

On minimal- α -spaces

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Abstract. An α -space is a topological space in which the topology is generated by the family of all α -sets (see [N]). In this paper, minimal- $\alpha\mathcal{P}$ -spaces (where \mathcal{P} denotes several separation axioms) are investigated. Some new characterizations of α -spaces are also obtained.

Keywords: α -space, αT_i -space, minimal- αT_i space, T_2 -closed space, minimal- T_2 space, ψ -space

Classification: Primary 54A10, 54D25; Secondary 54A05, 54D80

1. Introduction

The family of all topologies on a set X is a complete atomic lattice. There has been a considerable amount of interest in topologies which are minimal or maximal in this lattice with respect to certain topological properties.

Given a topological property \mathcal{P} , we say that a topology on a set X is \mathcal{P} -*minimal* if every weaker topology on X does not possess property \mathcal{P} .

Throughout this paper, the word “space” will mean topological space, the topology on a space X is denoted by $\tau(X)$, int_τ and cl_τ (or int_X and cl_X when no confusion is possible about the topology on X) will denote respectively the interior and the closure operators with respect to $\tau(X)$ and if σ is a topology on the underlying set of X , then σ is called an *expansion* (respectively a *compression*) of $\tau(X)$ if $\tau(X) \subseteq \sigma$ (resp. $\sigma \subseteq \tau(X)$).

A subset R of a space X is called *regular open* if $\text{int}_\tau(\text{cl}_\tau(R)) = R$. The family of all regular open sets of X is denoted by $RO(X)$ and forms a base for a topology $\tau_s(X)$ on X which is a compression of $\tau(X)$ and it is called the *semiregularization* of X . We say that X is *semiregular* if $\tau_s(X) = \tau(X)$.

The notion of an α -set was introduced in 1965 by Njåstad [N]. Given a space X , we say that $A \subseteq X$ is an α -*set* if $A \subseteq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))$. It is easy to prove that the family $\alpha\tau(X)$ of all α -sets of X is a topology on X which will be called the α -*topology* induced by $\tau(X)$ and it consists of all the subsets A of X such that there exists some open set $U \in \tau(X)$ such that $U \subseteq A \subseteq \text{int}_\tau(\text{cl}_\tau(U))$. The members of $\alpha\tau(X)$ will be called the α -*open* sets of X while their complements

* This research was supported by a grant of cofin. 2001 “Strutture Geometriche, Combinatorie e loro Applicazioni” and from the C.N.R. through I.N.D.A.M. (Italy).

will be called the α -closed sets of X . Evidently, every open set is an α -set and hence the α -topology is an expansion of $\tau(X)$. We say that X is an α -space if $\tau(X) = \tau_\alpha(X)$.

Obviously, the partial ordered set of all α -topologies contain both a maximum (the discrete topology) and a minimum (the trivial topology) element. If \mathcal{P} is a topological property, we denote by $\alpha\mathcal{P}$ the class of α -spaces which satisfy property \mathcal{P} . In this paper, we investigate the minimal- $\alpha\mathcal{P}$ property with particular regard to the properties $\mathcal{P} = T_0, T_1, T_2$ and $T_{\frac{1}{2}}$.

2. Basic facts on α -spaces

The α -topology $\alpha\tau(X)$ of a space $(X, \tau(X))$ has some interesting similarities with the notion of *semiregularization* $\tau_s(X)$ (see, for example, [PW]).

Lemma 1 ([N]). *For any α -open set A and any α -closed set C of a space X , we have:*

- (1) $\text{cl}_{\alpha\tau}(A) = \text{cl}_\tau(A)$;
- (2) $\text{int}_{\alpha\tau}(C) = \text{int}_\tau(C)$;
- (3) $\text{int}_{\alpha\tau}(\text{cl}_{\alpha\tau}(A)) = \text{int}_\tau(\text{cl}_\tau(A))$.

Proposition 2 ([N]). *For any space X , we have $\alpha(\alpha\tau(X)) = \alpha\tau(X)$.*

PROOF: Evidently $\alpha\tau(X) \subseteq \alpha(\alpha\tau(X))$. Let $B \in \alpha(\alpha\tau(X))$. Then, there is some $A \in \alpha\tau(X)$ such that $A \subseteq B \subseteq \text{int}_{\alpha\tau}(\text{cl}_{\alpha\tau}(A))$. So, there exists some $U \in \tau(X)$ such that $U \subseteq A \subseteq \text{int}_\tau(\text{cl}_\tau(U))$. Hence, $\text{int}_\tau(\text{cl}_\tau(U)) = \text{int}_\tau(\text{cl}_\tau(A))$ and, by Lemma 1(2), we have

$$U \subseteq A \subseteq B \subseteq \text{int}_{\alpha\tau}(\text{cl}_{\alpha\tau}(A)) = \text{int}_\tau(\text{cl}_\tau(A)) = \text{int}_\tau(\text{cl}_\tau(U)).$$

This proves that $B \in \alpha\tau(X)$. □

Definition 1 ([N]). A space X is called an α -space if $\tau(X) = \alpha\tau(X)$ or, equivalently, if $\tau(X) = \alpha(\sigma(X))$ for some topology $\sigma(X)$ on X .

Proposition 3. *Let $(X, \tau(X))$ be a space. The following are equivalent:*

- (1) X is an α -space;
- (2) $\tau(X) = \{U \cup \{p\} : U \in \tau(X), p \in \text{int}(\text{cl}(U))\}$;
- (3) $\{U \cup \{p\} : U \in \tau(X) \text{ is dense in } X \text{ and } p \in X\} \subseteq \tau(X)$.

PROOF: (1) \Rightarrow (2) Evidently, $\tau(X) \subseteq \{U \cup \{p\} : U \in \tau(X), p \in \text{int}_\tau(\text{cl}_\tau(U))\}$ (it suffices to take $p \in U$). Conversely, let $U \in \tau(X)$ and $p \in \text{int}_\tau(\text{cl}_\tau(U))$. Then $U \subseteq U \cup \{p\} \subseteq U \cup \text{int}_\tau(\text{cl}_\tau(U)) = \text{int}_\tau(\text{cl}_\tau(U))$. Since X is an α -space, $U \cup \{p\} \in \tau(X)$.

(2) \Rightarrow (3) Obvious because, when $\text{cl}_\tau(U) = X$, every $p \in \text{int}_\tau(\text{cl}_\tau(U)) = X$.

