Normally flat semiparallel submanifolds in space forms as immersed semisymmetric Riemannian manifolds

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Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday

Abstract. By means of the bundle of orthonormal frames adapted to the submanifold as in the title an explicit exposition is given for these submanifolds. Two theorems give a full description of the semisymmetric Riemannian manifolds which can be immersed as such submanifolds. A conjecture is verified for this case that among manifolds of conullity two only the planar type (in the sense of Kowalski) is possible.

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Classification: 53C21, 53C25, 53C42

Introduction

The geometry of a Riemannian manifold (M, g) depends essentially on its Levi-Civita connection ∇ and the curvature tensor R. If R is parallel with respect to ∇, i.e. if $\nabla R = 0$, then M is said to be *locally symmetric*. É. Cartan has developed the famous theory of such manifolds, both local and global, showing, in particular, that (M, g) is locally symmetric if and only if its geodesic reflection with respect to arbitrary point is local isometry.

Let $N^n(c)$ be a space form, i.e. a connected complete Riemannian manifold with constant curvature c, and let (M, g) be immersed isometrically into $N^n(c)$ as an m-dimensional submanifold M^m. The geometry of such an M^m in $N^n(c)$ depends essentially on its van der Waerden-Bortolotti connection $\bar{\nabla}$ (which is actually a pair of ∇ and of the normal connection $\nabla^\perp$) and the second fundamental (mixed) tensor h. The famous Gauss, Peterson-Mainardi-Codazzi, and Ricci equations establish the well-known relationships between h, R, $\bar{\nabla}$, and $R^\perp$ (here the latter is the curvature (mixed) tensor of $\nabla^\perp$).

A submanifold M^m in $N^n(c)$ is said to be *parallel*, if $\bar{\nabla} h = 0$. A consequence from Gauss equation is that the parallel immersion exists only for locally symmetric (M, g). Ferus [F3] and Strübing [Str] showed that a submanifold M^m in $N^n(c)$ is parallel if and only if the normal reflection in $N^n(c)$ with respect to M^m at its arbitrary point induces local isometry of M^m. Therefore they used the term *symmetric submanifolds* instead of *parallel submanifolds* which was introduced later by Takeuchi [T]. Ferus [F1], [F2] established that an irreducible symmetric ($\equiv$ parallel) submanifold in a Euclidean space E^n (the case $c = 0$) can be characterized as a symmetric R-space immersed standardly into E^n as a minimal submanifold of some hypersphere of E^n.

The differential systems $\nabla R = 0$ and $\bar{\nabla} h = 0$ have their integrability conditions, respectively, $\Omega \circ R = 0$ and $\bar{\Omega} \circ h = 0$, where the first ingredients are the curvature 2-form operators; the same integrability conditions can be written also as $R(X, Y) \circ R = 0$ and $\bar{R}(X, Y) \circ h = 0$. The manifold and submanifold satisfying these conditions, correspondingly, are called *semisymmetric manifold* (first in [S1], [S2]) and *semiparallel submanifold* (first in [D]), respectively (see e.g. [BKV]).

From the Gauss and Ricci equations it follows that every semiparallel submanifold is intrinsically a semisymmetric manifold. Lumiste [L2] has shown that a submanifold M^m in $N^n(c)$ is semiparallel if and only if it is a second order envelope of symmetric submanifolds (symmetric in the sense of Ferus). Therefore in [L1]–[L5] the semiparallel submanifolds were called, provisionally, semi-symmetric submanifolds (meant extrinsically).

The local classification of the semisymmetric Riemannian manifolds is given by Szabó [Sz]. The most interesting class is that of so called foliated semisymmetric manifolds, which can be characterized as foliated by locally Euclidean leaves of codimension two. Kowalski [K] has given for the dimension $m = 3$ a more detailed partition in this class, afterwards extended by Boeckx [B] for the arbitrary dimension m. So the *planar, hyperbolic, parabolic* and *elliptic* manifolds have been distinguished in this class (see [BKV, Chapter 7], where the manifolds of this class are characterized as *of conullity two*).

The results about semiparallel submanifolds are summarized recently in [L6]. The problem which will be considered in the present paper is: can every semisymmetric Riemannian manifold (M, g) be immersed isometrically into a space form $N^n(c)$ as a semiparallel submanifold, and if not, which of them admit such an immersion?

The first steps towards solving this problem have shown that the answer to the first question is negative. For instance, in [L7] there is established that among the three-dimensional Riemannian manifolds of conullity two only those of the planar type admit an isometric immersion into a E^n, the other types (hyperbolic, parabolic and elliptic) do not admit. In [L8] there is shown that the same phenomenon occurs when considering the semiparallel submanifolds M^m in E^n which are foliated into $(m-2)$-dimensional plane generators; they can be intrinsically of conullity two, but only of the planar type.

