Normal cones and C^*-m-convex structure

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Abstract. The notion of normal cones is used to characterize C^*-m-convex algebras among unital, symmetric and complete m-convex algebras.

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Introduction

In the first section, we consider unital, symmetric and complete m-convex algebras A which are Sym-s.b. We show (Theorem 2.1) that if the cone of positive elements is normal, then Pták's function is a C^*-norm and stronger than the initial topology; it is complete in the commutative case. The two topologies coincide in each of the following situations: A is a Q-algebra (Corollary 2.2), A is commutative and Fréchet (Corollary 2.4), A is barreled (Corollary 2.6). Noticing that the cone of positive elements is always normal in a C^*-m-convex algebra, we get Theorem 8.15 of [5]. We also obtain a generalization, to our context, of a result of Pták ([7, Theorem 8.4, p. 277]). Without Sym-s.b. condition but assuming commutativity we define, in Section 2, the notion of a normal cone for a family $(|.|_\lambda)_\lambda$ of seminorms; and show that it characterizes the cone of positive elements of a C^*-m-convex algebra. We also exhibit three independent conditions, on any cone which is stable by product, characterizing C^*-m-convex algebras among unital and complete m-convex ones.

1. Preliminaries

In a locally convex space E, a convex cone K is said to be normal if there is a family $(|.|_\lambda)_\lambda$ of seminorms, defining the topology of E, such that, for every λ, one has $|y|_\lambda \geq |x|_\lambda$ whenever $x, y \in K$ and $y - x \in K$. Let $(A, (|.|_\lambda)_\lambda)$ be an m-convex algebra ($l.m.c.a.$) which is unital and complete. It is known that $(A, (|.|_\lambda)_\lambda)$ is the projective limit of the normed algebras $(A_\lambda, \|.\|_\lambda)$, where $A_\lambda = A/N_\lambda$ with $N_\lambda = \{x \in A : |x|_\lambda = 0\}$; and $\|\overline{x}\|_\lambda = |x|_\lambda$. An element x of A is written $x = (x_\lambda)_\lambda = (\pi_\lambda(x))_\lambda$, where $\pi_\lambda : A \longrightarrow A_\lambda$ is the canonical surjection. The algebra $(A, (|.|_\lambda)_\lambda)$ is also the projective limit of the Banach algebras $\widehat{A_\lambda}$, the completions of A_λ's. The norm in $\widehat{A_\lambda}$ will also be denoted by $\|.\|_\lambda$. We will denote by $*$-*l.m.c.a.* any *l.m.c.a.* A with a continuous involution $x \longmapsto x^*$; it will be said symmetric if $e + xx^*$ is invertible for every x in A. The sets of hermitian elements and of positive elements, will be denoted by $\mathrm{Sym}(A)$ and A_+ respectively. It is known that A_+ is a convex cone in any symmetric, unital and complete *l.m.c.a.* ([4, Proposition 8.6, p. 39]). An involutive algebra is said to be symmetrically spectrally bounded (Sym-s.b.) if the spectrum of every hermitian element is bounded. In the sequel ρ and p_A will stand respectively for the spectral radius and Pták's function that is $\rho(x) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}\, x\}$ and $p_A(x) = \rho(xx^*)^{\frac{1}{2}}$.

2. Normality of A_+ and C^*-algebra structure

The cone of positive elements in a C^*-*l.m.c.a.* is always normal. The algebra $(C^\infty[0,1], (|.|_n)_n)$, where $|f|_n = \sum_{k=0}^{n} \frac{1}{k!} \sup\{|f^{(k)}(t)| : t \in [0,1]\}$, shows that this is not, in general, the case for symmetric $*$-*l.m.c.a.′s*. Actually, the normality of A_+ is strong enough as to ensure a C^*-algebra structure under suitable conditions. We first give a result on Pták's function.

