On self-homeomorphic dendrites

Janusz J. Charatonik, Paweł Krupski

Abstract. It is shown that for every numbers $m_1, m_2 \in \{3, \dots, \omega\}$ there is a strongly self-homeomorphic dendrite which is not pointwise self-homeomorphic. The set of all points at which the dendrite is pointwise self-homeomorphic is characterized. A general method of constructing a large family of dendrites with the same property is presented.

Keywords: dendrite, self-homeomorphic

Classification: 54F15, 54F50

1. Introduction

In [3, Section 2, p. 217] (see also [4, Section 2, p. 283]) the following four types of self-homeomorphic spaces are introduced and studied.

A topological space X is said to be:

— *self-homeomorphic* (concisely SH) provided that for each open set $U \subset X$ there is a set $W \subset U$ such that W is homeomorphic to X;

— *strongly self-homeomorphic* (concisely SSH) provided that for each open set $U \subset X$ there is a set $W \subset U$ with nonempty interior such that W is homeomorphic to X;

— *pointwise self-homeomorphic at a point* $x \in X$ provided that for each neighborhood U of x there is a set W such that $x \in W \subset U$ and W is homeomorphic to X; the space is said to be *pointwise self-homeomorphic* (concisely PSH) provided that it is pointwise self-homeomorphic at each of its points;

— *strongly pointwise self-homeomorphic at a point* $x \in X$ provided that for each neighborhood U of x there is a neighborhood W of x such that $x \in W \subset U$ and W is homeomorphic to X; the space is said to be *strongly pointwise self-homeomorphic* (concisely SPSH) provided that it is pointwise self-homeomorphic at each of its points.

The following diagram of implications applies to the above definitions (see [3, Theorem 2.5, p. 217]).

$$\begin{array}{ccc} X \in \mathrm{SPSH} & \Longrightarrow & X \in \mathrm{PSH} \\ \Downarrow & & \Downarrow \\ X \in \mathrm{SSH} & \Longrightarrow & X \in \mathrm{SH} \end{array} \tag{1.1}$$

Questions are asked in [3, Problems 6.21 and 6.23, p. 237] whether $X \in$ SH (or $X \in$ SSH) implies that $X \in$ PSH if X is a dendrite. A negative answer to both these questions is given in [7], where a dendrite $X(3,\omega)$ is constructed which is SSH (at each of its points) but not PSH (at some of its end points). In this paper we extend the result in several directions. First, it is shown that for every numbers $m_1, m_2 \in \{3,\dots,\omega\}$ there is a dendrite $X(m_1,m_2)$ which is SSH but not PSH. Second, the set of all points $\mathrm{PSH}(X(m_1,m_2))$ at which the dendrite is PSH is studied and characterized. Third, a general method of constructing a large family of dendrites with the same property is presented, and for each member of the family the set of its points at which the dendrite is PSH is characterized.

2. Preliminaries

A *mapping* means a continuous transformation. A *continuum* means a compact connected metric space. A countable family of metric spaces $\{M_n : n \in \mathbb{N}\}$ is called a *null-family* provided that $\lim \operatorname{diam} M_n = 0$.

A *dendrite* means a locally connected continuum containing no simple closed curve. Various characterizations of dendrites are collected in [2]. Compare also [6, Chapter X, Part 1, p. 166]. Given two points p and q of a dendrite X, we denote by pq the unique arc from p to q in X.

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [5, §51, I, p. 274]), and we denote by $\operatorname{ord}(p,X)$ order of the space X at a point $p \in X$. It is well-known (see e.g. [5, §51, p. 274–307]) that the function ord takes its values from the set $\{0,1,2,\dots,\omega,\aleph_0,2^{\aleph_0}\}$. Points of order 1 in a space X are called *end points* of X; the set of all end points of X is denoted by $E(X)$. Points of order 2 are called *ordinary points* of X. It is known that in a dendrite the set of all its ordinary points is a dense subset of the dendrite. For each $n \in \{3,4,\dots,\omega,\aleph_0,2^{\aleph_0}\}$ points of order n are called *ramification points* of X; the set of all ramification points is denoted by $R(X)$. For each dendrite X points of order $\aleph_0$ and $2^{\aleph_0}$ do not occur in X and the set $R(X)$ is at most countable [6, Theorems 10.20 and 10.23, p. 173 and 174]. Thus, for any ramification point p of a dendrite X the value $\operatorname{ord}(p,X)$ is in the set $\{3,\dots,\omega\}$.