(3) \Rightarrow (1) Suppose that $\{U \cup \{p\} : U \in \tau(X) \text{ is dense in } X \text{ and } p \in X\} \subseteq \tau(X)$ and let $A \in \alpha\tau(X)$, i.e. $A \subseteq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))$. The set $D = \text{int}_\tau(A) \cup \text{int}_\tau(X \setminus \text{int}_\tau(A))$ is open and dense because

$$\begin{aligned} \text{cl}(D) &= \text{cl}_\tau(\text{int}_\tau(A)) \cup \text{cl}_\tau(\text{int}_\tau(X \setminus \text{int}_\tau(A))) \\ &= \text{cl}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \cup \text{cl}_\tau(X \setminus \text{cl}_\tau(\text{int}_\tau(A))) \\ &= \text{cl}_\tau(\text{cl}_\tau(\text{int}_\tau(A)) \cup (X \setminus \text{cl}_\tau(\text{int}_\tau(A)))) \\ &= \text{cl}_\tau(X) \\ &= X. \end{aligned}$$

So, for any $p \in A$, by hypothesis we have that $D \cup \{p\} \in \tau(X)$.

Now, we consider the open set:

$$\begin{aligned} W_p &= (D \cup \{p\}) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \\ &= (\text{int}_\tau(A) \cup \text{int}_\tau(X \setminus \text{int}_\tau(A)) \cup \{p\}) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \\ &= (\text{int}_\tau(A) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))) \\ &\quad \cup (\text{int}_\tau(X \setminus \text{int}_\tau(A)) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))) \\ &\quad \cup (\{p\} \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))) \\ &= \text{int}_\tau(A) \cup \{p\} \end{aligned}$$

as

$$\begin{aligned} &(\text{int}_\tau(X \setminus \text{int}_\tau(A)) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))) \\ &= (X \setminus \text{cl}_\tau(\text{int}_\tau(A))) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \\ &\subseteq (X \setminus \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) = \emptyset. \end{aligned}$$

Thus, for every $p \in A$, $W_p = \text{int}_\tau(A) \cup \{p\} \in \tau(X)$ and so, also the union $\bigcup_{p \in A} W_p = \text{int}_\tau(A) \cup A = A$ is an open set of X . This proves that $\alpha\tau(X) \subseteq \tau(X)$ and hence that X is an α -space. \square

In [N] it is proved the following:

Proposition 4. *A space X is an α -space if and only if all the nowhere dense sets are closed sets.*

The following proposition improves the previous one.

Proposition 5. *A space (X, τ) is an α -space if and only if every nowhere dense set is discrete.*

PROOF: Suppose that X is an α -space and that N is a nowhere dense subset of X . By Proposition 4, N is a closed set and so $X \setminus N$ is an open dense set. For any $p \in N$, by Proposition 3(3), the set $U = (X \setminus N) \cup \{p\}$ is open in X . Since $U \cap N = \{p\}$, it follows that N is discrete.

Conversely, let U be an open dense set of X and $p \in X$. Then $X \setminus U$ is a nowhere dense set and by hypothesis, it is discrete. So, if $p \in X \setminus U$, there exists some $V \in \tau$ such that $V \cap (X \setminus U) = \{p\}$. Hence $U \cup \{p\} = U \cup V \in \tau(X)$. Since the case when $p \in U$ is trivial, by Proposition 3(3), it is proved that X is an α -space. □

It is shown in [Lo] that the operator α is not monotonic, i.e. that, in general, for two topologies $\tau(X)$ and $\sigma(X)$ on a set X , $\tau(X) \subseteq \sigma(X)$ does not imply $\alpha\tau(X) \subseteq \alpha\sigma(X)$. However, we have the following:

Lemma 6. *Let $\tau(X)$ and $\sigma(X)$ be topologies on a set X such that $\tau(X) \subseteq \sigma(X)$ and $\tau_s(X) = \sigma_s(X)$. Then $\alpha\tau(X) \subseteq \alpha\sigma(X)$.*

PROOF: Let $A \in \alpha\tau(X)$. Then there exists some $U \in \tau(X)$ such that $U \subseteq A \subseteq \text{int}_\tau(\text{cl}_\tau(U))$. So, being

$$\text{int}_\sigma(\text{cl}_\sigma(U)) = \text{int}_{\sigma_s}(\text{cl}_{\sigma_s}(U)) = \text{int}_{\tau_s}(\text{cl}_{\tau_s}(U)) = \text{int}_\tau(\text{cl}_\tau(U)),$$

we have $U \subseteq A \subseteq \text{int}_\sigma(\text{cl}_\sigma(U))$ with $U \in \tau \subseteq \sigma$, that is $A \in \alpha\sigma(X)$. □

Let us recall that a space $(X, \tau(X))$ is called:

- T_2 -closed (resp. $T_{2\frac{1}{2}}$ -closed) if it is closed in every Hausdorff (resp. $T_{2\frac{1}{2}}$) space containing X as a subspace,
- minimal- T_2 (resp. minimal- $T_{2\frac{1}{2}}$) if it is a T_2 - (resp. $T_{2\frac{1}{2}}$ -) space and there is no strictly coarser T_2 (resp. $T_{2\frac{1}{2}}$) topology on the same set X .

The following properties are well-known and will be used later.

Proposition 7. *A Hausdorff space is T_2 -closed if and only if every open ultrafilter on X is fixed.*

Proposition 8. *A space $(X, \tau(X))$ is Hausdorff if and only if its semiregularization $(X, \tau_s(X))$ is Hausdorff.*

Corollary 9. *A space $(X, \tau(X))$ is T_2 -closed if and only if its semiregularization $(X, \tau_s(X))$ is minimal- T_2 .*

Proposition 10. *Every regular closed subspace of a T_2 -closed space is T_2 -closed.*

Proposition 11. *A Hausdorff space X is minimal- T_2 if and only if it is semi-regular and T_2 -closed.*

Proposition 12. *If $\sigma(X)$ and $\tau(X)$ are two topologies on a set X such that $\tau_s(X) \subseteq \sigma(X) \subseteq \tau(X)$ then $\tau_s(X) = \sigma_s(X)$.*

PROOF: Let R be a regular open set of $(X, \tau(X))$. Then there exists some $U \in \tau(X)$ such that $R = \text{int}_\tau(\text{cl}_\tau(U))$. So, $R \in \tau_s(X) \subseteq \sigma(X)$. Since $\tau_s(X) \subseteq \sigma(X)$ and using a well-known property of the closure of the semiregularization (see [PW]), we have that

$$\text{cl}_\sigma(R) \subseteq \text{cl}_{\tau_s}(R) = \text{cl}_\tau(R)$$

while, being $\sigma(X) \subseteq \tau(X)$, it follows

$$\text{int}_\sigma(\text{cl}_\sigma(R)) \subseteq \text{int}_\tau(\text{cl}_\tau(R)) = R.$$

