

Multiplicity of positive solutions for some quasilinear Dirichlet problems on bounded domains in \mathbb{R}^n

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Abstract. We show that, under appropriate structure conditions, the quasilinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n , $1 < p < +\infty$, admits two positive solutions u_0, u_1 in $W_0^{1,p}(\Omega)$ such that $0 < u_0 \leq u_1$ in Ω , while u_0 is a local minimizer of the associated Euler-Lagrange functional.

Keywords: p -Laplacian, positive solutions, sub- and supersolutions, local minimizers, Palais-Smale condition

Classification: 35J20, 35J60, 35J70

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, with boundary of class C^2 and consider the quasilinear elliptic problem

$$(1.1) \quad \begin{cases} -\Delta_p u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the so-called p -Laplace operator with $1 < p < +\infty$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, i.e. continuous in u for a.e. $x \in \Omega$ and measurable in x for all $u \in \mathbb{R}$.

Questions concerning the effect of the nonlinear term $f(x, u)$ on the existence and multiplicity of solutions of (1.1) have been extensively investigated in recent years. A comprehensive review of the existing literature is beyond the present scope and the interested reader should consult the survey in [2]. Confining ourselves to the class of *positive* solutions, it is essential, however, to report the results which are closely related to the theme discussed in the present article. These pertain, in particular, to the model case provided by the function

$$f(x, u) = \lambda|u|^{r-2}u + |u|^{s-2}u,$$

where $1 < r \leq p < s$ and $\lambda > 0$ is a real parameter. As a matter of fact, with the aid of variational techniques it was shown in [3] that when $1 < r < p = 2 < s \leq 2^* := \frac{2n}{n-2}$, there exists a constant $\Lambda > 0$ such that problem (1.1) admits at least two positive solutions in $W_0^{1,p}(\Omega)$ for all $\lambda \in (0, \Lambda)$, at least one positive solution if $\lambda = \Lambda$ and no solution if $\lambda > \Lambda$. This multiplicity result was then extended via topological degree arguments in [1] for the quasilinear case $p \neq 2$ with $1 < r < p < s < p^* := \frac{np}{n-p}$, albeit for the special class of radial solutions. Note that when $1 < r < p < s < \infty$, the existence of one positive solution, without any symmetry assumptions on the domain Ω , was established in [8] via Sattinger’s iteration scheme [17]. Nevertheless, this method cannot yield more solutions. The issue of existence and multiplicity in the nonradial setting and with $p \neq 2$ was studied in [7] via an extension to p -Laplace equations of a theorem by Brezis and Nirenberg [10] which concerns the relationship between local minimizers of the associated Euler-Lagrange functional in the $W_0^{1,p}$ and C_0^1 topologies. More specifically, by applying arguments similar to those used in the semilinear case, it was shown in [7] that one positive solution can be obtained as a local minimizer of the above functional while a second positive solution can then be found by means of a variant of the Mountain-Pass Theorem.

In this paper we are concerned with the issue of multiplicity as above, but in the context of a much larger class of nonlinearities. Our approach remains variational in nature and combines several ideas from [3], [10] and [13]. In particular, we show the existence of two positive solutions u_0, u_1 which are ordered; i.e. $0 < u_0 \leq u_1$ in Ω . Note that this property has been established so far only in the semilinear case $p = 2$ where, in fact, due to the linearity of the principal part of (1.1), the ordering is strict (i.e. $0 < u_0 < u_1$ in Ω), [3].

Let us finally mention that the critical semilinear case $1 < r \leq p = 2 < s = 2^*$ was originally studied in the pioneering paper of Brezis and Nirenberg [9] and their results were then extended to the quasilinear case in [13] for $1 < r = p < s = p^*$ and in [6] for $1 < r < p < s = p^*$.

2. Existence and multiplicity of positive solutions

Throughout this section we are concerned with the problem of finding positive solutions for (1.1), assuming that the lower order nonlinearity f satisfies the structure conditions:

- (H1) $f(x, u)$ is nondecreasing in u with $f(x, 0) = 0$ for a.e. $x \in \Omega$.
- (H2) There exists $C > 0$ such that $|f(x, u)| \leq C(1 + |u|^{k-1})$ for a.e. $x \in \Omega$, where
 - (i) $k \in (1, p^*]$ if $p < n$, with $p^* := \frac{np}{n-p}$,
 - (ii) $k \in (1, +\infty)$ if $p \geq n$.
- (H3) $\liminf_{s \rightarrow 0^+} \frac{f(x, s)}{s^{p-1}} > \lambda_1$ for a.e. $x \in \Omega$, where λ_1 is the principal eigenvalue of $-\Delta_p$ on Ω with zero Dirichlet boundary conditions.

