On nonresonance impulsive functional nonconvex valued differential inclusions

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Abstract. In this paper a fixed point theorem for contraction multivalued maps due to Covitz and Nadler is used to investigate the existence of solutions for first and second order nonresonance impulsive functional differential inclusions in Banach spaces.

Keywords: impulsive functional differential inclusions, nonresonance problem, fixed point, Banach space

Classification: 34A37, 34A60, 34G20, 34K25

1. Introduction

This paper is concerned with the existence of solutions for nonresonance problems for first and second order functional differential inclusions with impulsive effects. More specifically, in Section 3, we consider the nonresonance problem for the first order impulsive functional differential inclusions in a Banach space with periodic boundary conditions

$$y'(t) - \lambda y(t) \in F(t, y_t), \quad t \in J = [0,T], \ t \neq t_k, \quad k = 1, \dots, m, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad y(0) = y(T), \tag{1.3}$$

where $\lambda \in \mathbb{R}$, $0 < r < \infty$, $F : J \times D \to \mathcal{P}(E)$ is a multivalued map, $D = \{\psi : [-r,0] \to E;\ \psi$ is continuous everywhere except for a finite number of points $\tilde{t}$ at which $\psi(\tilde{t}^-)$ and $\psi(\tilde{t}^+)$ exist and $\psi(\tilde{t}^-) = \psi(\tilde{t})\}$, $\phi \in D$, $\mathcal{P}(E)$ is the family of all subsets of E, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_k : E \to E \ \ (k = 1, 2, \dots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively, and E a real separable Banach space with norm $\|\cdot\|$.

For any continuous function y defined on $[-r,T] \setminus \{t_1, \dots, t_m\}$ and any $t \in J$, we denote by y_t the element of $C([-r,0],E)$ defined by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r,0]$. Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

In Section 4 we consider a nonresonance problem for the second order impulsive functional differential inclusions with more general boundary conditions in the Banach space E,

$$y''(t) - \lambda y(t) \in F(t, y_t), \quad t \in J = [0,T], \ t \neq t_k, \quad k = 1, \dots, m, \tag{1.4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.5}$$

$$\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.6}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad y(0) - y(T) = \mu_0, \quad y'(0) - y'(T) = \mu_1, \tag{1.7}$$

where λ, F, ϕ, I_k, $k = 1, \dots, m$ are as in the problem (1.1)–(1.3), $\mu_0, \mu_1 \in E$, and $\bar{I}_k : E \to E$, $(k = 1, 2, \dots, m)$.

Impulsive differential equations have become more important in recent years in some mathematical models of real world phenomena, especially in the biological or medical domain; see the monographs of Bainov and Simeonov [1], Lakshmikantham, Bainov and Simeonov [11], and Samoilenko and Perestyuk [13]. In [12] the Banach classical principle was used to obtain an existence and uniqueness result for a nonresonance first order scalar impulsive differential equation with periodic boundary conditions. Very recently by means of the known Schaefer's theorem, a class of nonresonance problems for first order impulsive functional differential equations with periodic boundary conditions was considered by Benchohra and Eloe in [2]. A variety of nonresonance problems for first and second order differential inclusions with convex valued right hand sides were considered with the aid of a fixed point theorem for condensing multivalued maps due to Martelli by Benchohra, *et al.* in [3] and [4].

We consider the case when $\lambda \neq 0$. Note that when the impulses are absent (i.e. for $I_k, \bar{I}_k \equiv 0$, $k = 1, \dots, m$), then the problems (1.1)–(1.3) and (1.4)–(1.7) are *nonresonance problems* since the linear part in the equations (1.1) and (1.4) is invertible. Our method involves reducing the existence of solutions for problems (1.1)–(1.3) and (1.4)–(1.7) to a search for fixed points of suitable multivalued maps on the Banach space $C([-r, T], E)$. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [6] (see also Deimling [7]). The results of the present paper generalize to the nonconvex valued right hand side those of [3] and [4].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C([-r, 0], E)$ is the Banach space of all continuous functions from $[-r, 0]$ into E with the norm

$$\|\phi\| = \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\}.$$

$AC^i(J,E)$ is the space of i-times differentiable functions $y : J \to E$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

Let (X,d) be a metric space. We use the notations:

$P(X) = \{Y \subset X : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$.