Obviously, being $R \in \sigma(X)$, we also have that $R \subseteq \text{int}_\sigma(\text{cl}_\sigma(R))$ and hence that $R = \text{int}_\sigma(\text{cl}_\sigma(R))$. Since $R \in \sigma(X)$, this means that R is a regular open set of $(X, \sigma(X))$. On the other hand, let S be a regular open set of $(X, \sigma(X))$. Then there exists some $V \in \sigma(X)$ such that $S = \text{int}_\sigma(\text{cl}_\sigma(V))$. Since $S \in \sigma(X) \subseteq \tau(X)$, we have that

$$\text{cl}_\tau(S) \subseteq \text{cl}_\sigma(S)$$

Hence, being $\tau_s(X) \subseteq \sigma(X)$ and by some well-known properties of the interior of the semiregularization it follows that

$$\text{int}_\tau(\text{cl}_\tau(S)) = \text{int}_{\tau_s}(\text{cl}_\tau(S)) \subseteq \text{int}_\sigma(\text{cl}_\sigma(S)) = S.$$

Obviously, being $S \in \tau(X)$, we also have that $S \subseteq \text{int}_\tau(\text{cl}_\tau(S))$ and hence that $S = \text{int}_\tau(\text{cl}_\tau(S))$, that is S is a regular open set of $(X, \tau(X))$. Thus, the topologies generated by these families of regular open sets, i.e. the semiregularization of $(X, \tau(X))$ and $(X, \sigma(X))$ coincide and we can conclude that $\tau_s(X) = \sigma_s(X)$. \square

Proposition 13. *Let \mathcal{U} be a free open ultrafilter on a Hausdorff space X and p be a fixed point in X . Then, there exists a Hausdorff topology $\tau_{\mathcal{U}}$ on X such that $\alpha_{\mathcal{U}}(X) \not\subseteq \alpha\tau(X)$.*

PROOF: Let us consider the family $\tau_{\mathcal{U}}(X) = \{U \in \tau(X) : p \in U \Rightarrow U \in \mathcal{U}\}$. It is a simple routine to verify that $\tau_{\mathcal{U}}(X)$ forms a topology on X such that $\tau_{\mathcal{U}}(X) \subseteq \tau(X)$.

The space $(X, \tau_{\mathcal{U}}(X))$ is T_2 . In fact, for every $x \neq y$ in X , since $(X, \tau(X))$ is Hausdorff, there are $U, V \in \tau(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. If $p \notin U \cup V, U, V \in \tau_{\mathcal{U}}(X)$ and we are done. Otherwise, if, for example, $p \in U$, we have $p \notin V$ and $V \in \tau(X)$. Furthermore, since \mathcal{U} is free with respect to $(X, \tau(X))$, there exist some $N \in \tau(X)$ and some $W \in \mathcal{U}$ such that $y \in N$ and $W \cap N = \emptyset$.

Thus $W \subseteq U \cup W$ implies $U \cup W \in \mathcal{U}$ and hence $U \cup W \in \tau_{\mathcal{U}}(X)$, while being $p \notin V \cap N \in \tau(X)$ it follows that $V \cap N \in \tau_{\mathcal{U}}(X)$. So, $U \cup W$ and $V \cap N$ are two open neighborhoods of x and y respectively in $\tau_{\mathcal{U}}(X)$ such that

$$\begin{aligned} (U \cup W) \cap (V \cap N) &= (U \cap (V \cap N)) \cup (W \cap (V \cap N)) \\ &\subseteq (U \cap V) \cup (W \cap N) \\ &= \emptyset \cup \emptyset = \emptyset \end{aligned}$$

and this proves that the space $(X, \tau_{\mathcal{U}}(X))$ is Hausdorff.

Since it is immediate to see that every neighborhood of p in $\tau_{\mathcal{U}}(X)$ belongs to \mathcal{U} and $(X, \tau_{\mathcal{U}}(X))$ is T_2 , it follows that p is the unique convergence point of \mathcal{U} with respect to $\tau_{\mathcal{U}}(X)$.

In order to show that $\alpha\tau_{\mathcal{U}}(X) \subseteq \alpha\tau(X)$, we observe first that, for every $U \in \tau_{\mathcal{U}}(X)$, it results:

$$(1) \quad \text{cl}_{\tau_{\mathcal{U}}}(U) = \begin{cases} \text{cl}_{\tau}(U) & \text{if } U \notin \mathcal{U}, \\ \text{cl}_{\tau}(U) \cup \{p\} & \text{if } U \in \mathcal{U}. \end{cases}$$

In fact, $\tau(X) \subseteq \tau_{\mathcal{U}}(X)$ implies, in any case, $\text{cl}_{\tau_{\mathcal{U}}}(U) \subseteq \text{cl}_{\tau}(U)$.

Now, consider the case $U \notin \mathcal{U}$ and suppose, by contradiction that there is some $x \in \text{cl}_{\tau_{\mathcal{U}}}(U)$ such that $x \notin \text{cl}_{\tau}(U)$. So, there exists some neighborhood N of x in $\tau(X)$ such that $N \cap U = \emptyset$.

Since \mathcal{U} is an open ultrafilter on $(X, \tau(X))$, $U \notin \mathcal{U}$ implies that $X \setminus \text{cl}_{\tau}(U) \in \mathcal{U}$. Now, $W = (X \setminus \text{cl}_{\tau}(U)) \cup N$ is an open neighborhood of x with respect to $\tau_{\mathcal{U}}(X)$ (because $W \in \tau(X)$, $x \in N \subseteq W$, and $X \setminus \text{cl}_{\tau}(U) \subseteq W$ implies $W \in \mathcal{U}$) but it results

$$W \cap U = \left((X \setminus \text{cl}_{\tau}(U)) \cup N \right) \cap U = \left((X \setminus \text{cl}_{\tau}(U)) \cap U \right) \cup (N \cap U) = \emptyset \cup \emptyset = \emptyset$$

which is a contradiction to $x \in \text{cl}_{\tau_{\mathcal{U}}}(U)$.

