Bases of minimal elements of some partially ordered free abelian groups

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Abstract. In the present paper, we will show that the set of minimal elements of a full affine semigroup $A \hookrightarrow \mathbb{N}_0^k$ contains a free basis of the group generated by A in $\mathbb{Z}^k$. This will be applied to the study of the group $\mathrm{K}_0(R)$ for a semilocal ring R.

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1. Introduction

A subsemigroup A of $\mathbb{N}_0^k$ is called *full affine* if and only if for any $a \in A$ and $b \in \mathbb{N}_0^k$, if $a + b \in A$ then $b \in A$. We may define a partial order on A by $a \leq c$ if and only if there is $b \in A$ such that $a + b = c$. Clearly, this order on A coincides with the order inherited from $\mathbb{N}_0^k$. In this paper, we will show that any full affine semigroup A contains in its set of minimal elements a free basis of the group generated by A in $\mathbb{Z}^k$.

Following [4], we will denote by $\mathcal{S}_\oplus(\mathcal{P}\text{-Mod } R)$ the semigroup of isomorphism classes of finitely generated projective modules over an associative ring with unit R (see Section 3). The result proved in this paper was motivated by the purpose to find a weak form of Krull-Schmidt theorem for this class of modules over semilocal rings. Indeed, A. Facchini and D. Herbera in [3] and [4], proved that the natural semigroup homomorphisms $\mathcal{S}_\oplus(\mathcal{P}\text{-Mod } R) \to \mathcal{S}_\oplus(\mathcal{P}\text{-Mod } R/J(R))$, where R is a semilocal ring, are precisely the order-unit embeddings of full affine semigroups into $\mathbb{N}_0^k$. We will use just the "only if" part of this statement to prove the weak form of Krull-Schmidt theorem over semilocal rings in Theorem 3.2.

2. Construction of a free basis

In the following, for an abelian partially ordered group G, $\leq$ and $\geq$ will mean the given partial order and G^+ will denote the positive cone of G, i.e. the set $\{g \in G;\ g \geq 0\}$.

Lemma 2.1. *Let G be a partially ordered abelian group such that G^+ is a finitely generated semigroup. Then the order satisfies descending chain condition (d.c.c.) on the positive cone.*

Proof: Let $g_1, \dots, g_k$ be a set of non-zero generators for the monoid G^+. One can easily see that in any infinite subset of $\mathbb{N}_0^k$ there can be found two different elements $(a_1, \dots, a_k)$, $(b_1, \dots, b_k)$ satisfying $a_i \le b_i,\ i = 1, \dots, k$. It follows that every non-zero element of G^+ can be expressed as a sum of $g_1, \dots, g_k$ in only finitely many ways otherwise we would have $a_1 g_1 + \dots + a_k g_k = b_1 g_1 + \dots + b_k g_k$ for some non-negative integers $a_i,\ b_i$ such that $a_i \le b_i$ for all $1 \le i \le k$ and $a_j < b_j$ for some $1 \le j \le k$. But then $g_j \le 0$, a contradiction.

Suppose that the lemma does not hold. Then we can find an infinite strictly decreasing chain $h_0 > h_1 > \dots$ of elements of G^+. So there are some non-zero $h'_1, h'_2, \dots$ in G^+ such that $h_i = h_{i+1} + h'_{i+1},\ i = 0, 1, \dots$. Then $h_0 = h'_1 + h_1 = h'_1 + h'_2 + h_2 = \dots = h'_1 + \dots + h'_i + h_i = \dots$. We can express each $h'_i,\ h_i$ as a sum of $g_1, \dots, g_k$ and we see that h_0 can be expressed as a sum of generators in infinitely many ways — a contradiction. □

Lemma 2.2. *Let G be a partially ordered abelian group such that the order on G^+ satisfies d.c.c. Then G^+ is generated (as a submonoid of G) by its minimal elements.*

Proof: Suppose that G^+ is not generated by its minimal elements. Let H be the submonoid of G^+ generated by minimal elements of G^+. Since the order on G^+ satisfies d.c.c., $G^+ \setminus H$ has a minimal element h, which cannot be minimal in G^+, so there is some $0 < h' < h$. This means that $h = h' + g$ for some $g \in G^+$ and, by minimality of h, both g and h' are in H, hence so is h, a contradiction. □

Corollary 2.3. *Let G be a partially ordered abelian group. Then G^+ is a finitely generated semigroup if and only if the order on G^+ satisfies d.c.c. and G^+ contains only finitely many minimal elements.*

We will say that an element $g \in G^+$ is an *order-unit* of G^+ if for all $h \in G^+$ there is a positive integer n such that $ng \ge h$.