The full classification and description of semiparallel submanifolds M^m in a space form $N^n(c)$ is missing yet. It is done only for some special cases, for instance for normally flat semiparallel submanifolds (i.e. with flat $\nabla^\perp$), investigated and classified for Euclidean space in [L3], [L4], [L5], and for general space forms in [DN].

The second part of the problem above for this case is solved in the present paper. It will be done by investigating the inner geometry of these normally flat semiparallel submanifolds M^m in $N^n(c)$. The results are formulated in Theorems 3.2 and 3.3. They give a new confirmation to a conjecture, that only the manifolds of conullity two of planar type (in the sense of Kowalski) can be immersed isometrically as semiparallel submanifolds.

1. Classification of semisymmetric Riemannian manifolds

The general classification of the semisymmetric Riemannian (M, g) is made by Szabó, locally in [Sz]. First he proved by means of infinitesimal and local holonomy groups that for every semisymmetric Riemannian manifold M there exists a dense open subset U such that around the points of U the manifold M is locally isometric to a direct product of semisymmetric manifolds $M_0 \times M_1 \times \cdots \times M_r$, where M_0 is an open part of a Euclidean space and the manifolds M_i, $i > 0$, are infinitesimally irreducible simple semisymmetric leaves. Here a semisymmetric M is called a *simple leaf* if at every its point x the primitive holonomy group determines a simple decomposition $T_xM = V_x^{(0)} + V_x^{(1)}$, where this group acts trivially on $V_x^{(0)}$ and there is only one other subspace $V_x^{(1)}$ which is invariant for this group. A simple leaf is said to be infinitesimally irreducible if at least at one point the infinitesimal holonomy group acts irreducibly on $V_x^{(1)}$.

The dimension $\nu(x) = \dim V_x^{(0)}$ is called the *index of nullity* at x and $u(x) = \dim M - \nu(x)$ the *index of conullity* at x.

The classification theorem by Szabó [Sz] asserts the following (according to the formulation given in [B], [BKV]).

Theorem 1.1. *Let M be an infinitesimally irreducible simple semisymmetric leaf and x a point of M. Then one of the following cases occurs:*

(a) $\nu(x) = 0$ *and* $u(x) > 2$: *M is locally symmetric and hence locally isometric to a symmetric space;*
(b) $\nu(x) = 1$ *and* $u(x) > 2$: *M is locally isometric to an elliptic, a hyperbolic or a Euclidean cone;*
(c) $\nu(x) = 2$ *and* $u(x) > 2$: *M is locally isometric to a Kählerian cone;*
(d) $\nu(x) = \dim M - 2$ *and* $u(x) = 2$: *M is locally isometric to a space foliated by Euclidean leaves of codimension two (or to a two-dimensional manifold, this for the case when* $\dim M = 2$*).*

Note that in case (d) the term *manifold of conullity two* is used for M in [BKV]. Kowalski considering the three-dimensional M introduced for this case (d) the geometric concept of *asymptotic foliation* (first in a preprint of 1991, then published in [K] and meanwhile generalized by Boeckx [B] to arbitrary dimension of M; see also [BKV]).

Namely, a codimension one submanifold of a Riemannian manifold M of conullity two is called the *asymptotic leaf* if it is generated by the codimension two Euclidean leaves of M and if its tangent spaces are parallel along each of these leaves (with respect to the Levi-Civita connection ∇ of M).

A codimension one foliation on such an M is called the *asymptotic foliation* if its integral manifolds are asymptotic leaves.

Let $O(M)$ be the bundle of orthonormal frames $(e_1, \ldots, e_m)$ on M. For the bundle $O^*(M)$ of the dual coframes $(\omega^1, \ldots, \omega^m)$ the following structure equations hold:

$$d\omega^i = \omega^j \wedge \omega^i_j, \;\; d\omega^i_j = \omega^k_j \wedge \omega^i_k + \Omega^i_j, \tag{1.1}$$

where ω^i_j and Ω^i_j are the connection 1-forms and the curvature 2-forms, correspondingly, of ∇. Here orthonormality yields $\omega^i_j + \omega^j_i = 0$, $\Omega^i_j + \Omega^j_i = 0$.