For a given integer $n \geq 3$, a *simple n-od* is a space homeomorphic to the cone over an n-point discrete space. The point of a simple n-od T which corresponds to the vertex of the cone (i.e. the only ramification point of T) is called a *vertex* of T. For an end point e of an n-od T the arc ve is called an *arm* of T.

Given a dendrite X we decompose it into disjoint subsets of points of a fixed order. Namely for each $n \in \{1,2,3,\dots,\omega\}$ we put

$$R_n(X) = \{p \in X : \operatorname{ord}(p,X) = n\}.$$

By a *free arc* A in a space X we mean an arc A with end points x and y such that $A \setminus \{x,y\}$ is an open subset of X. In particular, by a *maximal free arc* in a dendrite X we mean such an arc $st \subset X$ that $st \cap (E(X) \cup R(X)) = \{s,t\}$.

3. Generalized Pyrih's dendrite

An idea of the following construction is based on P. Pyrih's construction in [7, Section 2, p. 572–575].

Let a straight line segment $A = ab$ be fixed. For each $m \in \{3, \dots, \omega\}$ we define two auxiliary dendrites $G_0(A, m)$ and $G_1(A, m)$. Choose a countable dense set $D(A) \subset A \setminus E(A) = ab \setminus \{a, b\}$. To each point $x \in D(A)$ we attach an m-od $T(m, x)$ in such a way that:

(3.1) the point $x \in D(A)$ is the vertex of $T(m, x)$;
(3.2) $A \cap T(m, x) = \{x\}$;
(3.3) $T(m, x_1) \cap T(m, x_2) = \emptyset$ for $x_1, x_2 \in D(A)$ with $x_1 \neq x_2$;
(3.4) $\{T(m, x) : x \in D(A)\}$ is a null-family;
(3.5) $\operatorname{diam} T(m, x) < \frac{1}{2} \operatorname{diam} A$ for each $x \in D(A)$.

Thus the union

$$G_0(A, m) = A \cup \bigcup \{T(m, x) : x \in D(A)\} \tag{3.6}$$

is a dendrite.

Everything is the same in the definition of $G_1(A, m)$ except that condition (3.1) is replaced by

(3.7) the point $x \in D(A)$ is an end point of $T(m, x)$.

Again the union

$$G_1(A, m) = A \cup \bigcup \{T(m, x) : x \in D(A)\} \tag{3.8}$$

is a dendrite. Note that

(3.9) $R(G_0(A, m)) = R_{2+m}(G_0(A, m)) \subset A$ (with $2 + \omega = \omega$), and that
(3.10) $R(G_1(A, m)) = R_3(G_1(A, m)) \cup R_m(G_1(A, m))$, where
(3.11) $A \subset \operatorname{cl}(R_m(G_1(A, m)))$ and, if $m \neq 3$, then $R_3(G_1(A, m)) \subset A$.

Let

$$m_1, m_2 \in \{3, \dots, \omega\} \tag{3.12}$$

be fixed. We define a dendrite $X(m_1, m_2)$ as the inverse limit of an inverse sequence of dendrites X_n with bonding mappings being monotone retractions, as follows.

Let X_1 be a simple m_1-od. Define X_2 as a space obtained from X_1 replacing each maximal free arc A of X_1, which obviously is an arm of X_1, by the dendrite $G_0(A, m_2)$ in such a way that, if v_0 denotes the vertex of X_1, then

$$G_0(A_1, m_2) \cap G_0(A_2, m_2) = \{v_0\} \quad \text{for} \quad A_1 \neq A_2.$$

Thus $X_1 \subset X_2$ and X_2 is a dendrite. Let $f_1 : X_2 \to X_1$ be a monotone retraction, that is, $f_1 \,|\, X_1$ is the identity, and, for each maximal free arc $A \subset X_1$ and for each point $x \in D(A) \subset A$, if $T(m_2, x)$ is the m_2-od attached at the point x according to the definition (3.6) of $G_0(A, m_2)$, then $f_1 \,|\, T(m_2, x)$ is a constant mapping, with $f_1(T(m_2, x)) = \{x\}$.