Before we proceed, a few preliminary facts that will be used repeatedly in the sequel are in order. First, a basic ingredient in our approach is provided by the following proposition which concerns the boundary regularity of weak solutions of (1.1).

Theorem 1. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the quasilinear Dirichlet problem (1.1) where $f(x, u)$ conforms with (H2). Then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

A proof of Theorem 1 in the case where $1 < p \leq n$ and $f(x, u)$ is continuous in $\overline{\Omega} \times \mathbb{R}$ can be found in [13]. A different proof covering the present situation, as well as the full range of the exponent p , is provided in the Appendix.

Consider now the Euler-Lagrange functional associated with (1.1),

$$(2.1) \quad \Phi(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} F(x, u) dx,$$

where

$$(2.2) \quad F(x, u) := \int_0^u f(x, t) dt.$$

As is well known, on account of (H2), $\Phi(\cdot)$ defines a continuous functional from $W_0^{1,p}(\Omega)$ to \mathbb{R} which is also weakly lower semicontinuous unless $p < n$ and $k = p^*$. Moreover, it is easy to show that any minimizer $u \in W_0^{1,p}(\Omega)$ of $\Phi(\cdot)$ is a weak solution of (1.1). Even more, according to the following remarkable theorem, any local minimizer of $\Phi(\cdot)$ in the C_0^1 -topology must also be a local minimizer in the $W_0^{1,p}$ -topology.

Theorem 2. *Let (H2) hold and assume that there exist $w \in W_0^{1,p}(\Omega)$ and $\rho > 0$ such that*

$$(2.3) \quad \Phi(w) \leq \Phi(w + v) \text{ for every } v \in C_0^1(\overline{\Omega}) \text{ with } \|v\|_{C^1} \leq \rho.$$

Then there exists $\rho' > 0$ such that

$$(2.4) \quad \Phi(w) \leq \Phi(w + z) \text{ for every } z \in W_0^{1,p}(\Omega) \text{ with } \|z\|_{W^{1,p}} \leq \rho'.$$

As already mentioned in the introduction, this rather surprising result was first proved by Brezis and Nirenberg when $p = 2$ in [10] and then it was extended for all $p \in (1, +\infty)$ by Azorero, Alonso and Manfredi in [7]. It should be pointed out here, however, that this property may not hold for a general functional since it is the special structure of (2.1) which plays an essential role in the proof.

Finally, the following lemma is essentially a variant regarding the monotonicity of the $-\Delta_p$ operator and can be easily proved via Hölder's inequality.

Lemma 3. *Let $1 < p < +\infty$. Then for any $u, v \in W_0^{1,p}(\Omega)$ the following inequality holds*

$$\int_{\Omega} \left[|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right] (\nabla u - \nabla v) \, dx \geq (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|) \geq 0,$$

where $\|u\| := (\int_{\Omega} |\nabla u|^p \, dx)^{1/p}$.

Definition 4. A nonnegative function $w \in C^1(\overline{\Omega})$ is said to be a *strict supersolution* (resp. *strict subsolution*) for (1.1) if $-\Delta_p w > f(x, w)$ in Ω (resp. $-\Delta_p w < f(x, w)$ in Ω) and $w = 0$ on $\partial\Omega$.

Observe that, on account of the strong maximum principle of Vázquez [18] and (H1), a strict supersolution is necessarily positive everywhere in Ω .

Our first result is the following

Theorem 5. *Suppose that (H1), (H2) and (H3) hold. Assume further that a strict supersolution \bar{u} for (1.1) exists. Then, problem (1.1) admits a positive solution u_0 which is also a local minimizer of $\Phi(\cdot)$ in the $W_0^{1,p}$ -topology.*

