Covering ${}^{\omega}\omega$ by special Cantor sets

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Abstract. This paper deals with questions of how many compact subsets of certain kinds it takes to cover the space ${}^{\omega}\omega$ of irrationals, or certain of its subspaces. In particular, given $f \in {}^{\omega}(\omega \backslash \{0\})$, we consider compact sets of the form $\prod_{i\in\omega} B_i$, where $|B_i| = f(i)$ for all, or for infinitely many, i. We also consider "n-splitting" compact sets, i.e., compact sets K such that for any $f \in K$ and $i \in \omega$, $|\{g(i) : g \in K, g \restriction i = f \restriction i\}| = n$.

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1. Introduction

Let $\mathbb{P}$ denote the product ${}^{\omega}\omega$, and let $\mathbb{P}^+$ denote the product ${}^{\omega}(\omega \setminus \{0\})$. For $f \in \mathbb{P}^+$, we will call a compact set of the form $\prod_{i\in\omega} B_i$ an *f-cone* (resp., *weak f-cone*) if $|B_i| = f(i)$ for all i (resp., for infinitely many i). If f is constant n, we will also call the corresponding (weak) f-cone a *(weak) n-ary product*. Finally, we call a compact subset K of $\mathbb{P}$ *k-splitting* if each node of the tree $\{f\restriction n : f \in K, n \in \omega\}$ has exactly k successors.

In this paper, we consider how many of these kinds of compact sets it takes to cover $\mathbb{P}$, or some subset of $\mathbb{P}$. We denote by $c(\mathbb{P}, f)$ and $c^{\infty}(\mathbb{P}, f)$ the least number of f-cones, respectively weak f-cones, that it takes to cover $\mathbb{P}$. We denote the least number of n-splitting compact sets it takes to cover $\mathbb{P}$ by $c(\mathbb{P}, s(n))$.

While the emphasis of this paper is on covering $\mathbb{P}$, we will also discuss to some extent how many of the above kinds of compact sets it takes to cover an f-cone of the form $\Pi_f = \Pi_{i\in\omega} f(i)$, for $f \in \mathbb{P}^+$. Restricted to what we are calling cones, this was the topic of [GS], where the least number of g-cones needed to cover Π_f was denoted by $c(f, g)$. We follow this notation, and also denote by $c^{\infty}(f, g)$ the least number of weak g-cones it takes to cover Π_f. In this paper, we usually limit ourselves to mentioning results of this type if they follow easily from the methods we used in the results about covering $\mathbb{P}$.

It is known that it takes $\mathfrak{c}$ (= continuum) many n-ary products to cover $\mathbb{P}$, i.e., $c(\mathbb{P}, \vec{n}) = \mathfrak{c}$, where $\vec{n}$ denotes the function which is constant n; in fact

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$c(\overrightarrow{n+1}, \vec{n}) = \mathfrak{c}$; it is also easy to show that $c^\infty(\mathbb{P}, \vec{n}) = \mathfrak{c}$. But $c^\infty(\vec{n}, \vec{m})$ is different: we show that it is the same as the so-called *refinement number* $\mathfrak{r}$ (also called the *reaping number*), which is the least cardinal κ of a family $\mathcal{R} \subset [\omega]^\omega$ such that every $A \in [\omega]^\omega$ is either contained in or disjoint from some member of $\mathcal{R}$.

It follows from the above paragraph that $c(\mathbb{P}, f)$ (resp., $c^\infty(\mathbb{P}, g)$) is equal to $\mathfrak{c}$ whenever $\lim_{n\in\omega} f(n) \neq \infty$ (resp., if g is bounded)[1]. It is also the case that $c(\mathbb{P}, f)$ (resp., $c^\infty(\mathbb{P}, g)$) is the same for all f such that $\lim_{n\in\omega} f(n) = \infty$ (resp., such that g is unbounded). For the purposes of this paper, we denote the values of $c(\mathbb{P}, f)$ and $c(\mathbb{P}, g)$ for such f and g by $\mathfrak{v}$ and $\mathfrak{v}^\infty$, respectively.

We will see that the value of $\mathfrak{v}$ and $\mathfrak{v}^\infty$ may be less than $\mathfrak{c}$; in fact we will show that $\mathfrak{v} = \mathrm{cov}(\mathcal{N})$, the least cardinal of a collection of Lebesgue measure 0 sets covering the real line. Also note that for each $n \geq 2$, $c(\mathbb{P}, s(n)) \geq \mathfrak{v}$ (since $\mathfrak{v} = c(\mathbb{P}, f)$ where $f(i) = n^{i+1}$). Thus we have the following non-decreasing sequence of cardinals:

$$\mathfrak{c} = c(\mathbb{P}, s(1)) \geq \ c(\mathbb{P}, s(2)) \geq \dots c(\mathbb{P}, s(n)) \geq \dots \mathfrak{v} \geq \mathfrak{v}^\infty.$$

We will show that there are rather natural finite support c.c.c. iterations which produce models which split this sequence at any desired point (e.g., any one of these cardinals may be ω_1 with all cardinals to its left equal to ω_2).

In the final section, we consider covering dense G_δ subsets of $\mathbb{P}$ by some of these special compact sets.

2. ZFC results

It is easy to see that $\mathfrak{d}$, the least cardinal of a dominating family in $\mathbb{P}$, is also the least cardinal of a collection of compact sets covering $\mathbb{P}$ (see, e.g., [vD]). So all of our cardinals of the form $c(\mathbb{P}, f)$ (i.e., without the superscript ∞) are at least $\mathfrak{d}$. Sometimes it is provable in ZFC that a cardinal we are considering is equal to $\mathfrak{c}$. We first review this type of result, at least some of which are known or folklore.

The next two lemmas are key tools for this.