Lemma 2.4. *Let G be a partially ordered group such that G^+ is a finitely generated semigroup and has at least two minimal elements. Suppose that for any distinct minimal elements x, y of G^+ and any positive integers m, n, $mx \ne ny$. Then G^+ contains a non-zero element that is not an order-unit.*

Proof: We proceed by induction on the number of minimal elements of G^+.
I. G^+ has just two minimal elements. If $mx > y$ for some $m \in \mathbb{N}$, then for the least such m, mx would be a multiple of y.

II. Let $g_1, \dots, g_k$ be all minimal elements of G^+ and $k \ge 3$. Suppose that all of them are order-units. For distinct integers $1 < i, j \le k$ we can find non-negative integers $a_i, b_1, b_2, \dots, b_k,\ c_1, c_2 \dots, c_k$ such that

$$a_i g_i = \sum_{n=1}^{k} b_n g_n,\ \ a_i > 0, b_1 > 0,$$

$$c_1 g_1 = \sum_{n=2}^{k} c_n g_n,\ \ c_1 > 0, c_j > 0.$$

Let us multiply the first equation by c_1 and the second by b_1 and add together. We get $dg_i = \sum_{n=2}^{k} d_n g_n$, $d > 0$ and $d_j > 0$. Now let us equip G with order $\leq'$ whose positive cone is the monoid generated by $g_2, \dots, g_k$. It follows that all (minimal) elements of $(G, \leq')$ are order-units. This contradicts the induction hypothesis. □

A partial order is called *unperforated* if for all $g \in G$ and all positive integers n, if $ng \in G^+$ then $g \in G^+$.

Corollary 2.5. *Let G be an unperforated partially ordered abelian group such that G^+ is a finitely generated semigroup and has at least two minimal elements. Then G^+ contains a non-zero element that is not an order-unit.*

Proof: Let G be an unperforated partially ordered abelian group. Let x, y be two distinct minimal elements of G^+. Suppose that there are positive integers m, n such that $mx = ny$. For instance, let $m \leq n$. Then $0 = mx - ny \leq n(x - y)$. Since G is unperforated, we have $x - y \in G^+$, and so $y \leq x$, a contradiction. Now apply Lemma 2.4. □

The given partial order on G is called *directed* if for all $g, h \in G$ there is $k \in G$ such that $k \geq g$ and $k \geq h$, or, equivalently, $G = G^+ - G^+$.

If I is a directed convex subgroup of G then I is called an *ideal* of G.

Notation. Let G be a partially ordered abelian group. Let us denote by M the set of minimal elements of G^+, by $\mathcal{M}$ the power set of M and by $\mathcal{I}$ the set of ideals of G.

Lemma 2.6. *Let G be a partially ordered abelian group such that G^+ is a finitely generated semigroup. Then the assignment $\varphi : \mathcal{I} \to \mathcal{M}$ defined by $\varphi(I) = I^+ \cap M$ is an injective map. Moreover every ideal I is generated by $\sum_{m \in \varphi(I)} m$.*

Proof: Let us define the map from $\mathcal{M}$ to the set of subgroups of G defined by $\psi(A) = \langle A \rangle$, the subgroup generated by A, for any $A \subset M$. For any $I \in \mathcal{I}$ we have $\psi\varphi(I) \subset I$. Now, let $i \in I^+$. By assumption i is a sum of some minimal elements of G^+ say $i = m_1 + \cdots + m_k$. Every m_j for $1 \leq j \leq k$ is in I because I is convex. Thus $I^+ \subset \psi\varphi(I)$. Since I is directed we have $I = \langle I^+ \rangle \subset \psi\varphi(I)$. Thus $I = \psi\varphi(I)$ and φ is injective.