PROOF: Let φ_1 be the eigenfunction corresponding to the principal eigenvalue λ_1 of $-\Delta_p$ on Ω with zero Dirichlet boundary conditions, normalized so that $\|\varphi_1\|_{\infty} = 1$. Since $\lambda_1 > 0$ and $\varphi_1(\cdot) > 0$ in Ω (see [5]), in view of (H3), there exists $\varepsilon > 0$ such that $\underline{u} = \varepsilon\varphi_1$ is a strict subsolution of (1.1). Moreover, by virtue of the strong maximum principle ([18]) it is straightforward to check that if ε is chosen sufficiently small then

$$(2.5) \quad \underline{u} < \bar{u}, \quad \text{in } \Omega.$$

Let us now define

$$(2.6) \quad \widehat{f}(x, t) := \begin{cases} f(x, \bar{u}(x)), & \text{if } t > \bar{u}(x), \\ f(x, t), & \text{if } \underline{u}(x) \leq t \leq \bar{u}(x), \\ f(x, \underline{u}(x)), & \text{if } t < \underline{u}(x), \end{cases}$$

and consider the problem

$$(2.7) \quad \begin{cases} -\Delta_p u = \widehat{f}(x, u(x)), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

with the associated Euler-Lagrange functional

$$\widehat{\Phi}(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} \int_0^u \widehat{f}(x, t) \, dt \, dx.$$

From (H2) and (2.6), it is easily seen that $\widehat{\Phi}(\cdot)$ is bounded from below and weakly lower semicontinuous in $W_0^{1,p}(\Omega)$. Therefore, the infimum of $\widehat{\Phi}(\cdot)$ is achieved at some point $u_0 \in W_0^{1,p}(\Omega)$ which is a solution of (2.7). In particular, $u_0 \in C^1(\overline{\Omega})$ by Theorem 1. We claim that $\underline{u} \leq u_0 \leq \bar{u}$ in Ω . Indeed, let us define the set

$$\Omega_0 := \{x \in \Omega : u_0(x) < \underline{u}(x)\},$$

and assume that it is nonempty. Since Ω_0 is open, it must have positive measure. Furthermore, in view of (2.6),

$$(2.8) \quad -\Delta_p u_0 = f(x, \underline{u}(x)), \quad x \in \Omega_0,$$

while

$$(2.9) \quad -\Delta_p \underline{u} < f(x, \underline{u}(x)), \quad x \in \Omega_0.$$

Hence, by multiplying (2.8) and (2.9) with $\underline{u} - u_0$ and integrating over Ω_0 , we get

$$\int_{\Omega_0} |\nabla u_0|^{p-2} \nabla u_0 \nabla (\underline{u} - u_0) \, dx = \int_{\Omega_0} f(x, \underline{u}(x)) (\underline{u} - u_0) \, dx,$$

and

$$\int_{\Omega_0} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla (\underline{u} - u_0) \, dx < \int_{\Omega_0} f(x, \underline{u}(x)) (\underline{u} - u_0) \, dx,$$

which combined yield

$$\int_{\Omega_0} \left\{ |\nabla \underline{u}|^{p-2} \nabla \underline{u} - |\nabla u_0|^{p-2} \nabla u_0 \right\} \nabla (\underline{u} - u_0) \, dx < 0.$$

However, the last inequality contradicts Lemma 3 and so Ω_0 must be empty. The proof of $u_0 \leq \bar{u}$ in Ω is analogous. Because now $f(x, u)$ is nondecreasing in u for a.e. $x \in \Omega$, on account of (2.5), (2.6) and (2.7), we have

$$(2.10) \quad 0 < -\Delta_p \underline{u} < f(x, \underline{u}) \leq -\Delta_p u_0 = f(x, u_0) \leq f(x, \bar{u}) < -\Delta_p \bar{u}, \quad x \in \Omega,$$

and so, by the strong comparison principle in [13], we eventually deduce that

$$(2.11) \quad \underline{u}(x) < u_0(x) < \bar{u}(x), \quad x \in \Omega,$$

while

$$(2.12) \quad \frac{\partial \bar{u}}{\partial \nu}(x) < \frac{\partial u_0}{\partial \nu}(x) < \frac{\partial \underline{u}}{\partial \nu}(x), \quad x \in \partial\Omega,$$

where ν denotes the exterior unit normal at $x \in \partial\Omega$. Moreover, by virtue of the strong maximum principle in [18],

$$(2.13) \quad \frac{\partial \underline{u}}{\partial \nu}(x) < 0, \quad x \in \partial\Omega.$$

Note that this inequality holds under the assumption that the boundary $\partial\Omega$ satisfies the so-called interior sphere condition. However, this condition is automatically true here because $\partial\Omega$ was taken to be of class C^2 . In the sequel we shall show that there exists $\delta > 0$ such that

$$(2.14) \quad \underline{u}(x) + \delta \operatorname{dist}(x, \partial\Omega) \leq u_0(x) \leq \bar{u}(x) - \delta \operatorname{dist}(x, \partial\Omega), \quad x \in \Omega.$$