Lemma 2.1. *Let $g \in \mathbb{P}^+$ be a function such that $\lim_{n\in\omega} g(n) = \infty$. Then there is an uncountable compact set $K \subseteq \Pi_g$ such that if f_1 and f_2 are distinct elements of K, then there exists an integer N such that $f_1(n) \neq f_2(n)$ for all $n > N$.*

Proof: Let $(n_k)_{k\in\omega}$ be an increasing sequence of integers such that $g(i) > 2^{k+1}$ for each $i \geq n_k$. For $\sigma \in {}^\omega 2$, let $\tau_\sigma(r) = 0$ if $r \leq n_0$ and let $\tau_\sigma(r) = \sum_{j=0}^k \sigma(j)2^j$ for $n_k < r \leq n_{k+1}$. Let $K = \{\tau_\sigma : \sigma \in {}^\omega 2\}$. Then K has the required properties. □

[1] If $f \in \mathbb{P}$, we write $\lim_{n\in\omega} f(n) \neq \infty$ for the negation of $\lim_{n\in\omega} f(n) = \infty$. Therefore, $\lim_{n\in\omega} f(n) \neq \infty$ if and only if there exists $k \in \omega$ such that $f^{-1}(k)$ is infinite.

Lemma 2.2. *For any $N \in \omega$, ${}^{\omega}N$ contains a family $\mathcal{F}$ of $\mathfrak{c}$-many functions such that, for any N-element subset $\mathcal{F}'$ of $\mathcal{F}$, $|\pi_k(\mathcal{F}')| = N$ for infinitely many $k \in \omega$.*

Proof: Let $\mathcal{I}$ be an independent family of subsets of ω such that $|\mathcal{I}| = \mathfrak{c}$. Write $\mathcal{I}$ as the disjoint union of N subfamilies $\mathcal{I} = \bigcup_{k=1}^{N} \mathcal{I}_k$ where $|\mathcal{I}_k| = \mathfrak{c}$ for each k. For $k = 1, \cdots N$, write the elements of $\mathcal{I}_k$ as $\{I_k(\lambda) : \lambda < \mathfrak{c}\}$. For each $\lambda < \mathfrak{c}$ and $x \in \omega$, let $f_\lambda(x)$ be the number of sets $I_1(\lambda), \cdots, I_N(\lambda)$ which contain x.

We show that $\mathcal{F} = \{f_\lambda : \lambda < \mathfrak{c}\}$ has the desired property. Let $\mathcal{F}' = \{f_{\lambda_i} : i < N\}$ be an N-element subset of $\mathcal{F}$. Since $\mathcal{I}$ is an independent family, the set $B = \left(\bigcap_{i \leq j < N} (\omega \backslash I_j(\lambda_i))\right) \cap \bigcap_{j < i < N} I_j(\lambda_i)$ is infinite. For each $x \in B$, $f_{\lambda_i}(x) = i$, and so the result follows. $\square$

Theorem 2.3. *Let $f, g \in \mathbb{P}^+$, and let $L_{fg} = \{n : g(n) < f(n)\}$.*
Then:

(a) *If L_{fg} is finite, then $c(f, g) \leq \omega$;*
(b) *Suppose L_{fg} is infinite. Then $c(f, g) = \mathfrak{c}$ if $\lim_{n \in L_{fg}} g(n) \neq \infty$;*
(c) *If $\omega \setminus L_{fg}$ is infinite, then $c^\infty(f, g) = 1$;*
(d) *If $\lim_{n \in \omega} f(n) = \infty$ and g is bounded, then $c^\infty(f, g) = \mathfrak{c}$.*

Proof:

(a) If $L_{fg} \subseteq n$, then Π_f is covered by the countable collection of g-cones $\{\{\sigma\} \times \Pi_{i \geq n} f(i) : \sigma \in {}^{n}\omega\}$.
(b) Since $\lim_{n \in L_{fg}} g(n) \neq \infty$, there are $k \in \omega$ and an infinite subset A of L_{fg} such that $g(n) = k$ for every $n \in A$. Then $f(n) \geq k + 1$ for all $n \in A$, and so it follows from Lemma 2.2 that there is a collection $\mathcal{F}$ of $\mathfrak{c}$-many functions in Π_f such that, for any $(k+1)$-element subset $\mathcal{F}'$ of $\mathcal{F}$, $|\pi_n(\mathcal{F}')| = k + 1$ for infinitely many $n \in A$. Thus each g-cone contains at most k elements of $\mathcal{F}$, and the result follows.
(c) In this case, Π_f is a weak g-cone.
(d) Suppose $\lim_{n \in \omega} f(n) = \infty$ and g is bounded. Let $K \subset \Pi_f$ be as in Lemma 2.1, and let $g(i) < N$ for all $i \in \omega$. Then any weak g-cone contains at most N points of K. $\square$

Theorem 2.4. *Suppose $g \in \mathbb{P}^+$. Then:*

(a) *$c(\mathbb{P}, g) = \mathfrak{c}$ if $\lim_{n \in \omega} g(n) \neq \infty$;*
(b) *$c^\infty(\mathbb{P}, g) = \mathfrak{c}$ if g is bounded.*

Proof:

(a) Let $f \in \mathbb{P}^+$ be a function such that $g(n) < f(n)$ for all $n \in \omega$. By Theorem 2.3(b), $c(f, g) = \mathfrak{c}$, so $\mathbb{P}$ cannot be covered by fewer than $\mathfrak{c}$-many g-cones.
(b) Let $f \in {}^{\omega}(\omega \setminus \{0\})$ be a function such that $g(n) < f(n)$ for all $n \in \omega$. By Theorem 2.3(d), $c^\infty(f, g) = \mathfrak{c}$, so $c^\infty(\mathbb{P}, g) = \mathfrak{c}$. $\square$

Corollary 2.5. *Suppose $n \in \omega$. ${}^{\omega}(n+1)$ cannot be covered by fewer than $\mathfrak{c}$-many n-ary compact sets.*

Remark 2.6. Theorem 2.3(b), from which Theorem 2.4(a) and the above corollary follow, is Corollary 1.12 of [GS]. As pointed out in [GS], it can be easily derived from a result of Comfort and Negrepontis [CN].

Remark 2.7. If κ is any infinite cardinal, then there exists an independent family of cardinality 2^{κ} on κ (see [K]). It follows that Corollary 2.5 can be generalized to higher cardinals to give that for any infinite cardinal κ and any $n \in \omega$, the product ${}^{\kappa}(n+1)$ is not the union of fewer than 2^{κ} products $\prod_{\lambda<\kappa} F_{\lambda}$, where $|F_{\lambda}| \leq n$ for each λ.