Let M be of case (d). Then $O(M)$ and $O^*(M)$ can be adapted to this M so that $e_3, \ldots, e_m$ are tangent to codimension two Euclidean leaves and thus these leaves are determined by $\omega^1 = \omega^2 = 0$. Since this last differential system is totally integrable, $d\omega^1$ and $d\omega^2$ must vanish as the algebraic consequences of $\omega^1 = \omega^2 = 0$ (due to Frobenius theorem, second version; see [St]). This together with the fact that Euclidean leaves are totally geodesic, because M is a simple leaf, yields

$$\omega^1_u = a_u\omega^1 + b_u\omega^2, \;\; \omega^2_u = c_u\omega^1 + e_u\omega^2; \tag{1.2}$$

here (and also further) $u \in \{3, \ldots, m\}$.

In [BKV] there is shown that $\omega^1 : \omega^2$ determines an asymptotic foliation if and only if

$$c_u(\omega^1)^2 + (e_u - a_u)\omega^1\omega^2 - b_u(\omega^2)^2 = 0. \tag{1.3}$$

According to [K], [BKV] a foliated M is said to be *planar* if it admits infinitely many asymptotic foliations. If it admits just two (or one, or none, respectively) asymptotic foliations, it is said to be *hyperbolic* (or *parabolic*, or *elliptic*, respectively).

From (1.3) it is seen that a planar foliated M is characterized by $a_u - e_u = b_u = c_u = 0$, i.e. by the fact that (1.2) reduces to

$$\omega^1_u = a_u\omega^1, \;\; \omega^2_u = a_u\omega^2. \tag{1.4}$$

2. Normally flat semiparallel submanifolds

The space forms $N^n(c)$ will be considered further by their standard models, which are (see [W]):

for $c = 0$ the Euclidean space E^n,

for $c > 0$ the sphere $S^n(c) = \{x \in E^{n+1} | \langle \vec{ox}, \vec{ox} \rangle = r^2\}$ with a real radius $r = 1/\sqrt{c}$ and with the centre at the origin o,

for $c < 0$ one sheet $H^n(c)$ of $\{x \in E_1^{n+1} | \langle \vec{ox}, \vec{ox} \rangle = -r^2\}$ with imaginary $r = i/\sqrt{|c|}$ and with the centre at the origin o; here E_1^{n+1} is the Lorentz space, in which for a fixed frame $\{o; e_1, \dots, e_n, e_{n+1}\}$ there hold $\langle e_I, e_J \rangle = \delta_{IJ}$, $\langle e_i, e_{n+1} \rangle = 0$, where I, J run over $\{1, \dots n\}$, and $\langle e_{n+1}, e_{n+1} \rangle = -1$; this sheet $H^n(c)$ is usually separated by $x^{n+1} > 0$ and it is a model of the hyperbolic (or Lobachevski-Bolyai) space (see [N]).

Here and further x denotes both the point $x \in M^m$ and its radius vector $\vec{ox}$; note that dx for this vector does not depend on the origin o, which in the Euclidean case E^n can be fixed arbitrarily.

Let $O(N^n(c))$ be the bundle of orthonormal frames $(x; e_1, \dots, e_n)$ for $N^n(c)$, i.e. of orthonormal bases $(e_1, \dots, e_n, e_{n+1})$ in the vector spaces of E^{n+1} (or E_1^{n+1}) with $e_{n+1} = -\sqrt{|c|}x$, if $c \neq 0$. There hold the following derivation formulae (see e.g. [L6])

$$dx = e_I \omega^I, \quad de_I = e_J \omega_I^J - xc\omega^I, \quad \omega_I^J + \omega_J^I = 0,$$

where Einstein summation convention is used. From here exterior differentiation leads to the following structure equations

$$d\omega^I = \omega^J \wedge \omega_J^I, \quad d\omega_I^J = \omega_I^K \wedge \omega_K^J + c\omega^J \wedge \omega^I.$$

If an m-dimensional Riemannian manifold M is immersed isometrically into a space form $N^n(c)$ as a submanifold M^m of $N^n(c)$, then the subbundle $O(M^m, N^n(c))$ of frames adapted to M^m can be introduced if we require that $e_1, \dots, e_m$ are tangent and $e_{m+1}, \dots, e_n$ normal to M^m at an arbitrary point $x \in M^m$ (see e.g. [KN]). For this $O(M^m, N^n(c))$, the derivation formulae and structure equations above imply

$$\omega^\alpha = 0, \ \omega_i^\alpha = h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \tag{2.1}$$

where i, j run over $\{1, \dots, m\}$ and α, β run over $\{m+1, \dots, n\}$.