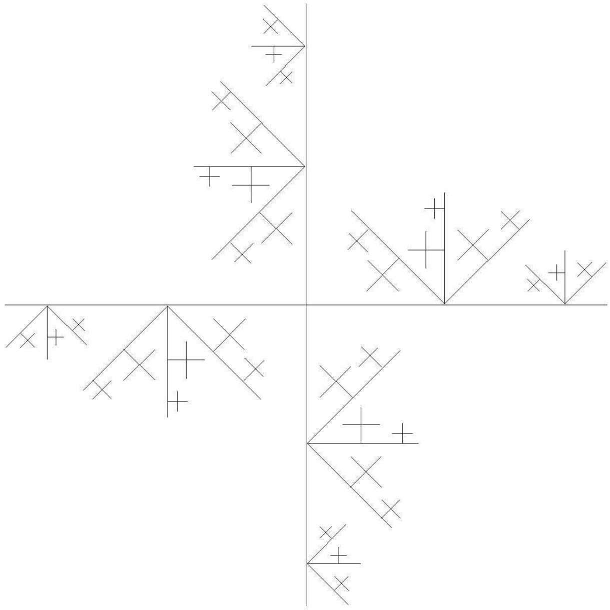

Figure: X_3 for $m_1 = 4, m_2 = 3$

Define X_3 as a space obtained from X_2 replacing each maximal free arc A of X_2, which obviously is an arm of $T(m_2, x)$, for some $x \in D(B)$, where B is a maximal free arc of X_1, by the dendrite $G_1(A, m_1)$ in such a way that $G_1(A_1, m_1) \cap G_1(A_2, m_1) = A_1 \cap A_2$ for every two distinct maximal free arcs A_1 and A_2 of X_2, that is, the intersection is either empty or it is a singleton $\{x\}$ for some $x \in D(B)$ as above (see Figure).

Thus $X_2 \subset X_3$ and X_3 is a dendrite. Let $f_2 : X_3 \to X_2$ be a monotone retraction, that is, $f_2 \,|\, X_2$ is the identity, and, for each maximal free arc $A \subset X_2$ and for each point $x \in D(A) \subset A$, if $T(m_1, x)$ is the m_1-od attached at the point x according to the definition (3.7) of $G_1(A, m_1)$, then $f_2 \,|\, T(m_1, x)$ is a constant mapping, with $f_2(T(m_1, x)) = \{x\}$.

The dendrite X_4 is constructed from X_3 in the same way as X_2 from X_1, i.e., with replacing each free arc A of X_3 by a dendrite $G_0(A, m_2)$, and the mapping $f_3 : X_4 \to X_3$ is again a monotone retraction. We continue this construction using interchangeably the auxiliary dendrites $G_0(A, m_2)$ or $G_1(A, m_1)$ to create X_{n+1} from X_n depending on n is even or odd, respectively, and defining $f_n : X_{n+1} \to X_n$ always as a monotone retraction. Then the inverse limit space

$$X(m_1, m_2) = \varprojlim \{X_n, f_n; n \in \mathbb{N}\} \tag{3.13}$$

is a dendrite by [6, Theorem 10.36, p. 180]. Moreover, condition (3.5) of the above construction guarantees that the assumptions of the Anderson-Choquet embedding theorem (see [6, Theorem 2.10, p. 23]) are satisfied, whence it follows that

$$X(m_1, m_2) \text{ is homeomorphic to } \operatorname{cl}(\bigcup\{X_n : n \in \mathbb{N}\}). \tag{3.14}$$

This completes the construction of the dendrite $X(m_1, m_2)$ for any pair m_1, m_2 as in (3.12).

Observe that if $m_1 = 3$ and $m_2 = \omega$ we get just the Pyrih's dendrite $X(3, \omega)$ as defined in [7, p. 574].