Note first that, since $\partial\Omega$ is compact, an immediate implication of (2.13) is the existence of positive constants β, σ such that

$$(2.15) \quad |\nabla \underline{u}(x)| > \beta > 0,$$

for all x in the annular region

$$\mathcal{R} := \{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) \leq \sigma\}.$$

Furthermore, (2.12), (2.13) and (2.15) imply that there exists a constant $\gamma > 1$ and a continuous function $\mu(\cdot)$ such that

$$(2.16) \quad \frac{\partial u_0}{\partial \nu}(x) = \mu(x) \frac{\partial \underline{u}}{\partial \nu}(x), \quad x \in \partial\Omega,$$

with

$$(2.17) \quad \mu(x) > \gamma > 1.$$

Since now $u_0 = \underline{u} = 0$ on $\partial\Omega$, the projections of $\nabla u_0(x)$ and $\nabla \underline{u}(x)$ on the hyperplane which is tangent to $\partial\Omega$ at x must be equal to zero. Consequently, (2.16) reduces to

$$(2.18) \quad \frac{\partial u_0}{\partial x_i}(x) = \mu(x) \frac{\partial \underline{u}}{\partial x_i}(x), \quad i = 1, \dots, n, \quad x \in \partial\Omega.$$

On the other hand, by the mean value theorem we can write, as in [13],

$$(2.19) \quad -\Delta_p u_0 + \Delta_p \underline{u} = - \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} (u_0 - \underline{u}) \right), \quad x \in \Omega,$$

where

$$a_{ij}(x) := |t_i \nabla u_0 + (1 - t_i) \nabla \underline{u}|^{p-4} \left(\delta_{ij} |t_i \nabla u_0 + (1 - t_i) \nabla \underline{u}|^2 + (p - 2) \left(t_i \frac{\partial u_0}{\partial x_i} + (1 - t_i) \frac{\partial \underline{u}}{\partial x_i} \right) \left(t_i \frac{\partial u_0}{\partial x_j} + (1 - t_i) \frac{\partial \underline{u}}{\partial x_j} \right) \right), \quad x \in \Omega,$$

and $t_i \in (0, 1)$, $i = 1, \dots, n$. By setting now

$$d_i(x) := |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial x_i} - |\nabla \underline{u}|^{p-2} \frac{\partial \underline{u}}{\partial x_i}, \quad i = 1, \dots, n, \quad x \in \overline{\Omega},$$

and using (2.17), (2.18), we have

$$(2.20) \quad d_i(x) = \frac{\mu^{p-1} - 1}{\mu - 1} |\nabla \underline{u}|^{p-2} \frac{\partial}{\partial x_i} (u_0 - \underline{u}), \quad i = 1, \dots, n, \quad x \in \partial\Omega.$$

But since

$$(2.21) \quad -\Delta_p u_0 + \Delta_p \underline{u} = - \sum_i \frac{\partial}{\partial x_i} d_i(x), \quad x \in \Omega,$$

on combining (2.15), (2.17), (2.19), (2.20) and (2.21), we deduce by continuity that the second order differential operator appearing on the righthand side of (2.19) is uniformly elliptic in the region \mathcal{R} . Hence, in view of (2.10), the extension of the classical Hopf’s lemma (see [18]) implies the existence of $\delta_1 > 0$ such that $u_0(x) - \underline{u}(x) \geq \delta_1 \text{dist}(x, \partial\Omega)$ for all $x \in \mathcal{R}$. In a similar fashion it can be shown that there exists $\delta_2 > 0$ such that $\overline{u}(x) - u_0(x) \geq \delta_2 \text{dist}(x, \partial\Omega)$ for all $x \in \mathcal{R}$. The validity of (2.14) for every $x \in \Omega$ then follows by using (2.11) and choosing $\delta > 0$ appropriately. Let now $u \in C_0^1(\Omega)$ with $\|u - u_0\|_{C_0^1} \leq \delta$. Then, $\underline{u} \leq u \leq \overline{u}$ in Ω by (2.14). At the same time, $\Phi = \widehat{\Phi}$ on the set

$$\{u \in C_0^1(\Omega) : \|u - u_0\|_{C_0^1} \leq \delta\}.$$

Therefore, u_0 is a local minimizer of $\Phi(\cdot)$ in $C_0^1(\Omega)$ and by Theorem 2, also a local minimizer of $\Phi(\cdot)$ in $W_0^{1,p}(\Omega)$. Consequently, u_0 is a positive solution of problem (1.1). □

Remark 6. The assumption for the existence of a strict supersolution \bar{u} in Theorem 5 appears to be very essential. Its importance can also be verified by consulting the proofs of the related theorems in [1], [3] and [8] where strict supersolutions are actually constructed. On the other hand, when a strict supersolution \bar{u} for (1.1) is known, it follows from (H3) that a strict subsolution \underline{u} can easily be found with $\underline{u} < \bar{u}$.