Theorem 2.1. *Let $(A, (|.|_\lambda)_\lambda)$ be a symmetric, unital and complete l.m.c.a. which is Sym-s.b. If A_+ is normal, then Pták's function is a C^*-norm and stronger than the topology of A.*

Proof: A_+ being normal, there is a family $(\|.\|_\mu)_\mu$ of seminorms, not necessarily submultiplicative, defining the topology of A such that $\|x\|_\mu \leq \|y\|_\mu$ for every μ whenever $x, y \in A_+$ and $y - x \in A_+$. Whence, A being symmetric, $\|h\|_\mu \leq 3\|e\|_\mu \rho(h)$ for every $h \in \mathrm{Sym}(A)$ and every μ. Indeed, since $\mathrm{Sp}\, h \subset R$, one has $-\rho(h) \leq h \leq \rho(h)$, i.e., $0 \leq h + \rho(h) \leq 2\rho(h)$; and one uses the normality of A_+. Since $x = h + ik$, with $h, k \in \mathrm{Sym}(A)$, one gets $\|x\|_\mu \leq 3\|e\|_\mu(\rho(h) + \rho(k))$, for every x in A. This implies, by Proposition 8.8 of [4], that $\|x\|_\mu \leq 3\|e\|_\mu p_A(x)$ for every x in A. On the other hand, p_A is a C^*-seminorm by Theorem 4.4 of [3].

□

Corollary 2.2. *Let $(A, (|.|_\lambda)_\lambda)$ be a Q-$*$-l.m.c.a. which is unital, complete and symmetric. If A_+ is normal, then A is a C^*-algebra.*

Proof: Follows from the fact that Pták's function is continuous ([4, Corollary 8.9]).

□

Example 2.3. The Q-property is essential in this corollary. Indeed, consider the algebra $l^\infty(N)$ with the usual operations, the family $(|.|_n)_n$ of seminorms where $|(x_p)_p|_n = \sup\{|x_p| : p \leq n\}$ and the involution given by $((x_p)_p)^* = (\overline{x_p})_p$.

Proposition 2.4. *If in Theorem 2.1 the algebra A is commutative, then Pták's function p_A is a Banach algebra norm.*

Proof: If $(x_n)_n$ is a Cauchy sequence for p_A, it is also Cauchy for the topology of A and hence converges to an element x in A. But A is the projective limit of

Banach algebras $\widehat{A_\lambda}$. The involution being continuous and due to commutativity, $\widehat{A_\lambda}$ can be endowed with a hermitian involution by $(\lim_n \pi_\lambda(a_n))^* = \lim_n \pi_\lambda(a_n^*)$. Denoting by $p_{\widehat{A_\lambda}}$ Pták's function in $\widehat{A_\lambda}$, one has, for every λ, $\lim_n p_{\widehat{A_\lambda}}(\pi_\lambda(x_n - x)) = 0$. Indeed

$$p_{\widehat{A_\lambda}}(\pi_\lambda(x_n - x)) \leq c_\lambda \widehat{p_\lambda}(\pi_\lambda(x_n - x)) = c_\lambda p_\lambda(x_n - x),$$

for a given $c_\lambda > 0$. But, $p_A(a) = \sup_\lambda p_{\widehat{A_\lambda}}(\pi_\lambda(a))$ for every a in A. Finally, we obtain, by standard technics, that $\lim_n p_A(x_n - x) = 0$. □

We have the following consequence.

Corollary 2.5. *Let $(A, (|.|_\lambda)_\lambda)$ be a commutative symmetric and Fréchet $*$-l.m.c.a. which is Sym-s.b. If A_+ is normal, then A is a C^*-algebra.*

Remark 2.6. In Proposition 2.4, commutativity is used to ensure that the extended involution, from A_λ to $\widehat{A_\lambda}$, remains hermitian. This is the case, for example, when we deal with any C^*-*l.m.c.a.* Indeed, A_λ is then complete by Theorem 2.4 of [1].

Corollary 2.7. *Let $(A, (|.|_\lambda)_\lambda)$ be a unital, symmetric and barreled complete l.m.c.a. which is Sym-s.b. If A_+ is normal, then A is a C^*-algebra.*

Proof: By Theorem 2.1, there is a family $(\|.\|_\mu)_\mu$ of seminorms, not necessarily submultiplicative, defining the topology of A and such that $\|x\|_\mu \leq p_A(x)$ for every x and every μ. Put $\|x\| = \sup\{\|x\|_\mu : \mu\}$, $x \in A$. Then $\|.\|$ is a norm which is finer than the topology of A. But $\{x \in A : \|x\| \leq 1\}$ is a barrel, which implies that $(A, (|.|_\lambda)_\lambda)$ is a normed algebra. Finally, it is a C^*-algebra, since $\|x\| \leq p_A(x)$ for every x ([5, Theorem 7.9]). □

As a consequence, we obtain a result of M. Fragoulopoulou.