Regarding Corollary 2.5, it turns out that it can take fewer than $\mathfrak{c}$-many weak n-ary products to cover ${}^{\omega}(n+1)$. Indeed, the number is the same as the refinement number $\mathfrak{r}$, as we now show.

Lemma 2.8. *If $\kappa(*)$ denotes the least cardinal of a family $\mathcal{A}$ of infinite subsets of ω satisfying condition $(*)$ below, then $\kappa(*) = \mathfrak{r}$.*

(1) *For all $B \subset \omega$, there exists $A \in \mathcal{A}$ with $A \subset B$ or $A \cap B = \emptyset$.*
(2) *For each finite non-degenerate partition $\mathcal{P}$ of ω, there exists $A \in \mathcal{A}$ and $P \in \mathcal{P}$ with $A \cap P = \emptyset$.*
(2_n) *Same as (2) restricted to partitions of cardinality $n \geq 2$.*
(3) *Same as (2), but concluding $A \subset P$ instead of $A \cap P = \emptyset$.*
(3_n) *Same as (3) for partitions of cardinality $n \geq 2$.*

Proof: $\kappa(1) = \mathfrak{r}$ by definition of $\mathfrak{r}$. Obviously, $\kappa(1) = \kappa(2_2) = \kappa(3_2)$.

Clearly, $\kappa(1) \leq \kappa(2_n) \leq \kappa(2) \leq \kappa(3)$ for all n, and $\kappa(1) \leq \kappa(3_n) \leq \kappa(3)$ for all n.

So it remains to show $\kappa(3) \leq \kappa(1)$. Let $\mathcal{A}$ be a family of minimum cardinality $\kappa(1)$ satisfying (1). We need to show there is a family of the same cardinality satisfying (3). Note that the union over $n \in \omega$, $n \geq 2$, of families satisfying (3_n) satisfies (3). So it suffices to show there is a family of size $\kappa(1)$ satisfying (3_n) for all $n \geq 2$.

(3_2) is clearly equivalent to (1), so it is true for $n = 2$. Given that it is true for n, we now show it is true for $n+1$. To this end, let $\mathcal{A}_n$ be a family of size $\kappa(1)$ satisfying (3_n). For each infinite $B \subset \omega$, there is a family $\mathcal{A}_B$ of infinite subsets of B of cardinality $\kappa(1)$ satisfying: for all $B' \subset B$, there exists $A \in \mathcal{A}_B$ with $A \subset B'$ or $A \subset B \setminus B'$. Now let $\mathcal{A}_{n+1} = \bigcup\{\mathcal{A}_B : B \in \mathcal{A}_n\}$. Then $\mathcal{A}_{n+1}$ has size $\kappa(1)$. Suppose $\mathcal{P} = \{P_i : i < n+1\}$ is a partition of ω. By the definition of $\mathcal{A}_n$, there exists $B \in \mathcal{A}_n$ such that either $B \subset P_i$ for some $i > 1$, or else $B \subset P_0 \cup P_1$. In the former case, we are easily done. In the latter case, there is $A \in \mathcal{A}_B$ with $A \subset P_0$ or $A \subset P_1$, so we are also done. □

Theorem 2.9. *Let $n, m \in \omega$ with $n > m \geq 1$. Then $c^{\infty}(\vec{n}, \vec{m}) = \mathfrak{r}$, i.e., the least cardinal of a family of weak m-ary products covering ${}^{\omega}n$ is equal to the refinement number $\mathfrak{r}$.*

Proof: Let $\mathcal{A}$ be a collection of cardinality $\mathfrak{r}$ satisfying condition (3) of the above lemma. For each $k < n$ and $A \in \mathcal{A}$, let $C(A, k) = {}^{\omega\setminus A}n \times {}^{A}\{k\}$. Then for each $f \in {}^{\omega}n$, there are $A \in \mathcal{A}$ and $k < n$ with $A \subset f^{-1}(k)$, whence $f \in C(A, k)$. Hence $c^{\infty}(\vec{n}, \vec{1}) \leq \mathfrak{r}$. It follows that $c^{\infty}(\vec{n}, \vec{m}) \leq \mathfrak{r}$ whenever $1 \leq m < n$.

It remains to show $\mathfrak{r} \leq c^{\infty}(\vec{n}, \vec{m})$. Let $\{\Pi_{i\in\omega} A_i^{\alpha} : \alpha < \lambda\}$ witness $c^{\infty}(\vec{n}, \vec{m}) = \lambda$. Without loss of generality, for each α there is an m-sized set $F_{\alpha} \subset n$ such that, for each i, $A_i^{\alpha} = F_{\alpha}$ or $A_i^{\alpha} = n$. Let $B^{\alpha} = \{i \in \omega : A_i^{\alpha} = F_{\alpha}\}$. To finish the proof, it suffices to show:

Claim. $\{B^{\alpha} : \alpha < \lambda\}$ *satisfies condition* (2_{m+1}) *of Lemma* 2.8.

To see this, suppose $f : \omega \to m + 1$ codes an $m + 1$-sized partition of ω. For some $\alpha < \lambda$, $f \in \Pi_{i\in\omega} A_i^{\alpha}$. Then for each $i \in B^{\alpha}$, $f(i) \in F_{\alpha}$. There is $k < m + 1$ such that $k \notin F_{\alpha}$. Then $B^{\alpha} \cap f^{-1}(k) = \emptyset$. □

Later, we will show that the values of the cardinals for the cases not covered by Theorems 2.3 and 2.4 cannot be determined in ZFC. The next result tells us that in any given model, $c(\mathbb{P}, f)$ for f having infinite limit is always the same cardinal. The proof is a minor adaptation of the proof of 6.26 on page 349 of [Go]. (Of course, by Theorem 2.4, $c(\mathbb{P}, f)$ is also always the same, namely $\mathfrak{c}$, for any f which does not satisfy $\lim f(n) = \infty$.)

Theorem 2.10. *Suppose that f and g are elements of $\mathbb{P}^{+}$ such that* $\lim f(n) = \lim g(n) = \infty$. *Then* $c(\mathbb{P}, f) = c(\mathbb{P}, g)$.