In (2.1) the coefficients h_{ij}^α are the components of the second fundamental (mixed) tensor h, symmetric with respect to i, j. Here $h_{ij} = e_\alpha h_{ij}^\alpha$ can be introduced as the components of the vector valued second fundamental tensor with values in the normal vector subspace $T_x^\perp M^m$ of M^m in $N^n(c)$ at arbitrary point $x \in M^m$. The structure equations give now

$$d\omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j, \quad d\omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta + \Omega_\alpha^\beta,$$

where

$$\Omega_i^j = \Omega_{ij} = -\langle h^*_{i[k}, h^*_{l]j}\rangle \omega^k \wedge \omega^l, \tag{2.2}$$

are the curvature 2-forms of the Levi-Civita connection ∇ (for the cases when $c \neq 0$ here $h^*_{ij} = h_{ij} - xc\delta_{ij}$ are the components of the outer (i.e. with respect to E^{n+1} or E_1^{n+1}) vector valued second fundamental form), but

$$\Omega_\alpha^\beta = \Omega^{\alpha\beta} = -\sum_i h^\alpha_{i[k} h^\beta_{l]i} \omega^k \wedge \omega^l \tag{2.3}$$

are the curvature 2-forms of the normal connection $\nabla^\perp$. (Note that (2.2) and (2.3) are the famous Gauss and Ricci equations, respectively; see [KN], [St].)

If we use exterior differentiation and then Cartan's lemma in (2.1), the result is

$$\bar\nabla h^\alpha_{ij} = h^\alpha_{ijk}\omega^k, \quad h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.4}$$

where

$$\bar\nabla h^\alpha_{ij} = dh^\alpha_{ij} - h^\alpha_{kj}\omega_i^k - h^\alpha_{ik}\omega_j^k + h^\beta_{ij}\omega_\beta^\alpha \tag{2.5}$$

are the components of the covariant differential $\bar\nabla h$ of h with respect to the van der Waerden-Bortolotti connection $\bar\nabla = \nabla \oplus \nabla^\perp$. This together with (2.1) shows that h^α_{ijk} are symmetric with respect to all its lower indices. (If we denote in (2.4) $h^\alpha_{ijk} = \bar\nabla_k h^\alpha_{ij}$ then $h^\alpha_{ijk} = h^\alpha_{ikj}$ are equivalent to the famous Peterson-Mainardi-Codazzi equations $\bar\nabla_k h^\alpha_{ij} = \bar\nabla_j h^\alpha_{ik}$, see [KN]; about the role of Peterson see [L9], [Ph], [MC].)

For h_{ij} and h^*_{ij} a straightforward computation shows that

$$\nabla h_{ij} + \sum_{k=1}^m e_k \langle h_{ij}, h_{kl}\rangle \omega^l = \nabla h^*_{ij} + \sum_{k=1}^m e_k \langle h^*_{ij}, h^*_{kl}\rangle \omega^l = h_{ijk}\omega^k, \tag{2.6}$$

where $h_{ijk} = e_\alpha h^\alpha_{ijk}$, $\nabla h_{ij} = dh_{ij} - h_{kj}\omega_i^k - h_{ik}\omega_j^k$, and the same for h^*_{ij}.

A submanifold M^m in $N^n(c)$ is *semiparallel* (see Introduction) if $\bar\Omega \circ h = 0$, where $\bar\Omega$ is the curvature 2-form operator of the van der Waerden-Bortolotti connection $\bar\nabla$. This condition in a more explicit form is

$$\sum_p (\Omega_{ip} h^\alpha_{pj} + \Omega_{jp} h^\alpha_{ip}) - \sum_\beta \Omega^{\alpha\beta} h^\beta_{ij} = 0. \tag{2.7}$$

Let the normal connection $\nabla^{\perp}$ of M^m in $N^n(c)$ be flat, i.e. let all $\Omega^{\alpha\beta} = 0$. Then from (2.3) it follows that h_{ij} is diagonalizable by a suitable orthogonal transformation of $e_1, \dots, e_m$ in the tangent vector space T_xM^m at each point $x \in M^m$. After that $h_{ij} = k_i\delta_{ij}$, where $k_1, \dots, k_m$ are the *principal curvature vectors* of such a normally flat M^m in $N^n(c)$. They can be considered as some generalizations of the classical principal curvatures of a surface in E^3 (or of a hypersurface in E^n, see [KN]). Now (2.2) reduces to

$$\Omega_{ij} = -\langle k_i^*, k_j^* \rangle \omega^i \wedge \omega^j, \tag{2.8}$$

where $k_i^* = k_i - xc$ are the *outer principal curvature vectors* (so that $h_{ij}^* = k_i^*\delta_{ij}$). From the derivation formulae, due to (2.1),

$$dx = e_i\omega^i, \quad de_i = e_j\omega_i^j + k_i^*\omega^i. \tag{2.9}$$

For the semiparallel normally flat submanifold M^m in $N^n(c)$ the condition (2.7) reduces to

$$(k_i^* - k_j^*)\langle k_i^*, k_j^* \rangle = 0 \Longleftrightarrow (k_i - k_j)(\langle k_i, k_j \rangle + c) = 0. \tag{2.10}$$

This gives the following statement.