By (3.14) we may assume that

$$X(m_1, m_2) = \operatorname{cl}(\bigcup\{X_n : n \in \mathbb{N}\}).$$

Now we will prove the needed properties of $X(m_1, m_2)$. We start with the following extension of [7, (iii), p. 574].

Theorem 3.15. *For every $m_1, m_2 \in \{3, \dots, \omega\}$ the dendrite $X(m_1, m_2)$ defined by* (3.13) *is* SSH.

Proof: To show that $X(m_1, m_2)$ is SSH, let U be an open subset of $X(m_1, m_2)$. Note that

(3.15.1) for each $n \in \mathbb{N}$ the difference $R(X(m_1, m_2)) \setminus R(X_n)$ is a dense subset of $X(m_1, m_2)$,

and that

(3.15.2) for each $n \in \mathbb{N}$ the union $D = \bigcup\{A : A$ is a maximal free arc in $X_n\}$ is a dense subset of X_n.

Conditions (3.4) and (3.5) imply that there is a number $n_0 \in \mathbb{N}$ such that some free arc A of X_{n_0} is contained in U. Further, it follows from (3.4) that there is a point $x \in D(A)$ such that if K is a component of $X(m_1, m_2) \setminus \{x\}$ satisfying $K \cap A = \emptyset$, then $K \subset U$. Observe that K is an open subset of $X(m_1, m_2)$. Take an even number $n_1 > n_0$ and note that we use the dendrites $G_1(A, m_1)$ to construct X_{n_1+1} from X_{n_1}. Thus (3.15.1) and (3.15.2) imply that there is a maximal free arc A_1 of X_{n_1} such that $A_1 \subset K$. Take a point $p \in D(A_1) \subset A_1$, and let $T(m_1, p)$

denote a copy of the m_1-od attached at the point p as a subset of $G_1(A_1, m_1)$ in the construction of X_{n_1+1}. Then

$$W = \{p\} \cup \{x \in X(m_1, m_2) : px \cap (T(m_1, p) \setminus \{p\}) \neq \emptyset\}$$

is the needed subset of U with nonempty interior which is homeomorphic to $X(m_1, m_2)$ by the construction. □

It is shown in [7, (iv), p. 575] that $X(3, \omega)$ contains a point at which it is not PSH. To find the set of *all* points of the dendrite $X(m_1, m_2)$ at which it is PSH we need some auxiliary results. Recall that v_0 denotes the vertex of the simple m_1-od $X_1 \subset X(m_1, m_2)$.

Observation 3.16. *For every $m_1, m_2 \in \{3, \ldots, \omega\}$ the following property of a point $v \in X(m_1, m_2)$ is topological:*

(3.16.1) *there exists an m_1-od $T(m_1) \subset X(m_1, m_2)$ having the point v as its vertex, such that each ramification point of $X(m_1, m_2)$ lying in $T(m_1) \setminus \{v\}$ is of order $2 + m_2$.*

Accept the following notation. Let V be the set consisting of the vertex v_0 of X_1 and of the vertices of all copies of X_1 attached in the sequential steps of the construction of $X(m_1, m_2)$. More precisely, a point $v \in X(m_1, m_2)$ is in the set V if and only if either $v = v_0$ or v is the vertex of an m_1-od $T(m_1, x) \subset G_1(A, m_1)$ satisfying (3.7) for some point $x \in D(A)$ and some maximal free arc A in certain X_{2n}. Observe that, since obviously v_0 satisfies (3.16.1), then

$$V = \{v \in X(m_1, m_2) : v \text{ satisfies condition (3.16.1)}\}.$$

Thus $V \subset R_{m_1}(X(m_1, m_2))$.

It follows from Observation 3.16 that $h(V) \subset V$ for any homeomorphism h of $X(m_1, m_2)$ into itself. Since v_0 satisfies (3.16.1), we get the following.

Statement 3.17. *Let $m_1, m_2 \in \{3, \ldots, \omega\}$. If $h : X(m_1, m_2) \to h(X(m_1, m_2)) \subset X(m_1, m_2)$ is a homeomorphism, then $h(v_0) \in V$.*

For a continuum X let PSH (X) denote the set of points $p \in X$ such that X is pointwise self-homeomorphic at p.