Consider $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

Definition 2.1. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x,y), \quad \text{for each } x, y \in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

N has *a fixed point* if there is $x \in X$ such that $x \in N(x)$. The fixed point set of the multivalued operator N will be denoted by $\operatorname{Fix} N$.

For more details on multivalued maps we refer to the books of Deimling [7], Gorniewicz [9] and Hu and Papageorgiou [10].

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [6] (see also Deimling [7, Theorem 11.1]).

Lemma 2.2. *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

3. Nonresonance first order impulsive FDIs

The main result of this section is devoted to the problem (1.1)–(1.3). Before stating and proving this result, we give the definition of a solution of the problem (1.1)–(1.3). In order to define its solution we shall consider the space,

$$\Omega := \Omega([-r,T]) = \{y : [-r,T] \longrightarrow E : y_k \in C(J_k,E), k = 0,\dots,m \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+), k = 1,\dots,m \text{ with } y(t_k^-) = y(t_k)\}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|,\ k = 0,\dots,m\}.$$

Here y_k denotes the restriction of y to $J_k = [t_k, t_{k+1}]$, $k = 0,\dots,m$. For each $y \in \Omega$ we define the set

$$S_{F,y} = \Big\{v \in L^1(J,E) : v(t) \in F(t,y_t) \text{ for a.e. } t \in J\Big\}.$$

Definition 3.1. A function $y \in \Omega \cap \bigcup_{k=0}^{m} AC(t_k, t_{k+1})$ is said to be a solution of (1.1)–(1.3), if y satisfies the differential inclusion $y'(t) - \lambda y(t) \in F(t, y_t)$ a.e. on $J \setminus \{t_1, \dots, t_m\}$, the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$ and $y(0) = y(T)$.

We have the following auxiliary result.

Lemma 3.2 ([4]). *$y \in \Omega \cap \bigcup_{k=0}^{m} AC(t_k, t_{k+1})$ is a solution to the problem (1.1)–(1.3) if and only if $y \in \Omega$ and there exists $v \in S_{F,y}$ such that y satisfies the following impulsive integral equation*

$$y(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ \displaystyle\int_0^T G(t,s)v(s)\,ds + \sum_{k=1}^{m} G(t, t_k) I_k(y(t_k^-)), & t \in J, \end{cases}$$

where

$$G(t,s) = (e^{-\lambda T} - 1)^{-1} \begin{cases} e^{-\lambda(T+s-t)}, & 0 \le s \le t \le T, \\ e^{-\lambda(s-t)}, & 0 \le t < s \le T. \end{cases}$$

Theorem 3.3. *Assume that:*

(H1) *$F : [0,T] \times C([-r,0], E) \longrightarrow P_{cl}(E)$ has the property that $F(\cdot, u) : [0,T] \to P_{cl}(E)$ is measurable for each $u \in C([-r,0], E)$;*

(H2) *$H_d(F(t,u), F(t,\overline{u})) \le l(t)\|u - \overline{u}\|$, for each $t \in [0,T]$ and $u, \overline{u} \in C([-r,0], E)$, where $l \in L^1([0,T], \mathbb{R})$;*

(H3) *$\|I_k(y) - I_k(\overline{y})\| \le c_k \|y - \overline{y}\|$, for each $y, \overline{y} \in E$, $k = 1, \dots, m$, where c_k are nonnegative constants.*

Let $h_0 = \sup_{(t,s) \in J \times J} |G(t,s)|$ and $l^* = \int_0^T l(t)\,dt$. If

$$\left[h_0 l^* + h_0 \sum_{k=1}^{m} c_k \right] < 1,$$

then the problem (1.1)–(1.3) has at least one solution on $[-r, T]$.

Proof: Transform the problem (1.1)–(1.3) into a fixed point problem. It is clear from Lemma 3.2 that the solutions of the problem (1.1)–(1.3) are fixed points of the multivalued operator, $N : \Omega \to \mathcal{P}(\Omega)$ defined by:

$$N(y) := \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ \displaystyle\int_0^T G(t,s)v(s)\,ds & \\ \displaystyle + \sum_{k=1}^{m} G(t, t_k) I_k(y(t_k^-)), & t \in J \end{cases} \right\},$$

where $v \in S_{F,y}$.

Remark 3.4. For each $y \in \Omega$ the set $S_{F,y}$ is nonempty since, by (H1), F has a measurable selection (see [5, Theorem III.6]).