Let us consider the case $U \in \mathcal{U}$. Evidently $p \in \text{cl}_{\tau_{\mathcal{U}}}(U)$ as for every open neighborhood N of p in $\tau_{\mathcal{U}}(X)$, it follows that $N \in \mathcal{U}$ and hence $N \cap U \in \mathcal{U}$ implies $N \cap V \neq \emptyset$. Conversely, suppose, by contradiction, that there is some $x \in \text{cl}_{\tau_{\mathcal{U}}}(U)$ such that $x \neq p$ and $x \notin \text{cl}_{\tau}(U)$. Then, there exists some open neighborhood N of p in $\tau(X)$ such that $N \cap U = \emptyset$ and, it must be $N \notin \tau_{\mathcal{U}}(X)$, that is $p \in N \notin \mathcal{U}$. Since $(X, \tau(X))$ is T_2 and $x \neq p$, there is some open neighborhood G of $x \in \tau(X)$ such that $p \notin G$. Thus, $p \notin N \cap G \in \tau(X)$ implies that $N \cap G \in \tau_{\mathcal{U}}(X)$. So, $N \cap G$ is an open neighborhood of x in $\tau_{\mathcal{U}}(X)$ such that $(N \cap G) \cap U \subseteq N \cap U = \emptyset$. A contradiction to $x \in \text{cl}_{\tau_{\mathcal{U}}}(U)$. Applying the usual duality rules to formula (1), we also obtain that, for every $U \in \tau_{\mathcal{U}}(X)$, it results:

$$(2) \quad \text{int}_{\tau_{\mathcal{U}}}(\text{cl}_{\tau_{\mathcal{U}}}(U)) = \begin{cases} \text{int}_{\tau}(\text{cl}_{\tau}(U)) \setminus \{p\} & \text{if } U \notin \mathcal{U}, \\ \text{int}_{\tau}(\text{cl}_{\tau}(U)) & \text{if } U \in \mathcal{U}. \end{cases}$$

In fact, if $U \notin \mathcal{U}$, as \mathcal{U} is an open ultrafilter on $(X, \tau(X))$, $X \setminus \text{cl}_\tau(U) \in \mathcal{U}$ and so we have that:

$$\begin{aligned} \text{int}_{\tau_{\mathcal{U}}}(\text{cl}_{\tau_c U}(U)) &= \text{int}_{\tau_{\mathcal{U}}}(\text{cl}_\tau(U)) \\ &= X \setminus \text{cl}_{\tau_{\mathcal{U}}}(X \setminus (\text{cl}_\tau(U))) \\ &= X \setminus \text{cl}_\tau(X \setminus (\text{cl}_\tau(U) \cup \{p\})) \\ &= (X \setminus \text{cl}_\tau(X \setminus \text{cl}_\tau(U))) \cap (X \setminus \{p\}) \\ &= \text{int}_\tau(\text{cl}_\tau(U)) \setminus \{p\}. \end{aligned}$$

If $U \in \mathcal{U}$, since \mathcal{U} is an open ultrafilter on $(X, \tau(X))$, it necessarily results $X \setminus \text{cl}_\tau(U) \notin \mathcal{U}$ and we have:

$$\begin{aligned} \text{int}_{\tau_{\mathcal{U}}}(\text{cl}_{\tau_{\mathcal{U}}}(U)) &= X \setminus \text{cl}_{\tau_{\mathcal{U}}}(X \setminus \text{cl}_{\tau_{\mathcal{U}}}(U)) \\ &= X \setminus \text{cl}_\tau(X \setminus \text{cl}_{\tau_{\mathcal{U}}}(U)) \\ &= \text{int}_\tau(\text{cl}_{\tau_{\mathcal{U}}}(U)) \\ &= \text{int}_\tau(\text{cl}_\tau(U) \cup \{p\}) \\ &= \text{int}_\tau(\text{cl}_\tau(U)), \end{aligned}$$

where the last equality is due to the fact that $p \in \text{cl}_\tau(U)$ (because for every neighborhood N of p in $\tau(X)$, we have $N \in \mathcal{U}$ and so, being $U \in \mathcal{U}$, it follows that $N \cap U \in \mathcal{U}$ and $N \cap U \neq \emptyset$).

Now, for every $A \in \alpha\tau_{\mathcal{U}}(X)$, there exists some $U \in \tau_{\mathcal{U}}(X)$ such that $U \subseteq A \subseteq \text{int}_{\tau_{\mathcal{U}}}(\text{cl}_{\tau_{\mathcal{U}}}(U))$ and by formula (2), it immediately follows, in both cases, that $U \subseteq A \subseteq \text{int}_\tau(\text{cl}_\tau(U))$ with $U \in \alpha\tau(X) \subseteq \tau(X)$ and so that $A \in \alpha\tau(X)$. Thus $\alpha\tau_{\mathcal{U}}(X) \subseteq \alpha\tau(X)$. To finish the proof, we will show that $\alpha\tau_{\mathcal{U}}(X) \neq \alpha\tau(X)$. Since \mathcal{U} is free, p is not an adherent point for \mathcal{U} and so there exist some neighborhood V of p in $\tau(X)$ and some $U \in \mathcal{U}$ such that $U \cap V = \emptyset$. Thus $V \notin \mathcal{U}$ and, being $p \in V$, by definition of $\tau_{\mathcal{U}}(X)$, it follows that $V \notin \tau_{\mathcal{U}}(X)$. Evidently $V \in \alpha\tau(X)$ (as $V \in \tau(X)$) but $V \notin \alpha\tau_{\mathcal{U}}(X)$. Suppose, by contradiction, that there exists some $W \in \tau_{\mathcal{U}}(X)$ such that $W \subseteq V \subseteq \text{int}_{\tau_{\mathcal{U}}}(\text{cl}_{\tau_{\mathcal{U}}}(W))$. Since $V \notin \mathcal{U}$, it follows a fortiori that $W \notin \mathcal{U}$ and so, by formula (2), we have that $W \subseteq V \subseteq \text{int}_\tau(\text{cl}_\tau(W)) \setminus \{p\}$ and thus that $p \notin V$. A contradiction. This proves that $\alpha\tau_{\mathcal{U}}(X) \subsetneq \alpha\tau(X)$ and concludes our proof. \square

3. Minimal- αT_0 -spaces

Definition 2. Let \mathcal{P} be a topological property. A space X is said to be an $\alpha\mathcal{P}$ -space if it is an α -space and property \mathcal{P} holds.

Definition 3. Let \mathcal{P} be a topological property. A space X is called *minimal- $\alpha\mathcal{P}$* if it is an $\alpha\mathcal{P}$ -space and there is no strictly coarser $\alpha\mathcal{P}$ -topology on the same set.

Since the unique T_1 -minimal-space is the cofinite topology which is an α -space (by Proposition 3), it is evident that the class of minimal- αT_1 -spaces coincides with the well-known class of minimal- T_1 .

One would suspect that if X is minimal- T_0 , then $(X, \alpha\tau(X))$ is minimal- αT_0 . Here is a counterexample.