Theorem 3.18. *For every $m_1, m_2 \in \{3, \ldots, \omega\}$ we have*

$$\text{PSH}\,(X(m_1, m_2)) = V.$$

Proof: If $v \in V$ and U is an open set containing v, then there is a small enough m_1-od $T(m_1) \subset T(m_1, x)$ with the vertex v such that $T(m_1) \subset U$. Take $W = \pi_{2n}^{-1}(T(m_1))$ for some n, where $\pi_k : X(m_1, m_2) \to X_k$ denotes the projection of the inverse limit (3.13). It follows from the construction of $X(m_1, m_2)$ that $v \in W \subset U$ and W is homeomorphic to $X(m_1, m_2)$. Thus one inclusion is shown.

To prove the other one take a point p at which $X(m_1, m_2)$ is PSH and suppose on the contrary that $p \notin V$. Then there is a homeomorphism $h : X(m_1, m_2) \to h(X(m_1, m_2)) \subset X(m_1, m_2)$ such that the copy $h(X(m_1, m_2))$ of $X(m_1, m_2)$ is small enough and such that the following conditions are satisfied (that are easy observations from the construction):

1) $p \in h(X(m_1, m_2))$;
2) $h(v_0) \in V \setminus \{v_0\}$;
3) the arc $A = ph(v_0) \subset h(X(m_1, m_2))$ is of the form $A = A_1 \cup A_2 \cup \cdots \cup \{p\}$ (of finitely or infinitely many sets A_i) such that each A_i is an arc, $A_i \cap A_{i+1}$ consists of a common end point of these two arcs, $h(v_0)$ is an end point of A_1, and $R(X(m_1, m_2)) \cap (A_i \setminus E(A_i))$ contains solely points of order $2 + m_2$ if i is odd, and of order 3 if i is even.

Let $a \in X(m_1, m_2)$ be such that $h(a) = p$, and let $B_i = h^{-1}(A_i)$ for each i. Then $v_0 a = h^{-1}(A) = B_1 \cup B_2 \cup \cdots \cup \{a\}$, where $R(X(m_1, m_2)) \cap (B_i \setminus E(B_i))$ contains solely points of order $2 + m_2$ if i is odd, and of order 3 if i is even. Then there exists an arm A' of the (initial) m_1-od X_1 that contains the arc B_1. Thus each ramification point of $X(m_1, m_2)$ lying on the arc $\mathrm{cl}(A' \setminus B_1)$ except of its end points is of order $2 + m_2$ in $X(m_1, m_2)$. It follows that each ramification point of $h(X(m_1, m_2))$ lying on the arc $\mathrm{cl}(h(A') \setminus A_1)$ except of its end points is also of order $2 + m_2$ in $h(X(m_1, m_2))$, while a subarc of the arc $\mathrm{cl}(h(A') \setminus A_1)$ contains points of order 3 by the construction. This contradiction completes that proof. □

4. General construction

The whole contents of the previous section, being an extension of results from [7], can be considered as a very special case of a more general approach, presented below.

For a given sequence

$$\sigma = (m_1, m_2, m_3, \dots), \quad \text{where} \quad m_n \in \{3, \dots, \omega\} \quad \text{for each} \quad n \in \mathbb{N}, \tag{4.1}$$

define a dendrite $X(\sigma)$ as the inverse limit of an inverse sequence of dendrites X_n with bonding mappings being monotone retractions, as follows.

Let X_1, X_2 and $f_1 : X_2 \to X_1$ be defined as above, in the previous section.