Proof: It suffices to prove that $c(\mathbb{P}, g) \leq \kappa$ implies $c(\mathbb{P}, f) \leq \kappa$. So, let $\{T_{\alpha} : \alpha < \kappa\}$ be κ-many g-cones covering $\mathbb{P}$. We need to show there are also κ-many f-cones covering $\mathbb{P}$.

Choose $n_0 \in \omega$ such that $n_0 > 0$ and $f(n) > g(1)$ for all $n \geq n_0$. Choose $n_1 > n_0$ such that $f(n) > g(2)$ for all $n \geq n_1$, etc.; so $f(n) > g(i)$ for all $n \geq n_{i-1}$ (for $i > 0$). Then define $t(m) = i$ where i is least such that $m < n_i$. Note that t has the following property: $g(t(m)) \leq f(m)$ for all $m \geq n_0$. Also note that $n_{t(n)} > n$.

Let b be a bijection from $Fn(\omega, \omega)$ to ω.

Given $\alpha < \kappa$, let $S_{\alpha}(n) = \bigcup\{\sigma(n) : b(\sigma) \in T_{\alpha}(t(n))\}$. We claim that S_{α} is an "almost f-cone" i.e., $|S_{\alpha}(n)| \leq f(n)$ for almost all n. To see this note that $|S_{\alpha}(n)| \leq |T_{\alpha}(t(n))| \leq g(t(n)) \leq f(n)$ whenever $n \geq n_0$. Since every almost f-cone is the union of finitely many f-cones, it suffices to show that this collection of κ many almost f-cones covers $\mathbb{P}$.

To this end suppose $h \in \mathbb{P}$. Define h' by $h'(k) = b(h \restriction n_k)$. Let $\alpha < \kappa$ such that h' is covered by T_{α}. We claim that h is covered by S_{α}. Fix n, and let $k = t(n)$ and let $\sigma = h \restriction n_k$. Since $n_k = n_{t(n)} > n$, $n \in \mathrm{dom}(\sigma)$. Also, $b(\sigma) = h'(k) \in T_\alpha(k)$. It follows that $h(n) = \sigma(n) \in S_\alpha(n)$. $\square$

A simpler argument gets an analogous result for $c^\infty(\mathbb{P}, f)$.

Theorem 2.11. *$c^\infty(\mathbb{P}, f) = c^\infty(\mathbb{P}, g)$ whenever f and g are not bounded.*

Proof: Let f and g be unbounded in $\mathbb{P}$. Let $\{K_\alpha : \alpha < \kappa\}$ be a cover of ${}^\omega\omega$ by weak f-cones. We need to show that ${}^\omega\omega$ may also be covered by κ-many weak g-cones. We may assume each K_α is a product of the form $\Pi_{i\in\omega} A_i(\alpha)$. Find $n_0 < n_1 < \dots$ with $g(n_k) \geq f(k)$, so that if $A = \{n_i : i \in \omega\}$, then $\omega \backslash A$ is infinite. Let $\omega \backslash A = \{m_i : i \in \omega\}$.

For $\alpha, \beta \in \kappa$, let $K_{\alpha\beta} = \Pi_{i\in\omega} B_i(\alpha\beta)$, where $B_{n_i}(\alpha\beta) = A_i(\alpha)$ and $B_{m_i}(\alpha\beta) = A_i(\beta)$. Then it is easy to see that the $K_{\alpha\beta}$'s cover ${}^\omega\omega$. Also, for each i such that $|A_i(\alpha)| \leq f(i)$, we have $|B_{n_i}(\alpha\beta)| \leq f(i) \leq g(n_i)$, whence each $K_{\alpha\beta}$ is a weak g-cone. $\square$

Are the cardinals defined by the previous two results equivalent to better known "small" cardinals? We now show that this is the case for one defined by Theorem 2.10: it is the same as $\mathrm{cov}(\mathcal{N})$, i.e., the least cardinal of a cover of the real line by Lebesgue measure zero sets.

Theorem 2.12. *Let $f \in \mathbb{P}^+$ have infinite limit. Then $c(\mathbb{P}, f) = \mathrm{cov}(\mathcal{N})$.*

Proof: By Theorem 2.39 in [BJ], $\mathrm{cov}(\mathcal{N})$ is equal to (to translate from [BJ] to our terminology) the least cardinal of a collection $\mathcal{C}$ of f-cones covering $\mathbb{P}$, where f is allowed to vary over the set of all $g \in \mathbb{P}^+$ with $\sum_{n=1}^\infty g(n)/n^2 < \infty$. It easily follows that $c(\mathbb{P}, n^2 + 1) \leq \mathrm{cov}(\mathcal{N}) \leq c(\mathbb{P}, \sqrt{n+1})$, so all are equal by Theorem 2.10. $\square$

For convenience, let $\mathfrak{v}$ and $\mathfrak{v}^\infty$ denote the cardinals defined by Theorems 2.10 and 2.11, respectively. We now know $\mathfrak{v} = \mathrm{cov}(\mathcal{N})$. We do not know, however, if $\mathfrak{v}^\infty$ is similarly ZFC-equivalent to some better known cardinal.

The situation for cardinals $c(f, g)$, where f is unbounded, is quite different. Goldstern and Shelah [GS] show, by a complicated forcing argument, that given a sequence $\langle f_\alpha, g_\alpha \rangle_{\alpha<\omega_1}$ of $\aleph_1$-many "sufficiently different" pairs $\langle f, g \rangle$ and "almost any" sequence $\langle \kappa_\alpha \rangle_{\alpha<\omega_1}$ of uncountable cardinals, there is a model in which $c(f_\alpha, g_a) = \kappa_\alpha$ for all $\alpha < \omega_1$.

Let us also mention the values of the cardinals we have discussed in the standard Sacks and Laver models (i.e., the models obtained by a countable support iteration over a model of CH of the poset for adding a Sacks (resp., Laver) real). Of course, $\mathfrak{c} = \omega_2$ in these models.