Example 14. Consider \mathbb{R} with $\tau(\mathbb{R}) = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Then \mathbb{R} is minimal- T_0 . If $\emptyset \neq U \in \tau(\mathbb{R})$, $\text{cl}_{\mathbb{R}}(U) = \mathbb{R}$, then $\text{int}_{\mathbb{R}}(\text{cl}_{\mathbb{R}}(U)) = \mathbb{R}$. So, if $a < b$ and $c = a - 1$, then $(-\infty, c) \cup \{a\}, (-\infty, c) \cup \{b\} \in \alpha\tau(\mathbb{R})$. In particular, it follows that $(\mathbb{R}, \alpha\tau(\mathbb{R}))$ is T_1 . If σ is the cofinite topology on \mathbb{R} , then $\sigma \subset \tau(\mathbb{R})$.

Lemma 15. Let X be a set, $p \in X$, and $\tau(X) = \{U : p \in U \text{ and } X \setminus U \text{ is finite}\} \cup \{\emptyset\}$. Then X is minimal- αT_0 .

PROOF: Clearly, X is T_0 . If $\emptyset \neq U \in \tau(X)$, then $p \in U$, $\text{cl}_X U = X$, and $\text{int}_X \text{cl}_X U = X$. Now $U \cup \{q\}$ is open for all $q \in X$, by Proposition 3, since X is αT_0 . Let $\sigma \subseteq \tau(X)$ and (X, σ) be αT_0 . If $\emptyset \neq U \in \tau(X)$, then $p \in U$, and $X \setminus U = \{q_1, \dots, q_n\}$ is finite. Since $\emptyset \neq V \in \sigma \subseteq \tau(X)$ implies $p \in V$, then for each $1 \leq i \leq n$, there is a $V_i \in \sigma$ such that $p \in V_i \subseteq X \setminus \{q_i\}$. For $T = \bigcap \{V_i : 1 \leq i \leq n\}$, $p \in T \in \sigma$ and $T \subseteq U$. As (X, σ) is αT_0 and $\text{cl}_{\sigma} T = X$, then $U = \bigcup \{T \cup \{q\} : q \in U \setminus T\} \in \sigma$. □

In the topology just considered, every point different from p is closed. Now, let us consider the case in which some point of X is not closed.

Lemma 16. Let X be an αT_0 -space, and $p \in X$ such that $\text{cl}_X \{p\} \neq \{p\}$. Then $\text{cl}_X(\{p\})$ is a regular-closed set and $p \in \text{int}_X(\text{cl}_X(\{p\}))$.

PROOF: Let $U = X \setminus \text{cl}_X(\{p\})$. It suffices to show that U is regular-open. If $p \in \text{cl}_X(U)$, then $\text{cl}_X(U) = X$ and $\text{int}_X(\text{cl}_X(U)) = X$. So, for $q \in \text{cl}_X(\{p\}) \setminus \{p\}$, $U \cup \{q\}$ is open and $(U \cup \{q\}) \cap \{p\} = \emptyset$. That is, $q \notin \text{cl}_X(\{p\})$, a contradiction. So, $p \notin \text{cl}_X(U)$. For $V = X \setminus \text{cl}_X(U)$, we have that $p \in V \subseteq \text{cl}_X(\{p\})$. Thus, $\text{cl}_X(V) = \text{cl}_X(\{p\})$. □

In order to obtain a characterization of minimal- T_0 -spaces, we need some other lemmas.

Lemma 17. Let X be an αT_0 -space, and $p, q \in X$ such that $\text{cl}_X(\{p\}) \neq \{p\}$, $\text{cl}_X(\{q\}) \neq \{q\}$, and $p \neq q$. Then $p \notin \text{cl}_X(\{q\})$ and $q \notin \text{cl}_X(\{p\})$.

PROOF: Since X is T_0 , then $\text{cl}_X(\{p\}) \neq \text{cl}_X(\{q\})$. Assume that $q \in \text{cl}_X(\{p\})$. Then $p \notin \text{cl}_X(\{q\})$ as $\text{cl}_X(\{p\}) \neq \text{cl}_X(\{q\})$. So, $\text{cl}_X(\{q\}) \subseteq \text{cl}_X(\{p\})$. By Lemma 16, $q \in \text{int}_X(\text{cl}_X(\{q\})) \subseteq \text{int}_X(\text{cl}_X(\{p\})) \subseteq \text{cl}_X(\{p\})$. As $p \notin \text{cl}_X(\{q\})$, it follows that $p \notin \text{int}_X(\text{cl}_X(\{q\}))$. Therefore, $q \notin \text{cl}_X(\{p\})$, a contradiction. □

Lemma 18. *Let X be a minimal- αT_0 -space, and $p, q \in X$ such that $\text{cl}_X(\{p\}) \neq \{p\}$, $\text{cl}_X(\{q\}) \neq \{q\}$, and $p \neq q$. Then $\text{int}_X(\text{cl}_X(\{p\})) = \{p\}$ and $\text{int}_X(\text{cl}_X(\{q\})) = \{q\}$.*

PROOF: Let $\tau = \tau(X)$ be the topology on X , $r \in \text{int}_X(\text{cl}_X(\{p\})) \setminus \{p\}$ and consider the topology $\sigma = \{U \in \tau : r \in U \text{ implies } q \in U\}$ on X . Clearly, $\sigma \subseteq \tau$. There are a number of cases to verify that (X, σ) is T_0 but each case is straightforward. Next we show that (X, σ) is an α -space. Let U be open and dense in (X, σ) and $t \in X \setminus U$. We want to show that $U \cup \{t\} \in \sigma$. If $r \in U$, then U is dense and open in (X, τ) and hence, $U \cup \{t\} \in \sigma$. Suppose that $r \notin U$. Our first goal is to show that U is dense in τ . If $\text{cl}_\tau(U) \neq \text{cl}_\sigma(U)$, then $r \in \text{cl}_\sigma(U) \setminus \text{cl}_\tau(U)$. Then $p \notin U$ and $\text{cl}_\tau(\{p\}) \cap U = \emptyset$. In particular, $\text{int}_\tau(\text{cl}_\tau(\{p\})) \cap U = \emptyset$. There is an $V \in \tau$ such that $p \in V$ and $r \notin V$. Now $p \in \text{int}_\tau(\text{cl}_\tau(\{p\})) \cap V \in \sigma$ and $\text{int}_\tau(\text{cl}_\tau(\{p\})) \cap V \cap U = \emptyset$. So, U is not dense in (X, σ) , a contradiction. Thus, U is dense in τ . Also the above proof shows that $p \in U$. If $q \notin U$, then $\text{cl}_\tau(\{q\}) \cap U = \emptyset$. In particular, $\text{int}_\tau(\text{cl}_\tau(\{q\})) \cap U = \emptyset$. But $q \in \text{int}_\tau(\text{cl}_\tau(\{q\})) \in \sigma$, a contradiction. Thus, $q \in U$. With both $p, q \in U$, we have that $U \cup \{r\} \in \sigma$. For $t \neq r$, then $U \cup \{t\} \in \sigma$ as $U \cup \{t\} \in \tau$ and $r \notin U \cup \{t\}$. □