Assume that a dendrite X_n is defined for some $n \in \mathbb{N}$. Define X_{n+1} as a space obtained from X_n replacing each maximal free arc A of X_n, either by the dendrite $G_0(A, m_{n+1})$ (if n is odd), or by the dendrite $G_1(A, m_{n+1})$ (if n is even), in such a way that

$$G_i(A_1, m_{n+1}) \cap G_i(A_2, m_{n+1}) = A_1 \cap A_2 \quad \text{for} \quad i \in \{0, 1\}$$
$$\text{and for every two distinct maximal free arcs} \quad A_1 \quad \text{and} \quad A_2 \quad \text{of} \quad X_n.$$

Thus $X_n \subset X_{n+1}$ and X_{n+1} is a dendrite. Let $f_n : X_{n+1} \to X_n$ be a monotone retraction. Thus the inverse sequence $\{X_n, f_n; n \in \mathbb{N}\}$ is defined, and its inverse limit

$$X(\sigma) = \varprojlim \{X_n, f_n; n \in \mathbb{N}\} \tag{4.2}$$

is a dendrite again by [6, Theorem 10.36, p. 180]. Similarly as in (3.14) we have

$$X(\sigma) \text{ is homeomorphic to } \operatorname{cl}\left(\bigcup\{X_n : n \in \mathbb{N}\}\right). \tag{4.3}$$

Using the above construction and repeating the arguments of the previous section (with necessary changes) one can show the following results.

Theorem 4.4. *Let an integer $k \geq 2$ and a finite sequence $(m_1, \ldots, m_k)$ with $m_i \in \{3, \ldots, \omega\}$ for each $i \in \{1, \ldots, k\}$ be fixed. Let $\sigma = (m_n : n \in \mathbb{N})$ be a periodic sequence of period k determined by*

$$m_n = m_i \text{ if } n \equiv i \pmod{k}. \tag{4.4.1}$$

Then the dendrite $X(\sigma)$ defined by (4.2) *is* SSH.

The following is an analog of Observation 3.16.

Observation 4.5. *For each sequence $\sigma = (m_n : n \in \mathbb{N})$ satisfying* (4.1) *the following property of a point $v \in X(\sigma)$ is topological:*

(4.5.1) *there exists an m_1-od $T(m_1) \subset X(\sigma)$ having the point v as its vertex, such that each ramification point of $X(\sigma)$ lying in $T(m_1) \setminus \{v\}$ is of order $2 + m_2$.*

Define

$$V(X(\sigma)) = \{v \in X(\sigma) : v \text{ satisfies condition (4.5.1)}\}.$$

An easy modification of the proof of Theorem 3.18 leads to the following result.

Theorem 4.6. *The equality* $\operatorname{PSH}(X(\sigma)) = V(X(\sigma))$ *holds for each sequence σ satisfying* (4.1).

Finally, let us remark that if the periodicity condition (4.4.1) of the sequence σ is replaced by demanding that all terms m_n of σ are different from each other, then the general method presented above can be used in construction of chaotic and/or rigid dendrites. For details see [1].

References

[1] Charatonik J.J., Charatonik W.J., *Strongly chaotic dendrites*, Colloq. Math. **70** (1996), 181–190.
[2] Charatonik J.J., Charatonik W.J., *Dendrites*, Aportaciones Mat. Comun. **22** (1998), 227–253.
[3] Charatonik W.J., Dilks A., *On self-homeomorphic spaces*, Topology Appl. **55** (1994), 215–238.
[4] Charatonik W.J., Dilks Dye A., Reed J.F., *Self-homeomorphic star figures*, Continuum Theory and Dynamical Systems. Papers of the conference/workshop on continuum theory and dynamical systems held at Lafayette, LA (USA), (Thelma West, ed.), Lect. Notes Pure Appl. Math. 149, M. Dekker, New York, 1993, pp. 283–290.
[5] Kuratowski K., *Topology*, vol. 2, Academic Press and PWN, New York, London, Warszawa, 1968.
[6] Nadler S.B., Jr., *Continuum Theory: An Introduction*, M. Dekker, 1992.
[7] Pyrih P., *An example of strongly self-homeomorphic dendrite not pointwise self-homeomorphic*, Comment. Math. Univ. Carolinae **40** (1999), 571–576.

(J.J. Charatonik)
Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4,
50-384 Wrocław, Poland

E-mail: jjc@hera.math.uni.wroc.pl

Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria,
04510 México, D.F., México

E-mail: jjc@math.unam.mx

(P. Krupski)
Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4,
50-384 Wrocław, Poland

E-mail: krupski@hera.math.uni.wroc.pl

Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria,
04510 México, D.F., México

E-mail: krupski@math.unam.mx

(*Received* October 1, 2001)