First the Sacks model. Here, all cardinals of the form $c(\mathbb{P}, f)$, $c^\infty(\mathbb{P}, f)$, $c(f, g)$, and $c^\infty(f, g)$ that we have not shown to be $\mathfrak{c}$ in ZFC are less than $\mathfrak{c}$, so equal to ω_1 except in the trivial cases where they are countable. To see this, recall that it is well-known that $\mathrm{cof}(\mathcal{N}) = \omega_1$ in the Sacks model (see, e.g., model 7.6.2, due to A. Miller, of [BJ]). So $\mathfrak{v}^{\infty} = \mathfrak{v} = \omega_1$ in the Sacks model. In fact, the stronger result that $c(\mathbb{P}, S(2)) = \omega_1$ follows from Theorem 2.3 of [NR], since Sacks forcing has what is termed there the "2-localization property". There is an ultrafilter with an ω_1-sized base in the Sacks model (see [BL]), hence $\mathfrak{r} = \omega_1 = c^{\infty}(\vec{n}, \vec{m})$ for $n > m \geq 1$. It is not difficult to deduce from the above that $c(f, g)$ and $c^{\infty}(f, g)$ are equal to ω_1 as well, outside of trivial cases, and those listed in Theorem 2.3(b) where the value is $\mathfrak{c}$.

In the Laver model, the cardinals of the form $c(\mathbb{P}, f)$ and $c^{\infty}(\mathbb{P}, f)$ are equal to $\omega_2 = \mathfrak{c}$; this follows from the fact that a Laver real dominates all ground model reals, and hence $\mathfrak{b} = \mathfrak{d} = \omega_2$ in the Laver model. Also, $\mathfrak{r} = \omega_2$ (since $\mathfrak{b} \leq \mathfrak{r}$) and so $c^{\infty}(f, g) = \mathfrak{c}$ if $g(n) < f(n)$ for almost all n and g is bounded. The remaining cases to be discussed are (1) $c(f, g)$ where $L_{gf} = \{n \in \omega : g(n) < f(n)\}$ is infinite and $\lim_{n \in L_{gf}} g(n) = \infty$, and (2) $c^{\infty}(f, g)$ when L_{fg} is co-finite and g is not bounded. Clearly, if the cardinal of case (1) is ω_1, which we shall presently show, then so is the cardinal of case (2).

Let V_{ω_2} denote the Laver model, and V_α, $\alpha < \omega_2$, the intermediate models. Suppose $f, g \in V_{\omega_2} \cap \mathbb{P}^+$, and L_{gf} infinite. It is easy to see that we may assume $L_{fg} = \omega$. Now $f, g \in V_\alpha$ for some $\alpha < \omega_2$. Let $h \in V_{\omega_2} \cap \Pi_f$. By Fact 6.26 in [G], there is a g-cone in V_α containing h. Since CH holds in V_α, it follows that $c(f, g)) = \omega_1$.

3. Forcing differences in covering numbers

In this section, we show that there are quite natural c.c.c. posets which can be used to show that the following non-decreasing sequence of cardinals

$$\mathfrak{c} = c(\mathbb{P}, s(1)) \geq \ c(\mathbb{P}, s(2)) \geq \ldots c(\mathbb{P}, s(n)) \geq \ldots \mathfrak{v} \geq \mathfrak{v}^{\infty}$$

can be split at any desired point, i.e., given two regular uncountable cardinals $\lambda < \kappa$, and any cardinal in the sequence other than the leftmost, there is a finite support c.c.c. extension of any ground model satisfying CH in which this cardinal and any cardinal to its right has value λ, while any cardinal to its left has value κ.[2]

First we describe three basic posets that will be used. Define the poset Q as follows: Q consists of all pairs $q = (F^q, k^q)$ satisfying

(i) $F^q \subset [{}^{\omega}\omega]^{<\omega}$ and $k^q \in \omega$;

(ii) $|\pi_i(F^q)| \leq 2^i$ for all $i \in \omega$.

Define $q' \leq q$ iff $k^{q'} \geq k^q$, $F^{q'} \supseteq F^q$, and $F^{q'} \restriction k^q = F^q \restriction k^q$ (where $F \restriction k = \{f \restriction k : f \in F\}$).

Next define the poset Q^{∞} as all triples $q = (F^q, k^q, H^q)$ satisfying

[2] The authors did not check this out in detail, but it appears that for $\lambda = \omega_1$ and $\kappa = \omega_2$, certain posets with the "k-localization property" (see [NR] and/or [R]) may be used to do the same job using countable support iterations.

(i) $F^q \subset [{}^\omega\omega]^{<\omega}$, $k^q \in \omega$, and $H^q \subset k^q$;

(ii) $|\pi_i(F^q)| \leq 2^i$ for all $i \in H^q$.

Define $q' \leq q$ iff $k^{q'} \geq k^q$, $H^{q'} \cap k^q = H^q$, $F^{q'} \supseteq F^q$, and $F^{q'}\restriction k^q = F^q \restriction k^q$.

Obviously, Q is going to add generic 2^n-cones and Q^∞ generic weak 2^n-cones. We now define a third type of poset which adds generic n-splitting compact sets. Let $R(n)$ consist of all pairs $r = (F^r, k^r)$ satisfying

(i) $F^r \subset [{}^\omega\omega]^{<\omega}$ and $k^r \in \omega$;

(ii) for each $j \in \omega$ and $\sigma \in {}^j\omega$, we have $|\{f(j) : f \in F^r, f\restriction j = \sigma\}| \leq n$.

Define $r' \leq r$ iff $k^{r'} \geq k^r$, $F^{r'} \supseteq F^r$, and $F^{r'}\restriction k^r = F^r\restriction k^r$.

What is going to make the argument work is the different centeredness properties of the posets. Recall that a subset A of a poset P is *n-linked* if every n-elements of A have a common extension, and is *centered* if it is n-linked for every $n \in \omega$. P is *σ-centered* if it is the countable union of centered subsets, and has *property K_n* if every uncountable subset A contains an uncountable A' which is n-linked.

Lemma 3.1. *Let Q, Q^∞, and $R(n)$ be the posets defined above. Then $R(n)$ has property K_n, Q has property K_n for every n, and Q^∞ is σ-centered.*

Proof: $Q^\infty$is σ-centered because for each $k \in \omega$, $H \subset k$, and $\Sigma \subset {}^\kappa\omega$, the set of all conditions q such that $k^q = k$, $H^q = H$, and $F^q\restriction k = \Sigma$ is centered.

