On D-property of strong Σ spaces

Raushan Z. Buzyakova

Abstract. It is shown that every strong Σ space is a D-space. In particular, it follows that every paracompact Σ space is a D-space.

Keywords: strong Σ space, D-space

Classification: 54D20, 54F99

In this paper we will show that any strong Σ space is a D-space. This result positively answers Borges and Matveev's question whether any paracompact Σ space is a D-space. The notion of D-space was introduced by Eric van Douwen [6].

A *neighborhood assignment* for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for any $x \in X$. A space X is a *D-space*, if for any neighborhood assignment φ for X there exists a closed discrete subset D of X such that $X = \bigcup_{d \in D} \varphi(d)$.

It is natural to ask which spaces possess the D-property. It is known that σ-compact spaces, metrizable spaces, semi-stratifiable spaces, and paracompact p-spaces are all D-spaces (see [5], [2]). In [5], DeCaux showed that every finite product of copies of the Sorgenfrey line is a D-space. The D-property of subspaces of generalized ordered spaces was studied in [8]. In a recent paper [10] of Fleissner and Stanley, the authors give conditions under which a subspace of a product of finitely many ordinals is a D-space. Several interesting questions on D-spaces were raised by E. van Douwen and W.F. Pfeffer in [7], which was the first published paper that contained results on D-spaces. Some other results and questions on D-spaces can be found in [5], [2], [3], [4], [8], [10].

The result in this article is obtained in an attempt to answer E.K. van Douwen's question whether each Lindelöf space is a D-space. However, this question remains unanswered. And, one of approaches to solve this problem could be to consider continuous images of Lindelöf D-spaces.

Question (A. V. Arhangelskii). *Is it true that a continuous image of a Lindelöf D-space is a D-space?*

We consider only Tychonoff spaces. In notation and terminology, we will follow [9].

A space X is a *strong* Σ space if there exist a σ-locally-finite family γ of closed sets in X and a cover $\mathcal{K}$ of X by compact subsets, such that for any open set U containing an element K of $\mathcal{K}$, $K \subseteq \Gamma \subseteq U$ for some $\Gamma \in \gamma$.

The class of strong Σ spaces is wide and it contains all metrizable spaces, σ-compact spaces, Lindelöf Σ spaces, paracompact Σ spaces, paracompact p-spaces, Moore spaces, spaces with countable network, as well as spaces with σ-discrete network (σ spaces). Thus, our result implies that the mentioned spaces are all D-spaces. In addition, a finite (countable) product of strong Σ spaces is a D-space as well, since the class of strong Σ spaces is closed with respect to countable products. Therefore, in particular, the product of a Lindelöf Σ space with a Moore space is still a D-space. However, as shown in [4], in general case the product of two D-spaces need not be a D-space.

Theorem. *Every strong Σ space X is a D-space.*

PROOF: Let $\mathcal{K}$ and γ be the families from the definition of a strong Σ space. Represent γ as $\bigcup\{\gamma_n\}$, where each γ_n is a locally-finite family of closed sets in X and $\gamma_n \subseteq \gamma_{n+1}$. Enumerate each $\gamma_n = \{\Gamma_\alpha^n\}$, where α ranges through some ordinal number.

Let φ be an arbitrary neighborhood assignment. We need to find a discrete closed subset D in X such that $X = \bigcup_{d\in D} \varphi(d)$. Recursively, we will define closed discrete sets D_n such that $D = \bigcup D_n$.

Step 0. Set $D_0 = \emptyset$.

Assume D_m is defined for all $0 < m < n$.

Step n. Recursively, we will define finite sets D_α^n such that $D_n = (\bigcup D_\alpha^n) \cup D_{n-1}$.
Sub-step 0. Set $D_0^n = \emptyset$.

Assume D_β^n is defined for all $0 < \beta < \alpha$.

Sub-step α. Let $U = \bigcup\{\varphi(d) : d \in (\bigcup_{\beta<\alpha} D_\beta^n) \cup D_{n-1}\}$. Take the first Γ in γ_n that satisfies the following requirement.

Requirement R_α^n: there exists $K \in \mathcal{K}$ which is not fully covered by U. And there exist $x_1, \dots, x_k \in K \setminus U$ such that $K \setminus U \subseteq \Gamma \setminus U \subseteq \varphi(x_1) \cup \dots \cup \varphi(x_k)$.

If no such Γ exists, sub-recursion stops. Put $D_\alpha^n = \{x_1, \dots, x_k\}$.
Let $D_n = (\bigcup D_\alpha^n) \cup D_{n-1}$. We need to show that D_n is closed and discrete in X. Take an arbitrary $x \in X$. We need to separate x from $D_n \setminus \{x\}$ by a neighborhood. Consider the family

$$\gamma_n' = \{\Gamma_\beta : \Gamma_\beta \text{ is the first in } \gamma_n \text{ satisfying Requirement } R_\alpha^n \text{ for some } \alpha\}.$$

Since $\gamma_n' \subseteq \gamma_n$, γ_n' is locally-finite too. Therefore, there exists a neighborhood of x that intersects only a finite number of elements in γ_n', and therefore, only finite number of sets D^n_α's. Since the D^n_α's are finite, x is not in the closure of $(\bigcup D^n_\alpha) \setminus \{x\}$. And x can be separated from $D_{n-1} \setminus \{x\}$ since the latter is closed and discrete by assumption.

The construction is complete. Put $D = \bigcup D_n$.

Let us show that $X = \bigcup_{d \in D} \varphi(d)$. Assume the contrary. Then there exists a K in $\mathcal{K}$ such that $K' = K \setminus \bigcup_{d \in D} \varphi(d) \neq \emptyset$. Since K' is compact we can find $x_1, \dots, x_k \in K'$ such that $K' \subseteq \varphi(x_1) \cup \dots \cup \varphi(x_k)$. Consider a compactum $K'' = K \setminus (\varphi(x_1) \cup \dots \cup \varphi(x_k))$. Find the smallest n such that $K'' \subseteq \bigcup_{d \in D_n} \varphi(d)$. Now take the first γ_l containing such a Γ that

$$K \subseteq \Gamma \subseteq \varphi(x_1) \cup \dots \cup \varphi(x_k) \cup \big(\bigcup_{d \in D_n} \varphi(d)\big).$$

Let $m = \max\{n, l\}$. Then $\gamma_l \subseteq \gamma_{m+1}$, and therefore, $\Gamma \in \gamma_{m+1}$. By the choice of n and l, Γ satisfies the *Requirement* starting not later than from Sub-step 1 of Step $m+1$. And eventually, Γ will be the first in γ_{m+1} satisfying the *Requirement*. Therefore, Γ must be covered by $\bigcup_{d \in D} \varphi(d)$, and so must K.

Let us show now that D is closed and discrete. Take an arbitrary $x \in X$. We need to show that x can be separated from $D \setminus \{x\}$ by a neighborhood of x. There exists an n such that $x \in \bigcup_{d \in D_n} \varphi(d)$. This means that x is separated from $D \setminus D_n$ by $\bigcup_{d \in D_n} \varphi(d)$ (follows from the construction of D_n's). And x can be separated from $D_n \setminus \{x\}$, since D_n is closed and discrete. □

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Mathematics Department, Brooklyn College, Brooklyn, 11210 USA

E-mail: RaushanB@brooklyn.cuny.edu

(*Received* December 20, 2001)