For the rest of the proof, let J be the ideal generated by $i = \sum_{m\in\varphi(I)} m$. Obviously $J \subset I$. Since J is convex we have $\varphi(I) = \varphi(J)$ and so $J = \psi\varphi(J) = \psi\varphi(I) = I$. □

We have seen that if G^+ is finitely generated, then every ideal of G is generated by some element of G^+. For any $g \in G^+$ let I_g denote an ideal generated by g. It can be easily seen that I_g is a group generated by the set $\{h \in G^+;\ h \le ng$ for some $n \in \mathbb{N}\}$. Hence, if the order on G is directed, then $I_g = G$ if and only if g is an order-unit of G^+.

Lemma 2.7. *Let G be an unperforated partially ordered abelian group, and let I be an ideal of G. Then G/I with the factor order is also an unperforated group.*

Proof: See [5, Proposition 1.20]. □

Theorem 2.8. *Let G be a directed unperforated partially ordered abelian group such that G^+ is a finitely generated semigroup. Then G is free and has a free basis of minimal elements of G^+.*

Proof: G is obviously a finitely generated torsion free group and hence it is free. Let us proceed by induction on the number of minimal elements of G^+. If G^+ has only one minimal element, this element is a generator of G.

Let us suppose that G^+ has minimal elements $g_1, \dots, g_k$. By Corollary 2.5 the set $S = \{I_h;\ h \in G^+, h \neq 0, I_h \neq G\}$ is nonempty. According to Lemma 2.6 the cardinality of S is bounded by the cardinality of $\mathcal{M}$ and so the system S is finite. Let I_g be the maximal element of S with respect to inclusion. Since $\varphi(I_g) \neq \{g_1, \dots, g_k\}$ we can apply the induction step. Thus there are $h_1, \dots, h_l \in \varphi(I_g)$ which form some free basis of I_g.

Now we claim that $\forall h \in G^+ \setminus I_g$ we have $\mathbb{Z}h \cap I_g = 0$. Suppose $nh = h_1 - h_2$ for some $h_1, h_2 \in I_g^+$ and n positive integer. Then $0 \le h \le nh \le h_1$ and $h \in I_g$, a contradiction.

By Lemma 2.7, G/I_g with the factor order is directed and unperforated and has finitely generated positive cone. Suppose that the cone has at least two minimal elements. By Corollary 2.5 there is some $g' \in G^+$ such that $g' + I_g$ is a non-zero non-order-unit element of $(G/I_g)^+$. So $I_g \subset I_{g+g'}$ and $I_{g+g'} \neq G$ since otherwise $g+g'$ would be an order-unit in G^+ and $g'+I_g = (g+g')+I_g$ would be an order-unit in $(G/I_g)^+$. But I_g was the maximal element of S, a contradiction. So the cone $(G/I_g)^+$ has only one minimal element, say $h+I_g$. By Lemma 2.1 and Lemma 2.2 we see that G/I_g is generated by $h+I_g$. Let us set $A = \{x \in G^+;\ x+I_g = h+I_g\}$. The set A is a nonempty subset of $G^+\setminus I_g$ and, by Lemma 2.1, it has some minimal element h_0. Suppose that there is $0 < x < h_0$. We have $0 \le x + I_g \le h_0 + I_g$ in the factor, hence by minimality of h_0 and $h + I_g$ we have $x \in I_g$. We see that h_0 has to be minimal in G^+, otherwise it would be a sum of elements strictly smaller then h_0 and thus it would be an element of I_g. Now it can be easily seen that the minimal elements $h_0, h_1, \dots, h_l$ form a free basis of G. □

3. Application

By the following we shall see that Theorem 2.8 indeed concerns full affine semigroups.