Lemma 2.1 (see [L6, Section 12]). *A normally flat submanifold M^m in $N^n(c)$ is semiparallel if and only if its every two outer principal curvature vectors are either equal or orthogonal. Here in the case of equality also the corresponding principal curvature vectors are equal, and vice versa.*

After using in (2.6) the outer principal curvature vectors the result is

$$dk_i^* = -\sum_{j=1}^{m} e_j \langle k_i^*, k_j^* \rangle \omega^j + K_i\omega^i + \sum_{j \neq i} L_{ij}\omega^j, \tag{2.11}$$

$$(k_i^* - k_j^*)\omega_i^j = L_{ij}\omega^i + L_{ji}\omega^j + \sum_{l \neq i}^{l \neq j} E_{ijl}\omega^l, \quad i \neq j, \tag{2.12}$$

where $K_i = h_{iii}$, $L_{ij} = h_{iij}$ $(i \neq j)$, $E_{ijl} = h_{ijk}$ (i, j, l have three distinct values and there is symmetry with respect to them) are some vectors normal to M^m and tangent to $N^n(c)$ in E^{n+1} (if $c > 0$) or in E_1^{n+1} (if $c < 0$) and the summation is denoted only by the sign $\sum$ with necessary hints.

Note that due to (2.6) in (2.11) and (2.12), k_i^* and k_j^* can be replaced by k_i and k_j, respectively.

Let there be exactly $r+1$ distinct vectors $k_{(0)}, k_{(1)}, \dots, k_{(r)}$ among the principal curvature vectors and let $k_{(\rho)}$ correspond to the directions of the tangent basic vectors e_{i_ρ}, where $\rho = 0, 1, \dots, r$.

From (2.11) it follows immediately that $K_{i_\rho} = K_{(\rho)}$, $L_{i_\rho j} = L_{(\rho)j}$.

Lemma 2.2. *In* (2.12) $E_{ijl} = 0$, $L_{(\rho)j_\rho} = 0$, *and* (2.12) *reduces to*

$$L_{(\rho)j_\tau} = \lambda_{(\rho)j_\tau}(k_{(\rho)} - k_{(\tau)}), \quad \omega^{j_\tau}_{i_\rho} = \lambda_{(\rho)j_\tau}\omega^{i_\rho} - \lambda_{(\tau)i_\rho}\omega^{j_\tau}, \tag{2.13}$$

where $\rho \neq \tau$.

Proof: Take in (2.12) $i = i_\rho$, $j = j_\rho$, $i_\rho \neq j_\rho$. Then

$$0 = L_{(\rho)j_\rho}\omega^{i_\rho} + L_{(\rho)i_\rho}\omega^{j_\rho} + \sum_{l\neq i_\rho}^{l\neq j_\rho} E_{i_\rho j_\rho l}\omega^l,$$

hence $L_{(\rho)j_\rho} = 0$ and $E_{i_\rho j_\rho l} = 0$. Due to symmetry, E_{ijl} is zero if two of i, j, l lead to the same $k_{(\rho)}$. It follows that if $r = 0$ or $r = 1$, then all E_{ijl} are zero. For $r > 1$ consider $E_{i_\rho j_\tau l_\varphi}$ with three distinct ρ, τ, φ. From (2.12) it follows, due to $k_{(\rho)} - k_{(\tau)} = k^*_{(\rho)} - k^*_{(\tau)} \neq 0$, that $\omega^{j_\tau}_{i_\rho}$ is a linear combination of ω^{i_ρ}, ω^{j_τ} and all ω^{l_φ}. Substituting this back into (2.12) one obtains that $E_{i_\rho j_\tau l_\varphi}$ is collinear to $k^*_{(\rho)} - k^*_{(\tau)}$ and, due to symmetry, also collinear to $k^*_{(\rho)} - k^*_{(\varphi)}$ and $k^*_{(\tau)} - k^*_{\varphi}$, thus there exist some functions λ, μ, and ν so that $E_{i_\rho j_\tau l_\varphi}$ is equal to

$$\lambda(k^*_{(\rho)} - k^*_{\tau)}) = \mu(k^*_{(\rho)} - k^*_{(\varphi)}) = \nu(k^*_{(\tau)} - k^*_{\varphi}).$$