Fix $n \in \omega$. For each $q \in Q$, find $l^q \geq k^q$ such that $n \cdot |F^q| \leq 2^{l^q}$. Then any subset of Q for which l^q and $F^q\restriction l^q$ are constant is n-linked. It follows that Q has property K_n. The proof for $R(n)$ is similar; consider l^q such that the projection of F^q onto ${}^{l^q}\omega$ is one-to-one. $\square$

Lemma 3.2. *Let $\kappa > \lambda$ be regular uncountable cardinals, and let the ground model V satisfy $\mathfrak{c} = \kappa$. Let V_0 (resp., V_1, V_2) be the model obtained by a finite support iteration over V of length λ of the poset Q^∞ (resp., $Q, R(n)$). Then for each $i < 3$, $V_i \models \mathfrak{c} = \kappa$, and:*

(a) $V_0 \models \mathfrak{v}^\infty = \lambda$;

(b) $V_1 \models \mathfrak{v} = \lambda$;

(c) $V_2 \models c(\mathbb{P}, s(n)) = \lambda$.

Proof: The posets are c.c.c., have size $\mathfrak{c}$, and the iteration is of length less than $\mathfrak{c}$. Therefore, $\mathfrak{c}$ remains κ.

Let G be a generic filter for one of these posets, and let $F^G = \bigcup\{F^q : q \in G\}$. Easy density arguments show that F^G is a 2^n-cone if the poset is Q, a weak 2^n-cone if Q^∞, and an n-splitting compact set if $R(n)$, and further that in every finite support iteration of infinite length, every ground model $f \in \mathbb{P}$ is covered by one of these generically added compact sets. Since every f in the extension appears at some stage λ, it follows that the λ-many generic compact sets added cover $\mathbb{P}$ in the final model.

Thus the pertinent covering numbers are $\leq \lambda$ in the models. And they cannot be less than λ, because the iteration adds λ-many Cohen reals, hence in the extension $\mathbb{P}$ cannot be the union of fewer than λ-many nowhere dense, in particular compact, sets. $\square$

Now we are ready to prove the theorem which shows that the sequence mentioned at the beginning of this section can be split at any point.

Theorem 3.3. *Let $\kappa > \lambda$ be regular uncountable cardinals. If ZFC is consistent, then there are c.c.c. forcing extensions of ZFC in which $\mathfrak{c} = \kappa$ and any one of the following holds:*

(a) $\mathfrak{v}^{\infty} = \lambda$ *and* $\mathfrak{v} = \kappa$;
(b) $\mathfrak{v} = \lambda$ *and* $c(\mathbb{P}, s(n)) = \kappa$ *for all* $n \geq 1$;
(c_n) (*where n is a positive integer*) $c(\mathbb{P}, s(n+1)) = \lambda$ *and* $c(\mathbb{P}, s(n)) = \kappa$.

Proof: To prove the theorem, we will show that if the model V of the previous lemma is obtained by adding κ-many Cohen reals over a model of CH, then in the model V_0 of the previous lemma, only countably many of the Cohen reals are in any given 2^n-cone, in V_1 only countably many are in any n-splitting compact set (for any n), and in V_2 only countably many are in any $(n-1)$-splitting compact set.

The arguments in the three cases are fairly similar, exploiting the different centeredness properties of the posets. We will give the argument for the model V_2, and only briefly outline the differences for V_0 and V_1.

So, we assume we have taken a model V of CH, added κ-many Cohen reals by the poset $Fn(\kappa \times \omega, \omega)$, followed by the finite support iteration of length λ of the poset $R(n)$, where $n \geq 2$. By the previous lemma, $c(\mathbb{P}, s(n)) = \lambda$, so it remains to show that $c(\mathbb{P}, s(n-1)) = \kappa$.

Let P_λ denote the iteration. We can represent an element of P_λ by

$$p = \langle p_0, \langle F^p_\beta, k^p_\beta \rangle_{\beta \in D_p} \rangle$$

where $p_0 \in Fn(\kappa \times \omega, \omega)$, D_p is a finite subset of $\lambda \backslash \{0\}$, and $F^p_\beta = \{\dot{f}^p_{\beta,i}\}_{i < m^p_\beta}$, where each $\dot{f}^p_{\beta,i}$ is a P_β-name for an element of $\mathbb{P}$. We may also assume that $p{\restriction}\beta$ decides the value of an integer $l^p_\beta \geq k^p_\beta$ and the value T^p_β of $F^p_\beta{\restriction} l^p_\beta$.

Let r_α be the α^{th} real added by the first coordinate; i.e., $r_\alpha = \bigcup G(\alpha, \cdot) : \omega \to \omega$, where G is the projection on the first coordinate of a P_λ-generic filter. We will complete the proof by showing that in V^{P_λ}, no uncountable subset of the r_α's is contained in any given $(n-1)$-splitting compact set. To this end, suppose $p \Vdash$ "$\dot{A}$ is an uncountable subset of ω_1". It will suffice to find an extension q of p forcing the existence of $\delta_0, \ldots, \delta_{n-1} \in \dot{A}$ and $k \in \omega$ such that $r_{\delta_i}{\restriction} k = r_{\delta_j}{\restriction} k$ and $r_{\delta_i}(k) \neq r_{\delta_j}(k)$ for all $i \neq j < n$.

Since $p \Vdash$ "$\dot{A}$ is uncountable", one can inductively define distinct $\delta(\alpha) \in \omega_1$ and extensions p^α of p such that $p^\alpha \Vdash$ "$\delta(\alpha) \in \dot{A}$". Let

$$p^\alpha = \langle p_0^\alpha, \langle F_\beta^\alpha, k_\beta^\alpha \rangle_{\beta \in D_\alpha} \rangle$$

and denote the values of l and T associated with p^α by l_β^α and T_β^α. By passing to an uncountable subset if necessary, we may assume that

(i) $\{D_\alpha : \alpha < \omega_1\}$ is a Δ-system with root D;

(ii) for each $\beta \in D$, there are k^β, l^β, and T^β such that $k_\beta^\alpha = k_\beta$, $l_\beta^\alpha = l_\beta$ and $T_\beta^\alpha = T_\beta$ for any $\alpha \in \omega_1$;

(iii) the set $\{\mathrm{dom}(p_0^\alpha) : \alpha < \omega_1\}$ is a Δ-system, with root E, and the p_0^α's agree on E;

(iv) There is $k \in \omega$ and $\sigma \in {}^k\omega$ such that $p_0^\alpha(\delta(\alpha), \cdot) = \sigma$ for all $\alpha < \omega_1$.