Lemma 3.1. *Let G be a directed partially ordered group. If G^+ is isomorphic to a full affine semigroup then G is unperforated and G^+ is a finitely generated semigroup.*

Proof: G is generated (as a group) by G^+ and so we may assume that G^+ is a full affine subsemigroup of $\mathbb{N}_0^k$ and G is the subgroup of $\mathbb{Z}^k$ generated by G^+. If $ng \in G^+$ for $g \in G$ and n positive, then $g \in \mathbb{N}_0^k$, hence $g \in G \cap \mathbb{N}_0^k = G^+$.

It remains to show, due to Corollary 2.3, that the order on G^+ satisfies d.c.c. and that it has only finitely many minimal elements. But the order on G^+ coincides with the product order inherited from $\mathbb{N}_0^k$ and, in this order, any subset of $\mathbb{N}_0^k$ has the required properties. □

On the other hand, if a partially ordered group G is unperforated and G^+ is a finitely generated semigroup, then G^+ is isomorphic to a full affine semigroup. For details see [1] where such semigroups are also called normal semigroups.

Recall that the ring R is called semilocal if $R/J(R)$ is a semisimple ring. Let us briefly recall the construction of the semigroup $\mathcal{S}_\oplus(\mathcal{P}\text{-Mod } R)$ for a ring R (for details see [4]). The elements of $\mathcal{S}_\oplus(\mathcal{P}\text{-Mod } R)$ are the isomorphism classes of finitely generated projective modules over R. Now we will define a semigroup structure on this set by $[P]+[Q] = [P \oplus Q]$. For a semilocal ring R, $\mathcal{S}_\oplus(\mathcal{P}\text{-Mod } R)$ is a cancellative semigroup and so it can be embedded into its group of fractions $\mathrm{K}_0(R)$. The order on $\mathrm{K}_0(R)$ is given by $\mathrm{K}_0(R)^+ = \mathcal{S}_\oplus(\mathcal{P}\text{-Mod } R)$. Clearly, $[R]$ is an order-unit of $\mathrm{K}_0(R)^+$ and $[P]$ is a minimal element of $\mathrm{K}_0(R)^+$ if and only if P is indecomposable. It can be shown (see e.g. [3], [4]) that if R is semilocal, then the partially ordered group $\mathrm{K}_0(R)$ can be embedded into the partially ordered group $(\mathbb{Z}^k, \mathbb{N}_0^k)$, where k is the cardinality of the representative set of simple $R/J(R)$ modules.

Hence, using Theorem 2.8 and Lemma 3.1, if R is a semilocal ring then $\mathrm{K}_0(R)$ is a free group and the set of minimal elements of $\mathrm{K}_0(R)^+$ contains a free basis of $\mathrm{K}_0(R)$, in other words:

Theorem 3.2. *Let R be a semilocal ring. There are non-zero finitely generated indecomposable projective R-modules, say $P_1, \dots, P_k$, such that for any finitely generated projective module Q there exist unique numbers $n_1, \dots, n_k$, $m_1, \dots, m_k \in \mathbb{N}_0$ such that $n_i m_i = 0$ for $i = 1, \dots, k$, and*

$$Q \oplus P_1^{(n_1)} \oplus \cdots \oplus P_k^{(n_k)} \simeq P_1^{(m_1)} \oplus \cdots \oplus P_k^{(m_k)}.$$

Let M be a module (over any ring) and let S be the endomorphism ring of M. It is a well known fact (see e.g. [2]) that the categories $proj-S$ of finitely generated projective modules over S and $add\,M$ of direct summands of finite direct sums of M are equivalent. Thus using Theorem 3.2 we obtain:

Corollary 3.3. *Let M be a module with semilocal endomorphism ring. There exist non-zero indecomposable modules in $add\,M$, say $M_1, \dots, M_k$, such that if Q is any module in $add\,M$ then there are unique non-negative integers $n_1, \dots, n_k$, $m_1, \dots, m_k$ such that $n_i m_i = 0$ for $i = 1, \dots, k$, and*

$$Q \oplus M_1^{(n_1)} \oplus \cdots \oplus M_k^{(n_k)} \simeq M_1^{(m_1)} \oplus \cdots \oplus M_k^{(m_k)}.$$

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