Corollary 2.8 ([5, Corollary 7.11])**.** *Let $(A, (|.|_\lambda)_\lambda)$ be a unital and complete l.m.c.a. The following assertions are equivalent.*

(i) *A is a Q-C^*-l.m.c.a.*
(ii) *A is a barreled C^*-l.m.c.a. which is Sym-s.b.*
(iii) *A is a Fréchet C^*-l.m.c.a. which is Sym-s.b.*
(iv) *A is a C^*-algebra.*

Proof: Due to the fact that A_+ is normal in any complete C^*-*l.m.c.a.* □

If in a hermitian Banach algebra $(A, \|.\|)$, there is $\alpha > 0$ such that $\rho(h) \geq \alpha\|h\|$, for every h in Sym(A), then A is a C^*-algebra for an equivalent norm ([6, Theorem 8.4]). In m-convex algebras, we have the following results.

Theorem 2.9. *Let $(A, (|.|_\lambda)_\lambda)$ be a symmetric and complete l.m.c.a. which is Sym-s.b. If, for every λ, there is $\alpha_\lambda > 0$ such that $\rho(h) \geq \alpha_\lambda |h|_\lambda$ for every $h \in \mathrm{Sym}(A)$, then Pták's function is a C^*-norm which is stronger than the topology of A.*

Theorem 2.10. *Let $(A, (|.|_\lambda)_\lambda)$ be a symmetric and complete $*$-l.m.c.a. which is Sym-s.b. The following assertions are equivalent.*

(i) *A is a C^*-algebra.*
(ii) *A is a Q-algebra and, for every λ, there is $\alpha_\lambda > 0$ such that $\rho(h) \geq \alpha_\lambda |h|_\lambda$ for every $h \in \mathrm{Sym}(A)$.*
(iii) *A is barreled and, for every λ, there is $\alpha_\lambda > 0$ such that $\rho(h) \geq \alpha_\lambda |h|_\lambda$ for every $h \in \mathrm{Sym}(A)$.*

3. Normality of A_+ and commutative C^*-m-convex structure

In a complete C^*-*l.m.c.a.*, the family $(|.|_\lambda)_\lambda$ of seminorms defining the topology satisfy $|y|_\lambda \geq |x|_\lambda$, for $x, y \in A_+$ such that $y - x \in A_+$. This fact suggests the following definition.

Definition 3.1. Let $(A, (|.|_\lambda)_\lambda)$ be an *l.m.c.a.*, C a convex cone in A and $(\|.\|_\lambda)_\lambda$ a family of seminorms on A. The cone C is said to be normal for the family $(\|.\|_\lambda)_\lambda$ if, for every λ, there is $\beta_\lambda > 0$ such that $\|y\|_\lambda \geq \beta_\lambda \|x\|_\lambda$ whenever $x, y \in C$ and $y - x \in C$.

Proposition 3.2. *Let $(A, (|.|_\lambda)_\lambda)$ be a commutative, unital complete and symmetric $*$-l.m.c.a. If A_+ is normal for $(|.|_\lambda)_\lambda$, then A is a C^*-l.m.c.a. for an equivalent family of seminorms.*

Proof: Consider the Banach algebras $\widehat{A_\lambda}$ of which A is the projective limit. We show that every $\widehat{A_\lambda}$ is a C^*-algebra for an equivalent norm. First, notice that every $\widehat{A_\lambda}$ is hermitian. Indeed, for $h \in \mathrm{Sym}(\widehat{A_\lambda})$, there is a sequence $(h_n)_n \subset \mathrm{Sym}(A)$ such that $h = \lim_n \pi_\lambda(h_n)$; hence $\mathrm{Sp}\, h \subset R$ for $\widehat{A_\lambda}$ is commutative. Now, one shows that $(\widehat{A_\lambda})_+ = \overline{\pi_\lambda(A_+)}$. Finally, $(\widehat{A_\lambda})_+$ is normal, since A_+ is so. We conclude by Corollary 2.2. $\square$

Let $(A, (|.|_\lambda)_\lambda)$ be a commutative, unital and complete *l.m.c.a.*, K a convex cone which is stable by product and $H = K - K$ the real sub-algebra spanned by K. We consider the following conditions which are satisfied by the cone of positive elements in a C^*-*l.m.c.a.*

(P_1) $A = H + iH$.
(P_2) K is normal for $(|.|_\lambda)_\lambda$.
(P_3) $(e + u)^{-1} \in K$, for every $u \in K$.