Now select any $\alpha_i < \omega_1$, $i < n$, such that $\delta(\alpha_i) \notin E$. Let $q_0' \in Fn(\kappa \times \omega, \omega)$ be an extension of $\bigcup_{i<n} p_0^{\alpha_i}$ such that $q_0'(\delta(\alpha_i), k) = i$. Note that by (iv) above, q_0' forces that $\{r_{\alpha_i} : i < n\}$ is not contained in any $(n-1)$-splitting compact set. Then the condition

$$q = \langle q_0', \langle \bigcup_{i<n} F_\beta^{\alpha_i}, k_\beta \rangle_{\beta \in D}, \langle F_\gamma^{\alpha_i}, k_\gamma^{\alpha_i} \rangle_{\gamma \in \bigcup_{i<n} D_{\alpha_i} \setminus D} \rangle$$

is the extension of p we were looking for.

For the model for (a), of course we iterate Q^∞ instead of $R(n)$. We also put together (with the help of the σ-centeredness of Q^∞) $2^k + 1$-many p^α's, where k is as in (iv) above. Say these correspond to α_i, $i < 2^k + 1$. Then extend to q so that its first coordinate q_0' forces that the projection of $\{r_{\alpha_i} : i < 2^k + 1\}$ on the k^{th} coordinate has cardinality $2^k + 1$. Finally, for (b), show that, for any given n, no uncountable subset of the Cohen reals is contained in an n-splitting compact set by using the K_{n+1} property of Q. $\square$

4. Covering G_δ sets

In this section we extend some of our earlier results to show that unions of fewer than $\mathfrak{c}$-many special Cantor sets cannot cover dense G_δ sets in various products. As usual, if A and B are non-empty sets, and σ is a function from a finite subset of A to B, we denote by $[\sigma]$ the set of all elements of ${}^AB$ which extend σ with the range of the extensions assumed to be clear from the context. If σ and $\widehat{\sigma}$ are finite partial functions from ω to B, denote by $\langle \sigma, \widehat{\sigma} \rangle$ the finite partial function whose domain is $\mathrm{Dom}(\sigma) \cup \mathrm{Dom}(\widehat{\sigma})$ and which is defined to be $\sigma(j)$ if $j \in \mathrm{Dom}(\sigma)$ and $\widehat{\sigma}(j)$ otherwise.

Lemma 4.1. *Suppose B is a subset of ω and U is a dense open subset of ${}^{\omega}B$. If $\{\tau, \sigma_0, \dots, \sigma_k\}$ is a finite set of finite partial functions from ω to B, then there exists a finite partial function τ' extending τ defined on an initial segment of ω such that $[\langle\sigma_i, \tau'\rangle] \subseteq U$ for each $i \leq k$.*

Proof: Let N be an integer larger than $\mathrm{Max}(\mathrm{Dom}(\tau) \cup \bigcup_{i=0}^{k} \mathrm{Dom}(\sigma_i))$. Since U is dense and open in ${}^{\omega}B$, there exists a finite partial function τ_0 extending $\langle\sigma_0, \tau\rangle$ whose domain is an interval $[0, N_0]$ where $N_0 \geq N$ such that $[\langle\sigma_0, \tau_0\rangle] \subset U$. Clearly we can choose τ_0 so that also $\tau_0 \restriction \mathrm{Dom}(\sigma_0) = \tau \restriction \mathrm{Dom}(\sigma_0)$, whence τ_0 extends τ. We now continue by induction. If N_i and τ_i are defined, let τ_{i+1} be a finite extension of τ_i whose domain is an interval $[0, N_{i+1}]$ where $N_{i+1} \geq N_i$ such that $[\langle\sigma_{i+1}, \tau_{i+1}\rangle] \subseteq U$. Then the finite partial function $\tau = \bigcup_{i \leq k} \tau_i$ has the required properties. □

Theorem 4.2. (a) *Suppose G is a dense G_δ subset of $\mathbb{P}$. If $f \in \mathbb{P}$, then G contains a product $\prod_{i\in\omega} A_i$ with $|A_i| = f(i)$ for infinitely many $i \in \omega$.*
(b) *Suppose $n \in \omega$ and G is a dense G_δ subset of ${}^{\omega}n$. Then G contains a product $\prod_{i\in\omega} A_i$ with $A_i = n$ for infinitely many $i \in \omega$.*
(c) *Suppose G is a dense G_δ subset of $\mathbb{P}$. If F is any finite subset of ω, then G contains a non-empty product $\prod_{i\in\omega} A_i$ such that $A_i = \omega$ for each $i \in F$.*

Proof:

(a) Write G as $\bigcap_{n\in\omega} U_n$, where each U_n is an open dense subset of $\mathbb{P}$ and $U_n \supseteq U_{n+1}$ for each $n \in \omega$. Let $n_0 \in \omega$ be arbitrary. Applying Lemma 4.1 with $\tau = \emptyset$ and $\{\sigma_i : i \leq k\} = \{(n_0, j) : j < f(n_0)\}$, there exists a finite partial function τ_0 defined on an interval $[0, n_1)$ such that $n_1 > n_0$ and $[\langle(n_0, j), \tau_0\rangle] \subseteq U_0$ for each $j < f(n_0)$. Applying Lemma 4.1 again to the $[\langle(n_0, j), \tau_0\rangle]$'s, we get a finite partial function τ_1 extending τ_0 defined on an interval $[0, n_2)$ such that $[\langle\{(n_0, j_0), (n_1, j_1)\}, \tau_0\rangle] \subseteq U_1$ for each $j_0 < f(n_0)$ and each $j_1 < f(n_1)$. Recursively, use Lemma 4.1 to get a finite partial function τ_{k+1} extending τ_k defined on an interval $[0, n_{k+2})$ such that $[\langle\{(n_0, j_0), \dots, (n_k, j_k)\}, \tau_0\rangle] \subseteq U_{k+1}$ whenever $j_i < f(n_i)$ for $i \leq k+1$. Let $\tau = \bigcup_{k\in\omega} \tau_k$. Then G contains the product $\prod_{i\in\omega} A_i$ where $A_i = \{0, \dots, f(i) - 1\}$ if $i = n_k$ for some k and $A_i = \{\tau(i)\}$ otherwise.
(b) This part follows from part (a) by letting $f = \vec{n}$.
(c) We may assume without loss of generality that F is an initial segment of ω. Write G as $\bigcap_{n\in\omega} U_n$, where each U_n is an open dense subset of $\mathbb{P}$ and $U_n \supseteq U_{n+1}$ for each $n \in \omega$ and list the elements of ${}^{\omega}F$ as $\{\sigma_k : k \in \omega\}$. By Lemma 4.1, there exists a finite partial function τ_0 defined on an interval $[0, n_0]$ such that $[\langle\sigma_0, \tau_0\rangle] \subseteq U_0$. Using Lemma 4.1, we can recursively find a finite partial function τ_k extending τ_{k-1} defined on an interval $[0, n_k]$, where $n_k \geq k$, such that $[\langle\sigma_i, \tau_j\rangle] \subseteq U_k$ for each