We shall show that N satisfies the assumptions of Lemma 2.2. The proof will be given in two steps.

Step 1: *$N(y) \in P_{cl}(\Omega)$ for each $y \in \Omega$.*

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ be such that $y_n \longrightarrow \tilde{y}$ in Ω. Then $\tilde{y} \in \Omega$ and for each $t \in J$

$$y_n(t) \in \int_0^T G(t,s)F(s,y_s)\,ds + \sum_{k=1}^{m} G(t,t_k)I_k(y(t_k^-)).$$

Because $\int_0^T G(t,s)F(s,y_s)\,ds + \sum_{k=1}^{m} G(t,t_k)I_k(y(t_k^-))$ is closed for each $t \in J$, then

$$y_n(t) \longrightarrow \tilde{y}(t) \in \int_0^T G(t,s)F(s,y_s)\,ds + \sum_{k=1}^{m} G(t,t_k)I_k(y(t_k^-)).$$

So $\tilde{y} \in N(y)$.

Step 2: *$H_d(N(y), N(\overline{y})) \leq \gamma\|y - \overline{y}\|$ for each $y, \overline{y} \in \Omega$ (where $\gamma < 1$).*

Let $y, \overline{y} \in \Omega$ and $h_1 \in N(y)$. Then there exists $v_1(t) \in F(t, y_t)$ such that for each $t \in J$

$$h_1(t) = \int_0^T G(t,s)v_1(s)\,ds + \sum_{k=1}^{m} G(t,t_k)I_k(y(t_k^-)).$$

From (H2) it follows that

$$H_d(F(t,y_t), F(t,\overline{y}_t)) \leq l(t)\|y_t - \overline{y}_t\| \quad t \in J.$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$\|v_1(t) - w\| \leq l(t)\|y_t - \overline{y}_t\|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|v_1(t) - w\| \leq l(t)\|y_t - \overline{y}_t\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see Proposition III.4 in [5]) there exists $v_2(t)$, which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}_t)$ and

$$\|v_1(t) - v_2(t)\| \leq l(t)\|y - \overline{y}\|, \quad \text{for each} \ \ t \in J.$$

Let us define for each $t \in J$

$$h_2(t) = \int_0^T G(t,s)v_2(s)\,ds + \sum_{k=1}^m G(t,t_k)I_k(\overline{y}(t_k^-)).$$

Then we have

$$\begin{aligned}\|h_1(t) - h_2(t)\| &\le \int_0^T |G(t,s)|\|v_1(s) - v_2(s)\|\,ds \\ &\quad + \sum_{k=1}^m |G(t,t_k)|\|I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))\| \\ &\le h_0 \int_0^T l(s)\|y_s - \overline{y}_s\|\,ds + h_0 \sum_{k=1}^m c_k \|y(t_k^-) - \overline{y}(t_k^-)\| \\ &\le h_0 l^* \|y - \overline{y}\|_\Omega + h_0 \sum_{k=1}^m c_k \|y - \overline{y}\|_\Omega.\end{aligned}$$

Then

$$\|h_1 - h_2\|_\Omega \le \left[h_0 l^* + h_0 \sum_{k=1}^m c_k\right] \|y - \overline{y}\|_\Omega.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \le \left[h_0 l^* + h_0 \sum_{k=1}^m c_k\right] \|y - \overline{y}\|_\Omega.$$

So, N is a contraction and thus, by Lemma 2.2, N has a fixed point y, which is a solution to (1.1)–(1.3). □

4. Nonresonance second order impulsive FDIs

In this section we shall give an existence result for the problem (1.4)–(1.7). Before stating and proving this result, we give the definition of a solution of the problem (1.4)–(1.7).

Definition 4.1. A function $y \in \Omega \cap \bigcup_{k=0}^m AC^1(t_k, t_{k+1})$ is said to be a solution of (1.4)–(1.7), if y satisfies the differential inclusion $y''(t) - \lambda y(t) \in F(t, y_t)$ a.e. on $J \setminus \{t_1, \dots, t_m\}$, the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1, \dots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$, $y(0) - y(T) = \mu_0$ and $y'(0) - y'(T) = \mu_1$.

We need the following auxiliary result.