Our next result provides the existence of a second positive solution u_1 of (1.1), with $u_0 \leq u_1$ in Ω , if more conditions on the structure of the nonlinearity $f(x, u)$ are imposed. In particular, our strategy involves the use of the Mountain-Pass Theorem for a modified functional $\Psi(\cdot)$ which satisfies the Palais-Smale condition and is unbounded from below under the assumptions:

(H2)' The same growth condition in (H2) holds but with $1 < k < p^*$ if $p < n$.

(H4) There exist $\varrho > 0$ and $\theta \in (0, \frac{1}{p})$ such that

$$(2.22) \quad F(x, u) \leq \theta f(x, u)u \quad \text{when } |u| \geq \varrho.$$

(H5) There exist $\eta > 0$ and $r > p$ such that $\liminf_{s \rightarrow +\infty} \frac{f(x,s)}{s^{r-1}} > \eta$ for a.e. $x \in \Omega$.

Theorem 7. *Suppose that (H1), (H2)', (H3), (H4) and H(5) hold. Assume further that (1.1) possesses a strict supersolution \bar{u} . Then problem (1.1) admits two solutions u_0, u_1 in $W_0^{1,p}(\Omega)$ such that $0 < u_0 \leq u_1$ in Ω .*

PROOF: Let u_0 be the solution obtained in Theorem 5 and consider the problem of finding $v \in W_0^{1,p}(\Omega)$ such that $v \not\equiv 0$ and

$$(2.23) \quad \begin{cases} -\Delta_p(u_0 + v) = f(x, u_0 + v), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

For this set

$$(2.24) \quad k(x, t) := \begin{cases} f(x, u_0 + t), & \text{if } t \geq 0, \\ f(x, u_0), & \text{if } t < 0, \end{cases}$$

$$(2.25) \quad K(x, v) := \int_0^v k(x, t) dt,$$

and define the functional

$$(2.26) \quad \Psi(v) := \frac{1}{p} \|u_0 + v\|^p - \int_{\Omega} K(x, v) dx.$$

We claim that $v = 0$ is a local minimizer of $\Psi(\cdot)$ in $W_0^{1,p}(\Omega)$. Indeed, if v^+ and v^- denote the positive and negative parts of v respectively, we have

$$\begin{aligned} \int_{\Omega} K(x, v) dx &= \int_{\{v \geq 0\}} K(x, v^+) dx + \int_{\{v < 0\}} K(x, v^-) dx \\ &= \int_{\Omega} \int_0^{v^+} k(x, t) dt dx + \int_{\Omega} \int_0^{v^-} k(x, t) dt dx \\ &= \int_{\Omega} \int_0^{v^+} f(x, u_0 + t) dt dx + \int_{\Omega} \int_0^{v^-} f(x, u_0) dt dx \\ &= \int_{\Omega} \int_{u_0}^{u_0+v^+} f(x, t) dt dx + \int_{\Omega} f(x, u_0)v^- dx. \end{aligned}$$

Thus,

$$\begin{aligned} \Psi(v) &= \frac{1}{p} \|u_0 + v^+\|^p + \frac{1}{p} \|u_0 + v^-\|^p - \frac{1}{p} \|u_0\|^p \\ &\quad - \int_{\Omega} \int_{u_0}^{u_0+v^+} f(x, t) dt dx - \int_{\Omega} f(x, u_0)v^- dx \\ &= \Phi(u_0 + v^+) + \int_{\Omega} \int_0^{u_0+v^+} f(x, t) dt dx + \frac{1}{p} \|u_0 + v^-\|^p - \frac{1}{p} \|u_0\|^p \\ &\quad - \int_{\Omega} \int_{u_0}^{u_0+v^+} f(x, t) dt dx - \int_{\Omega} f(x, u_0)v^- dx \\ &= \Phi(u_0 + v^+) + \frac{1}{p} \|u_0 + v^-\|^p - \frac{1}{p} \|u_0\|^p \\ &\quad + \int_{\Omega} \int_0^{u_0} f(x, t) dt dx - \int_{\Omega} f(x, u_0)v^- dx. \end{aligned}$$