Since $k^*_{(\rho)}$, $k^*_{(\tau)}$, and $k^*_{(\varphi)}$ are mutually orthogonal (see Lemma 2.1), from here, after scalar multiplication,

$$\lambda(k^*_{(\tau)})^2 = \mu(k^*_{(\varphi)})^2 = \nu(k^*_{(\rho)})^2 = 0,$$

where $(k^*_{(\rho)})^2$ is a short notation for the scalar square $\langle k^*_{(\rho)}, k^*_{(\rho)}\rangle$ etc. Here among $k^*_{(\rho)}$, $k^*_{(\tau)}$, and $k^*_{(\varphi)}$ only one can be zero and, if $c < 0$, only one non-zero with zero scalar square. (Indeed, if $(k^*_i)^2 = (k_i)^2 + c = 0$ and $(k^*_j)^2 = (k_j)^2 + c = 0$, then, due to (2.16), $(k_i - k_j)^2 = -c - 2(-c) - c = 0$ and so $k_i = k_j$.) Therefore at least one of the scalar squares above is non-zero and thus $E_{i_\rho j_\tau l_\varphi} = 0$. As a result, all $E_{ijk} = 0$.

The same substitution gives also all equalities in (2.13). This finishes the proof. □

Note that for the case $c = 0$ this deduction is made previously in [L3].

Now (2.11) reduces to

$$dk^*_{(\rho)} = -(k^*_{(\rho)})^2 \sum_{j_\rho} e_{j_\rho}\omega^{j_\rho} + K_{(\rho)}\omega^{i_\rho} + \sum_{\tau\neq\rho, j_\tau} (k^*_{(\rho)} - k^*_{(\tau)})\lambda_{(\rho)j_\tau}\omega^{j_\tau}. \tag{2.14}$$

Since

$$dk^*_{(\rho)} + (k^*_{(\rho)})^2 \sum_{j_\rho} e_{j_\rho}\omega^{j_\rho} = dk_{(\rho)} + (k_{(\rho)})^2 \sum_{j_\rho} e_{j_\rho}\omega^{j_\rho} - c \sum_{\tau\neq\rho, j_\tau} e_{j_\tau}\omega^{j_\tau},$$

the relation (2.14) is equivalent to

$$\begin{aligned}(2.15)\quad & dk_{(\rho)} = \\ & = -(k_{(\rho)})^2 \sum_{j_\rho} e_{j_\rho}\omega^{j_\rho} + K_{(\rho)}\omega^{i_\rho} + \sum_{\tau\neq\rho, j_\tau} [(k_{(\rho)} - k_{(\tau)})\lambda_{(\rho)j_\tau} + ce_{j_\tau}]\omega^{j_\tau}.\end{aligned}$$

Due to (2.10) and Lemma 2.1 for $\rho \neq \tau$

$$\langle k^*_{(\rho)}, k^*_{(\tau)}\rangle = 0 \Longleftrightarrow \langle k_{(\rho)}, k_{(\tau)}\rangle = -c. \tag{2.16}$$

If we differentiate here using (2.14) the result is

$$\langle K_{(\rho)}, k_{(\tau)}\rangle\omega^{i_\rho} - \sum_{j_\rho}(k^*_{(\rho)})^2\lambda_{(\tau)j_\rho}\omega^{j_\rho} = 0 \quad (\rho \neq \tau).$$

If $k_{(\rho)}$ is simple, then

$$\langle K_{(\rho)}, k_{(\tau)}\rangle = \lambda_{(\tau)i_\rho}(k^*_{(\rho)})^2 \quad (\rho \neq \tau). \tag{2.17}$$

If $k_{(\rho)}$ is nonsimple (i.e. of multiplicity $m_\rho > 1$), then (2.14) applied for two different values of i_ρ gives that $K_{(\rho)} = 0$, and thus

$$(k^*_{(\rho)})^2\lambda_{(\tau)i_\rho} = 0 \quad (\rho \neq \tau). \tag{2.18}$$

Lemma 2.3. *If $c < 0$, i.e. M^m is a normally flat semiparallel submanifold in $H^n(c) \subset E^{n+1}_1$, then there can be exactly one value $\rho = \rho_0$, for which $(k^*_{(\rho_0)})^2 = 0$. For every $\rho \neq \rho_0$ there is $(k^*_{(\rho)})^2 > 0$. Consequently, $\lambda_{(\tau)j_\rho} = 0$ if $m_\rho > 1$ and $\tau \neq \rho \neq \rho_0$, but $\lambda_{(\tau)j_{\rho_0}}$ $(\tau \neq \rho_0)$ can be non-zero.*

Proof: The first assertion is established already in the proof of Lemma 2.2.