Proposition 19. *Let X be a minimal- αT_0 -space, and $P = \{p \in X : \text{cl}_X(\{p\}) \neq \{p\}\}$ such that $|P| \geq 2$. Then P is dense in X and if $V \in \tau(X)$ and $V \setminus P \neq \emptyset$, then $P \subseteq V$.*

PROOF: Let $\tau = \tau(X)$ be the topology of the space X . First we show that $Q = \{q \in X : \{q\} \in \tau\}$ is dense. Clearly, $Q \supseteq P$. Fix $r \in X \setminus \text{cl}_X(Q)$. Note that the topology $\sigma = \{U \in \tau : r \in U \text{ implies } Q \subseteq U\} \subseteq \tau$. If we show that (X, σ) is αT_0 , it will follow that Q is dense. Since $\{q\} \in \sigma$ for all $q \in Q$, q can be T_0 -separated from all $p \in X \setminus \{q\}$ in σ . As $(X \setminus \text{cl}_X(Q)) \cup Q \in \sigma$ and $(X \setminus \text{cl}_X(Q)) \cup Q \cup \{t\} \in \sigma$ for $t \in \text{cl}_X(Q) \setminus Q$, a point $t \in \text{cl}_X(Q) \setminus Q$ can be T_0 -separated from all $p \in X \setminus \{t\}$ in σ . Let $s, t \in X \setminus \text{cl}_X(Q)$. As τ is T_0 , there is some $V \in \tau$ such that $s \in V$ and $t \notin V$ or vice versa. Now, $s \in V \cup Q \in \sigma$ and $t \notin V \cup Q$. This completes the proof that (X, σ) is T_0 . Next we show that (X, σ) is an α -space. Let U be an open and dense subset of (X, σ) . Note that $Q \subseteq U$. If $\text{cl}_\tau(U) \neq \text{cl}_\sigma(U)$, then $r \in \text{cl}_\sigma(U) \setminus \text{cl}_\tau(U)$. There is an $V \in \tau$ such that $r \in V$ and $V \cap Q = \emptyset$. As $r \notin P$, $\text{cl}_\tau(\{r\}) = \{r\}$, $\emptyset \neq V \setminus \{r\} \in \sigma$ and $V \setminus \{r\} \cap U = \emptyset$, a contradiction as U is dense in (X, σ) . This shows that $\text{cl}_\tau(U) = X$. As τ is an α topology, for $t \in X, U \cup \{t\} \in \tau$. As $Q \subseteq U, U \cup \{t\} \in \sigma$. This completes the proof that (X, σ) is an αT_0 -space. Finally, we show that $Q = P$. Assume that $q \in Q \setminus P$. The topology $\sigma = \{U \in \tau : q \in U \text{ implies } P \subseteq U\} \subseteq \tau$. Similar to the above, it is straightforward to show that (X, σ) is T_0 and an α -space, a contradiction as τ is minimal- αT_0 . Thus, $P = Q$. □

Proposition 20. *Let X be a minimal- αT_0 -space, and $P = \{p \in X : \text{cl}_X(\{p\}) \neq \{p\}\}$ such that $|P| \geq 2$. Then $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$.*

PROOF: This is an obvious consequence of Proposition 19. □

Proposition 21. *Let X be a set and $P \subsetneq X$ such that $|P| \geq 2$. Let $\tau(X)$ be generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$. Then X is a minimal- αT_0 -space.*

PROOF: Let $\tau = \tau(X)$ be the topology on X and $\sigma \subseteq \tau$ be an αT_0 topology. For $p \in P, (X \setminus P) \cup \{p\} = \text{cl}_\tau(\{p\}) \subseteq \text{cl}_\sigma(\{p\})$. Thus, for all $p \in P, \text{cl}_\sigma(\{p\}) \neq \{p\}$. By Lemma 17, for $q \in P \setminus \{p\}, q \notin \text{cl}_\sigma(\{p\})$. Hence $\text{cl}_\sigma(\{p\}) = (X \setminus P) \cup \{p\} = \text{cl}_\tau(\{p\})$. Thus, $P \setminus \{p\} \in \sigma$ for all $p \in P$. By Lemma 16, $p \in \text{int}_\sigma(\text{cl}_\sigma(\{p\}))$. As $|P| \geq 2$, let $q \in P \setminus \{p\}$. Then $\text{int}_\sigma(\text{cl}_\sigma\{q\}) \cap P \setminus \{p\} = \{q\} \in \sigma$ and $P \in \sigma$. As $X = \text{cl}_\tau(P) \subseteq \text{cl}_\sigma(P), X = \text{int}_\sigma(\text{cl}_\sigma(P))$. Since (X, σ) is an αT_0 -space, it follows that for $q \in X \setminus P, P \cup \{q\} \in \sigma$. This shows that $\sigma = \tau$. □

Finally, we obtain the characterization of αT_0 -spaces.

Theorem 22. *Let X be an αT_0 -space and $P = \{p \in X : \text{cl}_X(\{p\}) \neq \{p\}\}$. Then X is minimal- αT_0 iff $P \neq \emptyset$ and*

- (i) *if $P = \{p\}$, then $\tau(X) = \{U : p \in U \text{ and } X \setminus U \text{ is finite}\} \cup \{\emptyset\}$, or*
- (ii) *if $|P| \geq 2$, then $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$.*

PROOF: (\Leftarrow) It follows from Lemma 18 and Proposition 20. □
 (\Rightarrow) It follows from Lemma 15 and Proposition 21. □

4. Minimal- αT_2 -spaces

Proposition 23. *Let X be an αT_2 -space. Then X is a minimal- αT_2 -space if and only if $(X, \tau_s(X))$ is minimal- T_2 and $\tau(X) = \alpha \tau_s(X)$.*

PROOF: (\Rightarrow) Suppose that X is a minimal- αT_2 -space. Thus $\alpha \tau(X) = \tau(X)$. Then the space $(X, \tau(X))$ is T_2 -closed. In fact, if, by contradiction, it is not, by Proposition 7, there exists some free open ultrafilter \mathcal{U} on $(X, \tau(X))$ and by Proposition 13, there exists a strictly coarser αT_2 topology $\tau_{\mathcal{U}}$. A contradiction to the α -minimality of $(X, \tau(X))$. Hence, by Corollary 9, $(X, \tau_s(X))$ is minimal- T_2 . Furthermore, being $\tau_s(X) \subseteq \tau(X)$ and, obviously, $(\tau_s(X))_s = \tau_s(X)$, by Lemma 6, we have that $\alpha \tau_s(X) \subseteq \alpha \tau(X)$, i.e. $\alpha \tau_s(X) \subseteq \tau(X)$ where $\alpha \tau_s(X)$ is Hausdorff by Proposition 11 as $\tau(X)$ is Hausdorff and the T_2 axioms are expansive. Since $(X, \tau(X))$ is minimal- αT_2 , we conclude that $\alpha \tau_s(X) = \tau(X)$.