Moreover, since u_0 solves (1.1),

$$\int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \nabla v^- dx = \int_{\Omega} f(x, u_0)v^- dx,$$

and so

$$\Psi(v) = \Phi(u_0 + v^+) - \Phi(u_0) + \frac{1}{p} \|u_0 + v^-\|^p - \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \nabla v^- dx.$$

On the other hand, by the strict convexity of the mapping $\xi \mapsto |\xi|^p$ for any $p > 1$, the following inequality holds

$$(2.27) \quad |\xi_2|^p \geq |\xi_1|^p + p |\xi_1|^{p-2} \xi_1 \cdot (\xi_2 - \xi_1), \quad \xi_1, \xi_2 \in \mathbb{R}^n,$$

which yields

$$\Psi(v) \geq \Phi(u_0 + v^+) - \Phi(u_0) + \frac{1}{p} \|u_0\|^p.$$

But since u_0 is a local minimizer of $\Phi(\cdot)$ in $W_0^{1,p}(\Omega)$, this implies

$$\Psi(v) \geq \frac{1}{p} \|u_0\|^p = \Psi(0),$$

if $\|v\|$ is small enough, thereby proving the claim. At the same time it is easily checked that, on account of $(H2)'$ and $(H4)$, the functional $\Psi(\cdot)$ satisfies the Palais-Smale condition (see [4]). Moreover, by using $(H5)$, $\Psi(tu_0) \rightarrow -\infty$ as $t \rightarrow +\infty$ and so there exists $t_0 > 0$ such that $\Psi(t_0u_0) < 0$. Hence, by applying the Ghoussoub-Preiss version of the Mountain-Pass Theorem [12] we get the existence of a second critical point $v_0 \neq 0$ of $\Psi(\cdot)$. In particular, $v_0 \in C^1(\bar{\Omega})$ by virtue of Theorem 1. We shall now show that $v_0 \geq 0$. Indeed, since $\Psi'(v_0) = 0$, we have

$$\int_{\Omega} |\nabla u_0 + \nabla v_0|^{p-2} (\nabla u_0 + \nabla v_0) \nabla z \, dx = \int_{\Omega} k(x, v_0) z \, dx, \quad z \in W_0^{1,p}(\Omega),$$

and by choosing $z = v_0^-$,

$$\begin{aligned} (2.28) \quad \int_{\Omega} |\nabla u_0 + \nabla v_0^-|^{p-2} (\nabla u_0 + \nabla v_0^-) \nabla v_0^- \, dx &= \int_{\Omega} k(x, v_0^-) v_0^- \, dx \\ &= \int_{\Omega} f(x, u_0) v_0^- \, dx. \end{aligned}$$

At the same time, since $\Phi'(u_0) = 0$,

$$(2.29) \quad \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \nabla v_0^- \, dx = \int_{\Omega} f(x, u_0) v_0^- \, dx,$$

and so, on combining (2.28) and (2.29),

$$(2.30) \quad \int_{\Omega} \left\{ |\nabla u_0 + \nabla v_0^-|^{p-2} (\nabla u_0 + \nabla v_0^-) - |\nabla u_0|^{p-2} \nabla u_0 \right\} \nabla v_0^- \, dx = 0,$$

which, by applying Lemma 3, yields

$$(2.31) \quad \|u_0 + v_0^-\| = \|u_0\|.$$

On the other hand, by applying (2.27) with $\xi_1 = \nabla u_0 + \nabla v_0^-$, $\xi_2 = \nabla u_0$ and using (2.31), we get

$$(2.32) \quad \int_{\Omega} |\nabla u_0 + \nabla v_0^-|^{p-2} (\nabla u_0 + \nabla v_0^-) \nabla v_0^- \, dx \geq 0,$$

while by doing the same with $\xi_1 = \nabla u_0$, $\xi_2 = \nabla u_0 + \nabla v_0^-$,

$$(2.33) \quad \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \nabla v_0^- \, dx \leq 0.$$

Thus, from (2.29), (2.30), (2.32) and (2.33) we conclude that

$$\int_{\Omega} f(x, u_0) v_0^- \, dx = 0,$$

which, since $f(x, u_0) > 0$, implies $v_0^- = 0$; i.e. $v_0 \geq 0$. Hence, $u_1 := u_0 + v_0$ is a second positive solution of (1.1). The proof is complete. \square

Appendix

PROOF OF THEOREM 1: We proceed by examining separately three different ranges of the exponent p and showing first that $u \in L^\infty(\Omega)$.