We show that these conditions characterize the cone of positive elements in a C^*-*l.m.c.a.* Let us begin with the Banach case.

Proposition 3.3. *Let $(A, \|.\|)$ be a commutative, unital, Banach algebra and K a convex cone which is stable by product. If conditions (P_1), (P_2) and (P_3) are fulfilled, then A is a C^*-algebra for an equivalent norm.*

PROOF: We may suppose K closed for the closure $\overline{K}$ of K also satisfies (P_1), (P_2) and (P_3). By Theorem 2, p. 260, of [2], one has

(1) $K = H \cap \operatorname{Splsa}(A)$; where $\operatorname{Splsa}(A) = \{x \in A : \operatorname{Sp} x \subset R_+\}$,

(2) $\operatorname{Sp} h \subset R$, for every $h \in H$.

Now K, being normal, is salient and so $H \cap iH = \{0\}$. Then, the map $(h+ik)^* = h - ik$ defines, on A, a hermitian involution. The cone A_+, of positive elements, for this involution, is exactly K. The conclusion follows from Corollary 2.2. □

This result extends to m-convex algebras as follows.

Proposition 3.4. *Let $(A, (|.|_\lambda)_\lambda)$ be a commutative, unital; complete l.m.c.a. and K a convex cone which is stable by product. If conditions (P_1), (P_2) and (P_3) are fulfilled, then A is a C^*-l.m.c.a. for an equivalent family of seminorms.*

PROOF: For every λ, the closed convex cone $K_\lambda = \overline{\pi_\lambda(K)}$ is stable by product and satisfies (P_2) and (P_3). The real subalgebra $H_\lambda = K_\lambda - K_\lambda$ is closed in $\widehat{A_\lambda}$ ([2, Theorem 2, p. 260]). On the other hand, one shows that, for every λ, there is $\beta_\lambda > 0$ such that $|h|_\lambda \leq \beta_\lambda |h + ik|_\lambda$, for $h, k \in A_\lambda$. So the subalgebra $H_\lambda + iH_\lambda$ is closed in $\widehat{A_\lambda}$ and hence $\widehat{A_\lambda} = H_\lambda + iH_\lambda$. We conclude by Proposition 3.3. □

Remark 3.5. Without condition (P_1), one obtains that $\varprojlim(H_\lambda + iH_\lambda)$ is a C^*-*l.m.c.a.*, containing $H + iH$ and contained in A. An application of this fact is the following.

Let $(A, (|.|_\lambda)_\lambda)$ be a commutative, unitary and complete *l.m.c.a.* The set $P = \{x \in A : \operatorname{Sp} x \subset R_+\}$ is a convex cone which is stable by product. Put $H = P - P$ the real subalgebra spanned by P.

Proposition 3.6. *If the cone P is normal for $(|.|_\lambda)_\lambda$, then the complex algebra $H + iH$ is a C^*-l.m.c.a.*

PROOF: It is sufficient to show that $H + iH$ is closed in the C^*-*l.m.c.a.* $B = \varprojlim(H_\lambda + iH_\lambda)$. First, H is closed, since $H = \{x \in A : \operatorname{Sp} x \subset R\}$. Let $(h_n + ik_n)_n$ be a sequence, in $H + iH$, converging to x in B. Since, for every λ, $H_\lambda + iH_\lambda$ is a C^*-algebra for an equivalent norm $\|.\|_\lambda$, there is $\alpha_\lambda > 0$ such that $\|h_\lambda + ik_\lambda\|_\lambda \geq \alpha_\lambda \|h_\lambda\|_\lambda$, for $h_\lambda, k_\lambda \in H_\lambda$. So the sequences $(h_n)_n$ and $(k_n)_n$ are Cauchy, in A, and hence converge, to h and k in H, respectively. Finally $x = h + ik$. □

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