Lemma 4.2. *$y \in \Omega \cap \bigcup_{k=0}^{m} AC^1(t_k, t_{k+1})$ is a solution to the problem (1.4)–(1.7) if and only if $y \in \Omega$ and there exists $v \in S_{F,y}$ such that y satisfies the impulsive integral equation*

$$y(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \displaystyle\int_0^T M(t,s)v(s)\,ds + M(t,0)\mu_1 + L(t,0)\mu_0 & \\ \displaystyle + \sum_{k=1}^{m} [M(t,t_k)I_k(y(t_k)) + L(t,t_k)\bar{I}_k(y(t_k))], & t \in J, \end{cases}$$

where

$$M(t,s) = \frac{-1}{2\sqrt{\lambda}(e^{\sqrt{\lambda}T} - 1)} \begin{cases} e^{\sqrt{\lambda}(T+s-t)} + e^{\sqrt{\lambda}(t-s)}, & 0 \le s \le t \le T, \\ e^{\sqrt{\lambda}(T+t-s)} + e^{\sqrt{\lambda}(s-t)}, & 0 \le t < s \le T, \end{cases}$$

and

$$L(t,s) = \frac{\partial}{\partial t} M(t,s) = \frac{1}{2(e^{\sqrt{\lambda}T} - 1)} \begin{cases} e^{\sqrt{\lambda}(T+s-t)} - e^{\sqrt{\lambda}(t-s)}, & 0 \le s \le t \le T, \\ e^{\sqrt{\lambda}(s-t)} - e^{\sqrt{\lambda}(T+t-s)}, & 0 \le t < s \le T. \end{cases}$$

Proof: We omit the proof since it is similar to the results in [8]. □

Theorem 4.3. *Assume in addition to (H1)–(H3), the condition*

(H4) $\|\bar{I}_k(y) - \bar{I}_k(\overline{y})\| \le d_k \|y - \overline{y}\|$, *for each $y, \overline{y} \in E$, $k = 1, \dots, m$, where d_k are nonnegative constants.*

Let $m_0 = \sup_{(t,s) \in J \times J} |M(t,s)|$, $l_0 = \sup_{(t,s) \in J \times J} |L(t,s)|$. If

$$\left[m_0 l^* + m_0 \sum_{k=1}^{m} c_k + l_0 \sum_{k=1}^{m} d_k \right] < 1,$$

then the problem (1.4)–(1.7) has at least one solution on $[-r, T]$.

Proof: Transform the problem (1.4)–(1.7) into a fixed point problem. It is clear from Lemma 4.2 that the solutions of the problem (1.4)–(1.7) are fixed points of the multivalued operator, $\bar{N} : \Omega \to \mathcal{P}(\Omega)$ defined by:

$$\bar{N}(y) := \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ \displaystyle\int_0^T M(t,s)v(s)\,ds + M(t,0)\mu_1 & \\ \displaystyle + L(t,0)\mu_0 + \sum_{k=1}^{m} [M(t,t_k)I_k(y(t_k)) & \\ + L(t,t_k)\bar{I}_k(y(t_k))], & t \in J \end{cases} \right\},$$

where $v \in S_{F,y}$.

Remark 4.4. For each $y \in \Omega$ the set $S_{F,y}$ is nonempty since, by (H1), F has a measurable selection (see [5, Theorem III.6]).

We shall show that $\bar{N}$ satisfies the assumptions of Lemma 2.2. The proof will be given in two steps.

Step 1: *$\bar{N}(y) \in P_{cl}(\Omega)$ for each $y \in \Omega$.*
The proof is similar to that of Step 1 of Section 3.

Step 2: *$H(\bar{N}(y), \bar{N}(\overline{y})) \leq \gamma \|y - \overline{y}\|$ for each $y, \overline{y} \in \Omega$ (where $\gamma < 1$).*

Let $y, \overline{y} \in \Omega$ and $h_1 \in \bar{N}(y)$. Then there exists $v_1(t) \in F(t, y_t)$ such that for each $t \in J$

$$\begin{aligned} h_1(t) = & \int_0^T M(t,s) v_1(s)\, ds + M(t,0)\mu_1 + L(t,0)\mu_0 \\ & + \sum_{k=1}^{m} [M(t,t_k) I_k(y(t_k)) \\ & + L(t,t_k)\bar{I}_k(y(t_k))]. \end{aligned}$$