$i \leq k$ and each $j < k$. Let $\tau = \bigcup_{k\in\omega} \tau_k$. Then G contains the product $\prod_{i\in\omega} A_i$ where $A_i = \omega$ if $i \in F$ and $A_i = \{\tau(i)\}$ otherwise.

$\square$

Remark 4.3. In Theorem 4.2(a), it is too much to expect that $|A_i| = f(i)$ for all $i \in \omega$. If G is the set of elements of ${}^\omega\omega$ which have at least one coordinate 0, then G is a dense G_δ — in fact, it is open — in ${}^\omega\omega$, and yet it contains no product where each factor has two elements.

Corollary 4.4. (1) *A dense G_δ-set in $\mathbb{P}$ cannot be covered by fewer than $\mathfrak{c}$-many members of $B(<\omega)$.*

(2) *If $n \in \omega$, a dense G_δ-set in ${}^\omega(n+1)$ cannot be covered by fewer than $\mathfrak{c}$-many n-ary Cantor sets.*

Proof: Part (a) follows from Theorem 4.2(a) and Theorem 2.4. Part (b) follows from Theorem 4.2(b) and Corollary 2.5. $\square$

We next show that Theorem 4.2 cannot be strengthened to get that a dense G_δ subset of ${}^\omega\omega$ contains a product with infinitely many infinite factors.

Example 4.5. *There exists a dense G_δ subset G of $\mathbb{P}$ such that G contains no product of the form $\prod_{n\in\omega} A_n$ where A_n is infinite for infinitely many values of n.*

Proof: Let $T = \{2^n : n \in \omega\}$. Rather than working with ${}^\omega\omega$, we will work with ${}^\omega T$. For each $k \in \omega$, let $U_k = \{f \in {}^\omega T : \sum_{i\leq n} f(i) = 2^r$ for some $n \geq k$ and $r \in \omega\}$. Then it is easy to see that each set U_k is a dense, open subset of ${}^\omega T$, so the set $G = \bigcap_{k\in\omega} U_k$ is a dense G_δ. Note that an element f of ${}^\omega T$ is in G if and only if there are infinitely many $k \in \omega$ such that $\sum_{i\leq k} f(i)$ is a power of 2. Now suppose that G contained a product $\prod_{n\in\omega} A_n$ where A_n is infinite for infinitely many values of n, say A_n is infinite for $n \in \{\lambda_k : k \in \omega\}$, where $0 < \lambda_0 < \lambda_1 < \cdots$. We may assume that if $n \notin \{\lambda_k : k \in \omega\}$, then A_n is a singleton. We are done if we can find a function $g \in G$ such that $\sum_{i\leq k} g(i)$ is a power of 2 for only finitely many values of k. If $n \notin \{\lambda_k : k \in \omega\}$, then $g(n)$ is the unique element of A_n. We define $g(\lambda_k)$ inductively. Let $g(\lambda_0)$ be any element of A_{λ_0} which is larger than $\sum_{\substack{i<\lambda_1 \\ i\neq\lambda_0}} g(i)$. If $g(\lambda_i)$ is defined for $i < n$, let $g(\lambda_n)$ be any element of A_{λ_n} which is larger than $\sum_{\substack{i<\lambda_{n+1} \\ i\neq\lambda_n}} g(i)$. We now show that if $k \geq \lambda_0$, then $\sum_{i\leq k} g(i)$ is not a power of 2. To see this, suppose $\lambda_n \leq k < \lambda_{n+1}$. Let $a = \sum_{\substack{i<k \\ i\neq\lambda_n}} g(i)$. Then $\sum_{i\leq k} g(i) = a + g(\lambda_n) = a + 2^r$ where 2^r is an element of A_{λ_n} satisfying $2^r > a \geq 1$. It follows that $\sum_{i\leq k} g(i)$ is not a power of 2. $\square$

References

[BJ] Bartoszyński T., Judah H., *Set Theory: On the Structure of the Real Line*, A K Peters, 1995.

[BL] Baumgartner J., Laver R., *Iterated perfect-set forcing*, Ann. Math. Logic **17** (1979), no. 3, 271–288.

[CN] Comfort W.W., Negrepontis S., *Theory of Ultrafilters*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[vD] van Douwen E.K., *The integers and topology*, in: Handbook of Set-theoretic Topology, K. Kunen and J.E. Vaughan, Eds., North-Holland, Amsterdam, 1984, pp. 111–167.

[Go] Goldstern M., *Tools for your forcing construction*, in: Set Theory of the Reals, H. Judah, Ed., Israel Math. Conf. Proceedings 6 (1993), pp. 305–360.

[GS] Goldstern M., Shelah S., *Many simple cardinal invariants*, Arch. Math. Logic **32** (1993), no. 3, 203–221.

[K] Kunen K., *Set Theory*, Studies on Logic and the Foundations of Mathematics, North-Holland, 1980.

[NR] Newelski L., Roslanowski A., *The ideal determined by the unsymmetric game*, Proc. Amer. Math. Soc. **117** (1993), no. 3, 823–831.

[R] Roslanowski A., *Mycielski ideals generated by uncountable systems*, Colloq. Math. **66** (1994), no. 2, 187–200.

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