(I) Let $1 < p < n$. Clearly, $u \in L^{p^*}(\Omega)$ by Sobolev's inequality. We now distinguish two cases:

Case 1: $1 < k < p^*$.

Then, by virtue of Theorem 7.1, Chapter IV, of [14], we immediately conclude that $u \in L^\infty(\Omega)$.

Case 2: $k = p^*$.

Here Theorem 7.1, Chapter IV, of [14] cannot be applied directly. Hence, motivated by [10], we proceed by decomposing $f(\cdot, u(\cdot))$ as follows:

$$f(x, u(x)) = a(x)|u(x)|^{p-2}u(x) + b(x),$$

where

$$(2.34) \quad a(x) := \begin{cases} \frac{f(x, u(x))}{|u(x)|^{p-2}u(x)}, & |u(x)| > 1, \\ 0, & |u(x)| \leq 1, \end{cases}$$

and

$$(2.35) \quad b(x) := \begin{cases} 0, & |u(x)| > 1, \\ f(x, u(x)), & |u(x)| \leq 1. \end{cases}$$

Then, in view of (H2), it is easily seen that $b \in L^\infty(\Omega)$ while $a(\cdot)$ satisfies the growth estimate

$$|a(x)| \leq C_1(1 + |u(x)|^{p^*-p}),$$

which, since $u \in L^{p^*}(\Omega)$, implies that $a \in L^{n/p}(\Omega)$.

For any $m \in \mathbb{N}$ we set

$$u_m := \begin{cases} m & \text{if } u \geq m, \\ u & \text{if } |u| < m, \\ -m & \text{if } u \leq -m. \end{cases}$$

Clearly, if $r \geq 2$ then $|u_m|^{r-2}u_m \in W_0^{1,p}(\Omega)$ and so by multiplying (1.1)₁ with $|u_m|^{r-2}u_m$ and integrating over Ω we get

$$\begin{aligned} & (r-1) \sum_{i=1}^n \int_{\Omega} \left(|\nabla u_m|^{p-2} \frac{\partial u_m}{\partial x_i} \right) \left(|u_m|^{r-2} \frac{\partial u_m}{\partial x_i} \right) \\ &= \int_{\Omega} a |u|^{p-2} u |u_m|^{r-2} u_m + \int_{\Omega} b |u_m|^{r-2} u_m, \end{aligned}$$

which, since $uu_m \geq 0$ a.e. in Ω , gives

$$(2.36) \quad (r-1) \int_{\Omega} |\nabla u_m|^p |u_m|^{r-2} \leq \int_{\Omega} a^+ |u|^{p-1} |u_m|^{r-1} + \int_{\Omega} b |u_m|^{r-2} u_m,$$

where a^+ denotes the positive part of $a(\cdot)$. At the same time, it can be easily verified that

$$(2.37) \quad \int_{\Omega} |\nabla u_m|^p |u_m|^{r-2} = \left(\frac{p}{p+r-2} \right)^p \int_{\Omega} \left| \nabla \left(|u_m|^{\frac{r-2}{p}} u_m \right) \right|^p.$$

Moreover, by Sobolev’s inequality

$$(2.38) \quad \left\| |u_m|^{\frac{p+r-2}{p}} \right\|_{L^{p^*}} \leq C_S \left\| \nabla |u_m|^{\frac{p+r-2}{p}} \right\|_{L^p} = C_S \left\| \nabla \left(|u_m|^{\frac{r-2}{p}} u_m \right) \right\|_{L^p}$$

where C_S is the best Sobolev constant. Hence, on combining (2.36), (2.37) and (2.38), we deduce that

$$(2.39) \quad \left\| |u_m|^{\frac{p+r-2}{p}} \right\|_{L^{p^*}}^p \leq c_1 \left(\int_{\Omega} a^+ |u|^{p-1} |u_m|^{r-1} + \int_{\Omega} b |u_m|^{r-2} u_m \right)$$

where $c_1 > 0$ is a constant depending only on p, r and C_S . Fix now $k > 0$ and let $\Omega_1 := \{x \in \Omega : a^+(x) \leq k\}$ and $\Omega_2 := \{x \in \Omega : a^+(x) > k\}$. Since $|u_m| \leq |u|$ a.e. in Ω , (2.39) gives

$$\begin{aligned} (2.40) \quad & \left\| |u_m|^{\frac{p+r-2}{p}} \right\|_{L^{p^*}}^p \\ & \leq kc_1 \int_{\Omega_1} |u|^{p+r-2} + c_1 \int_{\Omega_2} a^+ |u|^{p+r-2} + c_1 \int_{\Omega} b |u_m|^{r-2} u_m. \end{aligned}$$