From (H2) it follows that

$$H_d(F(t, y_t), F(t, \overline{y}_t)) \leq l(t)\|y_t - \overline{y}_t\| \quad t \in J.$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$\|v_1(t) - w\| \leq l(t)\|y_t - \overline{y}_t\|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|v_1(t) - w\| \leq l(t)\|y_t - \overline{y}_t\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see Proposition III.4 in [5]) there exists $v_2(t)$, which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}_t)$ and

$$\|v_1(t) - v_2(t)\| \leq l(t)\|y - \overline{y}\|, \text{ for each } t \in J.$$

Let us define for each $t \in J$

$$\begin{aligned} h_2(t) = & \int_0^T M(t,s) v_2(s)\, ds + M(t,0)\mu_1 + L(t,0)\mu_0 \\ & + \sum_{k=1}^{m} [M(t,t_k) I_k(\overline{y}(t_k)) \\ & + L(t,t_k)\bar{I}_k(\overline{y}(t_k))]. \end{aligned}$$

Then we have

$$
\begin{aligned}
\|h_1(t) - h_2(t)\| \leq & \int_0^T |M(t,s)| \|v_1(s) - v_2(s)\| \, ds \\
& + \sum_{k=1}^{m} |M(t,t_k)| \|I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))\| \\
& + \sum_{k=1}^{m} |L(t,t_k)| \|\bar{I}_k(y(t_k^-)) - \bar{I}_k(\overline{y}(t_k^-))\| \\
& \leq m_0 \int_0^T l(s) \|y_s - \overline{y}_s\| \, ds + m_0 \sum_{k=1}^{m} c_k \|y(t_k^-) - \overline{y}(t_k^-)\| \\
& + l_0 \sum_{k=1}^{m} d_k \|y(t_k^-) - \overline{y}(t_k^-)\| \\
& \leq m_0 l^* \|y - \overline{y}\|_\Omega + m_0 \sum_{k=1}^{m} c_k \|y - \overline{y}\|_\Omega + l_0 \sum_{k=1}^{m} d_k \|y - \overline{y}\|_\Omega.
\end{aligned}
$$

Then

$$
\|h_1 - h_2\|_\Omega \leq \left[m_0 l^* + m_0 \sum_{k=1}^{m} c_k + l_0 \sum_{k=1}^{m} d_k \right] \|y - \overline{y}\|_\Omega.
$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$
H_d(\bar{N}(y), \bar{N}(\overline{y})) \leq \left[m_0 l^* + m_0 \sum_{k=1}^{m} c_k + l_0 \sum_{k=1}^{m} d_k \right] \|y - \overline{y}\|_\Omega.
$$

So, $\bar{N}$ is a contraction and thus, by Lemma 2.2, $\bar{N}$ has a fixed point y, which is a solution to (1.4)–(1.7). □

References

[1] Bainov D.D., Simeonov P.S., *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[2] Benchohra M., Eloe P., *On nonresonance impulsive functional differential equations with periodic boundary conditions*, Appl. Math. E.-Notes **1** (2001), 65–72.

[3] Benchohra MK., Henderson J., Ntouyas S.K., *On nonresonance impulsive functional differential inclusions with periodic boundary conditions*, Intern. J. Appl. Math. **5** (4) (2001), 377–391.

[4] Benchohra M., Henderson J., Ntouyas S.K., *On nonresonance second order impulsive functional differential inclusions with nonlinear boundary conditions*, Canad. Appl. Math. Quart., in press.

[5] Castaing C., Valadier M., *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, vol. 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[6] Covitz H., Nadler S.B., Jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11.
[7] Deimling K., *Multivalued Differential Equations*, Walter de Gruyter, Berlin-New York, 1992.
[8] Dong Y., *Periodic boundary value problems for functional differential equations with impulses*, J. Math. Anal. Appl. **210** (1997), 170–181.
[9] Gorniewicz L., *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications, 495, Kluwer Academic Publishers, Dordrecht, 1999.
[10] Hu Sh., Papageorgiou N., *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1997.
[11] Lakshmikantham V., Bainov D.D., Simeonov P.S., *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
[12] Nieto J.J., *Basic theory for nonresonance impulsive periodic problems of first order*, J. Math. Anal. Appl. **205** (1997), 423–433.
[13] Samoilenko A.M., Perestyuk N.A., *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

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