If $\rho \neq \rho_0$ then $(k^*_{(\rho)})^2 > 0$. Indeed, $0 < (k_{(\rho_0)} - k_{(\rho)})^2 = -c - 2(-c) + (k_{(\rho)})^2 = (k^*_{(\rho)})^2$.

The last assertion follows now immediately from (2.18). □

2.1 Normally flat semiparallel M^m of type (p, q, s). A normally flat semiparallel submanifold M^m in $N^n(c)$ is said to be *of type* (p, q, s) if

(1) there are p different non-zero nonsimple principal curvature vectors $k_{(1)}, \dots, k_{(p)}$ with multiplicities $m_1, \dots, m_p$, correspondingly,

(2) the following $k_{(p+1)}, \dots, k_{(p+q)}$ are q different non-zero simple principal curvature vectors and

(3) there exists a zero principal curvature vector $k_{(0)} = 0$ of multiplicity s; here, of course, $s = m - m^* - q$, where $m^* = m_1 + \dots + m_p$.

Let us further specify the scopes of the indices as follows. Let ρ, τ, ... be used only as running over $\{1, \dots, p\}$, so that all $k_{(\rho)}$ are from now on non-zero nonsimple. Therefore $K_{(\rho)} = 0$ for all values of ρ. Moreover, let $a, b, \dots$ run over $\{m^* + 1, \dots, m^* + q\}$, so that all $k_{(a)}$ are non-zero simple and therefore can be denoted further simply by k_a. Finally, let $i_0, j_0, \dots$ run through the remaining set $\{m^* + q + 1, \dots, m\}$ of s values. Here, of course, $k^*_{(\rho)} = k_{(\rho)} + cx$ are also non-zero nonsimple, $k^*_a = k_a + cx$ are non-zero simple.

If $c \geq 0$ then

$$(k^*_{(\rho)})^2 = (k_{(\rho)})^2 + c > 0, \quad (k^*_a)^2 = (k_a)^2 + c > 0;$$

thus (2.18) implies $\lambda_{ai_\rho} = \lambda_{(\tau)i_\rho} = 0$ $(\tau \neq \rho)$. Moreover, due to (2.16) $k_{(0)} = 0$ is possible only if $c = 0$ and then due to (2.11) $K_{(0)} = 0$ and $L_{(0)j} = 0$, therefore (2.13) gives $\lambda_{(0)j_\tau} = \lambda_{(0)a} = 0$.

Here different kind of space forms $N^n(c)$ must be considered separately.

2.2 Let $N^n(c)$ be a Euclidean space E^n. Then (2.15) gives

(2.19)
$$dk_{(\rho)} = -(k_{(\rho)})^2 \sum_{j_\rho} e_{j_\rho}\omega^{j_\rho} + \sum_a (k_{(\rho)} - k_a)\lambda_{(\rho)a}\omega^a + k_{(\rho)} \sum_{i_0} \lambda_{(\rho)i_0}\omega^{i_0},$$

$$dk_a = (-(k_a)^2 e_a + K_a)\omega^a + \sum_{b \neq a} (k_a - k_b)\lambda_{ab}\omega^b + k_a \sum_{i_0} \lambda_{ai_0}\omega^{i_0}, \tag{2.20}$$

but the last relations (2.13) reduce to

$$\omega^{j_\tau}_{i_\rho} = 0 \ (\rho \neq \tau), \quad \omega^a_{i_\rho} = \lambda_{(\rho)a}\omega^{i_\rho}, \quad \omega^{i_0}_{i_\rho} = \lambda_{(\rho)i_0}\omega^{i_\rho}, \tag{2.21}$$

$$\omega^b_a = \lambda_{ab}\omega^a - \lambda_{ba}\omega^b \ (a \neq b) \quad \omega^{i_0}_a = \lambda_{ai_0}\omega^a. \tag{2.22}$$

Remark that these results have been deduced in [L4] and [L5] and are used there to obtain a full geometric description of normally flat semiparallel submanifolds M^m in E^n as special warped products whose fibres are the products of spheres.

Here these results are useful for further characterization of inner geometry of such M^m in E^n.