Because Ω is bounded, there exists a constant $c_2 > 0$, depending only on Ω , p and r , such that

$$\|u_m\|_{L^{r-1}} \leq c_2 \|u_m\|_{L^{p+r-2}},$$

and so

$$(2.41) \quad \int_{\Omega} b|u_m|^{r-2}u_m \leq c_2 \|b\|_{L^\infty} \|u_m\|_{L^{p+r-2}}^{r-1}.$$

On the other hand, by virtue of Hölder’s inequality,

$$(2.42) \quad \begin{aligned} \int_{\Omega_2} a^+|u|^{p+r-2} &\leq \left(\int_{\Omega_2} |a^+|^{\frac{n}{p}}\right)^{\frac{p}{n}} \left(\int_{\Omega_2} |u|^{\frac{n}{n-p}(p+r-2)}\right)^{\frac{n-p}{n}} \\ &\leq \|a^+\|_{L^{\frac{n}{p}}(\Omega_2)} \|u\|_{L^{\frac{p^*}{p}(p+r-2)}}^{p+r-2}. \end{aligned}$$

Thus, in view of (2.41) and (2.42), inequality (2.40) yields

$$\begin{aligned} &\|u_m\|_{L^{\frac{p^*}{p}(p+r-2)}}^{p+r-2} \\ &\leq kc_1 \int_{\Omega} |u|^{p+r-2} + c_1 \|a^+\|_{L^{\frac{n}{p}}(\Omega_2)} \|u\|_{L^{\frac{p^*}{p}(p+r-2)}}^{p+r-2} + c_3 \|b\|_{L^\infty} \|u_m\|_{L^{p+r-2}}^{r-1}, \end{aligned}$$

and by choosing $k > 0$ large enough so that $c_1 \|a^+\|_{L^{\frac{n}{p}}(\Omega_2)} \leq \frac{1}{2}$,

$$\|u_m\|_{L^{\frac{p^*}{p}(p+r-2)}}^{p+r-2} \leq 2kc_1 \int_{\Omega_1} |u|^{p+r-2} + 2c_3 \|b\|_{L^\infty} \|u_m\|_{L^{p+r-2}}^{r-1}.$$

Assuming now that $u \in L^{p+r-2}(\Omega)$, if we allow $m \rightarrow \infty$ in the last inequality, we get

$$(2.43) \quad \|u\|_{L^{\frac{p^*}{p}(p+r-2)}}^{p+r-2} \leq 2kc_1 \|u\|_{L^{p+r-2}}^{p+r-2} + 2c_3 \|b\|_{L^\infty} \|u\|_{L^{p+r-2}}^{r-1},$$

which implies that $u \in L^{\frac{p^*}{p}(p+r-2)}(\Omega)$. Hence, by starting from $r = p^* - p + 2$ and bootstrapping (2.43) we easily deduce that $u \in L^s(\Omega)$ for every $s \in [p, +\infty)$. Therefore, $u \in W_0^{1,p}(\Omega) \cap L^s(\Omega)$ for every $s \in [p^*, +\infty)$ and so, by virtue of Theorem 7.1, Chapter IV, of [14], we deduce again that $u \in L^\infty(\Omega)$.

(II) Suppose now $p = n$. Then, $u \in L^q(\Omega)$ for any $q \in [1, +\infty)$ by the Sobolev embedding. Hence, on account of (H2), $f(\cdot, u(\cdot)) \in L^q(\Omega)$ for any $q \in [1, +\infty)$ and so by a standard bootstrap procedure in the spirit of Moser [16] we infer that $u \in L^\infty(\Omega)$ (see e.g. the proof of Proposition 2.1 in [11]).

(III) Finally, let $p > n$. Then, $u \in L^\infty(\Omega)$ directly by the Sobolev embedding. The assertion of the proposition now follows by applying Theorem 1 of [15]. \square

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