(\Leftarrow) Let us suppose that $(X, \tau_s(X))$ is minimal- T_2 and $\tau(X) = \alpha \tau_s(X)$. Then $(X, \tau(X))$ is H -closed by Corollary 9. Now, let $\sigma(X)$ be an αT_2 topology on X such that

$$\sigma(X) \subseteq \tau(X).$$

For every R regular open set of $(X, \tau(X))$, $X \setminus R$ is a regular closed set and hence, by Proposition 10, it is a T_2 -closed subspace of the Hausdorff space $(X, \sigma(X))$. Thus $X \setminus R$ is a closed set of $(X, \sigma(X))$ and $R \in \sigma(X)$. This proves that

$$\tau_s(X) \subseteq \sigma(X).$$

Hence, by Proposition 12, $\tau_s(X) = \sigma_s(X)$ and so, applying Lemma 6 to $\tau_s(X) \subseteq \sigma(X)$, we have that $\alpha\tau_s(X) \subseteq \alpha\sigma(X)$. Since $\sigma(X)$ is an α topology and, by hypothesis $\alpha\tau_s(X) = \tau(X)$, it follows that $\tau(X) \subseteq \sigma(X)$ and so that $\tau(X) = \sigma(X)$. This shows that $(X, \tau(X))$ is a minimal- αT_2 -space and concludes our proof. \square

Corollary 24. *Let $(X, \tau(X))$ be a space with a dense set D of isolated points. Then:*

- (1) *the α -topology $\alpha\tau(X)$ coincides with the topology generated by $\tau(X) \cup \{D \cup \{x\} : x \in X\}$, i.e. it is a simple extension of the subspace D ;*
- (2) *X is an α -space if and only if $X \setminus D$ is discrete;*
- (3) *if X is a semiregular α -space, it results:*

$$\{\sigma : \sigma \text{ is a topology on } X \text{ such that } \sigma_s = \tau(X)\} = \{\tau(X)\}.$$

PROOF: Straightforward applications of Propositions 3, 5 and 6. \square

Example 25. Let us consider the set $X = \mathbb{R} \times [0, +\infty[$. It is easy to verify that

$$\tau_s(X) = \{U \subseteq X : (x, 0) \in U \Rightarrow \exists \epsilon > 0 \text{ such that }]x - \epsilon, x + \epsilon[\times]0, \epsilon[\subseteq U\}$$

defines a Tychonoff topology on X and that $D = \mathbb{R} \times]0, +\infty[$ is a dense set of isolated points. Now, if we consider another topology on X ,

$$\begin{aligned} \tau(X) = \{ & U \subseteq X : (x, 0) \in U \setminus \mathbb{Q} \times \{0\} \Rightarrow \exists \epsilon > 0 \\ & \text{such that }]x - \epsilon, x + \epsilon[\times]0, \epsilon[\subseteq U \\ & \text{and } (x, 0) \in U \cap \mathbb{Q} \times \{0\} \Rightarrow \exists \epsilon > 0 \text{ such that} \\ &]x - \epsilon, x + \epsilon[\times]0, \epsilon[\cup (]x - \epsilon, x + \epsilon[\cap \mathbb{Q} \times \{0\}) \subseteq U \}, \end{aligned}$$

it is a simple routine to check that $\tau_s(X)$ is the semiregularization of $\tau(X)$ and that $\tau(X) \neq \tau_s(X)$. Furthermore, since $C = X \setminus (\mathbb{R} \times]0, +\infty[)$ is a closed nowhere dense subset with respect to both $\tau(X)$ and $\tau_s(X)$, by Corollary 24 follows that $\alpha\tau(X) = \alpha\tau_s(X)$ coincides with the topology generated by $\tau(X) \cup \{(\mathbb{R} \times]0, +\infty[) \cup \{(x, 0)\} : x \in \mathbb{R}\}$ and so that $\tau(X) \neq \alpha\tau(X)$. Thus, we have that

$$\tau_s(X) \subsetneq \tau(X) \subsetneq \alpha\tau(X) = \alpha\tau_s(X).$$

Example 26 (A Tychonoff α -space which is not minimal- T_2). Let us recall that two sets are *almost disjoint* if their intersection is a finite set. It is a simple routine to show (by using Zorn’s Lemma) that there exists a maximal almost disjoint family \mathcal{M} of subsets of \mathbb{N} . The space generated on the set $\psi = \mathbb{N} \cup \mathcal{M}$ by the base

$$\mathcal{B} = \left\{ \{n\} : n \in \mathbb{N} \right\} \cup \left\{ \{M\} \cup S : M \in \mathcal{M} \text{ and } S \text{ is a cofinite subset of } M \right\}$$

is a 0-dimensional (and hence Tychonoff) but not normal (and hence not minimal- T_2). This space is known in literature as the ψ -space (see 1N, [PW]). Since, every closed nowhere dense set of ψ is discrete, by Proposition 5, it is evident that ψ is an α -space.

Example 27 (A minimal- T_2 , α -space). Let us consider the set

$$Z = \left\{ \left(\frac{1}{n}, 0 \right) : n \in \mathbb{N} \right\} \cup \left\{ \left(\frac{1}{n}, \frac{1}{m} \right) : n \in \mathbb{N}, m \in \mathbb{Z} \right\}$$

with the topology $\tau(Z)$ induced by the usual topology on \mathbb{R}^2 . Let $X = Z \cup \{a, b\}$ and define a topology $\tau(X)$ on X by saying that a subset $U \subset Z$ is open if $U \cap Z \in \tau(Z)$ and if $a \in U$ (respectively, $b \in U$) there exists some $k \in \mathbb{N}$ such that $\left\{ \left(\frac{1}{n}, \frac{1}{m} \right) : n \in \mathbb{N}, m \geq k \right\} \subseteq U$ (respectively, $\left\{ \left(\frac{1}{n}, -\frac{1}{m} \right) : n \in \mathbb{N}, m \geq k \right\} \subseteq U$). It is well-known (see 4.8(d), [PW]) that the space X is Urysohn, not compact and minimal- T_2 . Furthermore, since every its nowhere dense subset (namely, $\left\{ \left(\frac{1}{n}, 0 \right) : n \in \mathbb{N} \right\} \cup \{a, b\}$) is discrete, it follows from Proposition 5 that X is an α -space.

