4) of Lemma 2.7 gives

$$p\Big(\frac{\Delta^{r+1}f(a,\dots,a,z)}{(r+1)!}\Big)|z-a|^{r+1} \leq M_p\left(\frac{D(a,z)}{R}\right)^{r+1}.$$

But $D(a,z) \leq C|a-z|$ for some $C > 0$. Now, taking $R_1 < \min\{R/C, R\}$, we obtain

$$p\Big(\frac{\Delta^{r+1}f(a,\dots,a,z)}{(r+1)!}\Big)|z-a|^{r+1} \leq M_p\left(\frac{|z-a|}{R_1}\right)^{r+1},$$

for all $a, z \in K$ and $r \geq 0$.

Finally if we define $f_a : B(a,R_1) \to E$ for each $a \in K$, as at the beginning of the proof, then (3) of Lemma 2.7 shows that $f(z) = f_a(z)$ for $z \in B(a,R_1) \cap K$. Hence the technical lemma applies again. □

Definition 2.10. Let $K \subseteq \mathbb{C}$ be a perfect compact set, E a quasi-complete LC-space and $R > 0$. We define $\Delta(R,K,E)$ as the subspace of $\mathcal{B}(K,E)$ of all maps $f : K \to E$ such that $\bigcup_{r\geq 0} \frac{\Delta^r(f)(K^{(r+1)})}{r!}R^r$ is bounded in E. We also define $J(R,K,E)$ as the subspace of $\mathcal{B}(K,E)$ of all maps $f : K \to E$ such that $\bigcup_{r\geq 0} \frac{f^{(r)}(K)}{r!} R^r$ is bounded.

Clearly, from Remark 2.6 follows that $\Delta(R,K,E) \subseteq J(R,K,E)$ and if K is uniformly C^1-regular these subspaces are equal. Furthermore, Proposition 2.9 has the following corollary:

Corollary 2.11. *The following assertions are true:*

(a) $\mathcal{O}(K,E) = \bigcup_{R>0} \Delta(R,K,E)$;
(b) *if K is uniformly C^1-regular, then* $\mathcal{O}(K,E) = \bigcup_{R>0} J(R,K,E)$.

Example 2.12. Let $\Omega = \widehat{\mathbb{C}} - K$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. We let $H(\Omega)$ denote the Fréchet space of all holomorphic maps $g : \Omega \to \mathbb{C}$ such that $g(\infty) = 0$. It is well known that $H(\Omega)$ can be identified with the strong dual of $\mathcal{O}(K)$ ([3]).

We define $\phi : K \to H(\Omega)$ by $\phi(a)(\xi) = (\xi - a)^{-1}$ ($\xi \in \Omega, a \in K$). It is easy to see that $\Delta^1(\phi)(a_0,a_1) = \phi(a_0)\phi(a_1)$, and in general we have $\Delta^r(\phi)(a_0,\dots,a_r) = r! \prod_{i=0}^r \phi(a_i)$.

Now we define the sequence H_m ($m \geq 1$) of compact sets $H_m = \{z \in \Omega : d(z,K) \geq \frac{1}{m}\}$. The seminorms $p_m(g) = \sup\{|g(z)| : z \in H_m\}$ form a fundamental system of seminorms in $H(\Omega)$. We let E_m denote the Banach space obtained by completing $H(\Omega)$ with respect to p_m, so that $H(\Omega) \simeq \varprojlim E_m$ via the maps $i_m : H(\Omega) \to E_m$. Then, we have for each m, r

$$p_m\Big(\frac{\Delta^r(\phi)(a_0,\dots,a_r)}{r!}\Big) = p_m\Big(\prod_{i=0}^{r} \phi(a_i)\Big) = m^{r+1}. \tag{5}$$

Hence, for a fixed m, the map $i_m \circ \phi : K \to E_m$ can be extended to a holomorphic map in a neighborhood K, thanks to Proposition 2.9. Nevertheless, the same formula (5) and the same argument shows that it is impossible to get an extension of $\phi : K \to H(\Omega)$. Consequently, $\mathcal{O}(K,H(\Omega)) \neq \varprojlim \mathcal{O}(K,E_m)$, since $\phi \in \varprojlim \mathcal{O}(K,E_m)$ but $\phi \notin \mathcal{O}(K,H(\Omega))$. Note that ϕ is a "virtual holomorphic map", in the terminology of [3].

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