Let us note that, by 24(3), in both of the above spaces, we have that $\{\sigma : \sigma \text{ is a topology on } X \text{ such that } \sigma_s = \tau(X)\} = \{\tau(X)\}$. Thus, the question remains: *is there some semiregular α -space $(X, \tau(X))$ such that $\{\tau(X)\} \subsetneq \{\sigma : \sigma \text{ is a topology on } X \text{ such that } \sigma_s = \tau(X)\}$?*

We now provide an example of a Tychonoff α -space X such that

$$\{\tau(X)\} \subsetneq \{\sigma : \sigma \text{ is a topology on } X \text{ such that } \sigma_s = \tau(X)\}.$$

Example 28. Recall that a measurable subset A of \mathbb{R} has density 1 if

$$A = \left\{ x \in \mathbb{R} : \lim_{h \rightarrow 0} \frac{m(A \cap [x - h, x + h])}{2h} = 1 \right\}.$$

The set of $\{A \subseteq \mathbb{R} : A \text{ measurable with density } 1\}$ forms a topology $\delta(\mathbb{R})$, called the density topology, on the set \mathbb{R} . The space $(\mathbb{R}, \delta(\mathbb{R}))$ is a Tychonoff space without isolated points, strictly finer than the usual topology $\tau(\mathbb{R})$, and has the property that every nowhere dense subset is closed and discrete (see 2.7 in [T]). In

particular, $(\mathbb{R}, \delta(\mathbb{R}))$ is an α -space by Proposition 4. We need one additional fact about $(\mathbb{R}, \delta(\mathbb{R}))$. Note that for $x \in U \in \delta(\mathbb{R})$, $U \cap (x, \infty) \neq \emptyset$ and $U \cap (-\infty, x) \neq \emptyset$. In particular, the $\delta(\mathbb{R})$ open neighborhood filter is contained in two distinct $\delta(\mathbb{R})$ open ultrafilters on \mathbb{R} . Before continuing with this example, we need a result about absolutes of spaces.

For a Hausdorff space X and let $EX = \{\mathcal{U} : \mathcal{U} \text{ is a convergent, open ultrafilter on } X\}$. For $U \in \tau(X)$, let $O(U) = \{\mathcal{U} \in EX : U \in \mathcal{U}\}$. For $U, V \in \tau(X)$, it is easy to verify (see [PW]) that $O(\emptyset) = \emptyset$, $O(X) = EX$, $O(U \cap V) = O(U) \cap O(V)$, $O(U \cup V) = O(U) \cup O(V)$, $EX \setminus O(U) = O(X \setminus \text{cl}_X(U))$, and $O(U) = O(\text{int}_X \text{cl}_X(U))$. EX with the topology generated by $\{O(U) : U \in \tau(X)\}$ is an extremally disconnected Tychonoff space, called the *absolute* of X . The function $k : EX \rightarrow X$ defined by $k(\mathcal{U})$ is the unique convergent point of \mathcal{U} is called a covering function and has the properties that k is irreducible, θ -continuous, perfect and onto. If X is regular, then k is also continuous. If $D \subseteq EX$ such that $k[D] = X$, then D is dense in EX .

Proposition 29. *Let X be a regular α -space and Y a subspace of EX such that for each $x \in X$, $|k^{\leftarrow}(\{x\})| = 2$. Then Y is an α -space.*

PROOF: Let $\bar{k} = k|_Y$. The function $\bar{k} = k|_Y : Y \rightarrow X$ is continuous and onto. So, Y is dense in EX and extremally disconnected. Let N be a nowhere dense subset of Y . Suppose $U = \text{int}_{EX}(\text{cl}_{EX}(N)) \neq \emptyset$. Then, as Y is dense, $\emptyset \neq U \cap Y \subseteq \text{cl}_{EX}(N) \cap Y = \text{cl}_Y(N)$, contradicting that N is nowhere dense in Y . Thus, N is nowhere dense in EX . By 6.5d(2) in [PW], $k[N] = \bar{k}[N]$ is nowhere dense in X and hence discrete in X . Let $p \in N$. There is an open set V in X such that $V \cap \bar{k}[N] = \{k(p)\}$. So, $N \cap \bar{k}^{\leftarrow}[V] = \bar{k}^{\leftarrow}(\{p\})$. But $\bar{k}^{\leftarrow}(\{p\})$ has only two points and there is an open set W in Y such that $N \cap \bar{k}^{\leftarrow}[V] \cap W = \{p\}$. This shows that N is discrete in Y and Y is an α -space. \square

We are ready to apply Proposition 29 to the regular α -space $(\mathbb{R}, \delta(\mathbb{R}))$. First note that the covering function $k : E\mathbb{R} \rightarrow (\mathbb{R}, \delta(\mathbb{R}))$ has the property that $|k^{\leftarrow}(r)| \geq 2$ for each $r \in \mathbb{R}$ since each open neighborhood filter is contained in two distinct $\delta(\mathbb{R})$ open ultrafilters on \mathbb{R} . Let $Y \subseteq E\mathbb{R}$ have the property that $|k^{\leftarrow}(r) \cap Y| = 2$ for each $r \in \mathbb{R}$, and let $Z \subset Y$ be such that $|k^{\leftarrow}(r) \cap Z| = 1$ for each $r \in \mathbb{R}$. Now, both Y and Z are dense in $E\mathbb{R}$. By Proposition 29, Y is a Tychonoff α -space. Let σ be the topology on Y generated by $\tau(Y) \cup \{Z\}$. Clearly, $\tau(Y) \subsetneq \sigma$, and it is straightforward to show that $\sigma_s = \tau(Y)$. The space Y is the desired space.

5. A problem

Since the property to be $T_{2\frac{1}{2}}$ -closed does not pass to the regular open subspaces, we cannot use a result like Proposition 10 to prove a proposition similar to Proposition 23. So, we leave the following as an unsolved problem:

Conjecture. Let X be an $\alpha T_{2\frac{1}{2}}$ -space. Then X is a minimal- $\alpha T_{2\frac{1}{2}}$ -space if and only if $(X, \tau_s(X))$ is semiregular, $T_{2\frac{1}{2}}$ -closed and $\tau(X) = \alpha\tau_s(X)$.

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(Received October 7, 2002)