2.3 Let $N^n(c)$ be a non-Euclidean space. Then $c \neq 0$, and due to (2.16) $k_{(0)} = 0$ is impossible, so that $s = 0$ and thus there are no indices $i_0, j_0, \dots$.

More precisely, if $N^n(c)$ is a sphere $S^n(c)$, then (2.15) gives

$$dk_{(\rho)} = -(k_{(\rho)})^2 \sum_{j_\rho} e_{j_\rho} \omega^{j_\rho} + \sum_a (k_{(\rho)} - k_a)\lambda_{(\rho)a}\omega^a + c(dx - \sum_{i_\rho} e_{i\rho}\omega^{i_\rho}), \tag{2.23}$$

$$dk_a = (-(k_a)^2 e_a + K_a)\omega^a + \sum_{b \neq a}(k_a - k_b)\lambda_{ab}\omega^b + c(dx - e_a\omega^a), \tag{2.24}$$

but the last relations (2.13) reduce to

$$\omega_{i_\rho}^{j_\tau} = 0 \ \ (\rho \neq \tau), \quad \omega_{i_\rho}^a = \lambda_{(\rho)a}\omega^{i_\rho}, \quad \omega_a^b = \lambda_{ab}\omega^a - \lambda_{ba}\omega^b \ \ (a \neq b). \tag{2.25}$$

Let $N^n(c)$ be a hyperbolic space $H^n(c)$, i.e. $c < 0$ and thus Lemma 2.3 holds: it is possible that $(k^*_{(\rho_0)})^2 = 0$ for one value ρ_0 from $\{1, \dots, r\}$, and therefore (2.18) makes it possible for $\lambda_{(\tau)i_{\rho_0}}$ $(\rho_0 \neq \tau)$ to be non-zero; note that here τ is used in its former meaning, like in (2.18), i.e. running over $\{1, \dots, r\}$.

Then the last relations (2.13) reduce to

$$\omega_{i_\rho}^{j_\tau} = 0 \ \ (\rho, \tau, \rho_0 - \text{ three different}), \quad \omega_{i_\rho}^a = \lambda_{(\rho)a}\omega^{i_\rho}, \tag{2.26}$$

$$\omega_{i_{\rho_0}}^{j_\tau} = -\lambda_{(\tau)i_{\rho_0}}\omega^{j_\tau} \ \ (\rho_0 \neq \tau), \quad \omega_{i_{\rho_0}}^a = \lambda_{(\rho_0)a}\omega^{i_{\rho_0}} - \lambda_{ai_{\rho_0}}\omega^a, \tag{2.27}$$

and also the last relation (2.25) holds:

$$\omega_a^b = \lambda_{ab}\omega^a - \lambda_{ba}\omega^b \ \ (a \neq b). \tag{2.28}$$

If $1 \leq \rho_0 \leq p$, then for $\rho \neq \rho_0$ from (2.14) it follows, due to $K_{(\rho)} = K_{(\rho_0)} = 0$ and (2.18), that

$$\begin{aligned} dk_{(\rho)} = -(k_{(\rho)})^2 \sum_{j_\rho} e_{j_\rho}\omega^{j_\rho} + \sum_{j_{\rho_0}} (k_{(\rho)} - k_{(\rho_0)})\lambda_{(\rho)j_{\rho_0}}\omega^{j_{\rho_0}} \\ + \sum_a (k_{(\rho)} - k_a)\lambda_{(\rho)a}\omega^a + c(dx - \sum_{j_\rho} e_{j_\rho}\omega^{j_\rho}), \end{aligned} \tag{2.29}$$

$$dk_{(\rho_0)} = \sum_a (k_{(\rho_0)} - k_a)\lambda_{(\rho_0)a}\omega^a + c(dx - \sum_{i_{\rho_0}} e_{i_{\rho_0}}\omega^{i_{\rho_0}}), \tag{2.30}$$

$$(2.31)\quad dk_a = (-(k_a)^2 e_a + K_a)\omega^a + \sum_{i_{\rho_0}} (k_a - k(\rho_0))\lambda_{ai_{\rho_0}}\omega^{i_{\rho_0}} + \sum_{b\neq a}(k_a - k_b)\lambda_{ab}\omega^b + c(dx - e_a\omega^a).$$

But if $p+1 \le \rho_0 \le q$, then it is suitable to denote ρ_0 by a_0, and there hold (2.23) and (2.24), the latter for $a \neq a_0$, and

$$(2.32)\qquad dk_{a_0} = K_{a_0}\omega^{a_0} + \sum_{b\neq a_0}(k_{a_0} - k_b)\lambda_{a_0 b}\omega^b + c(dx - e_{a_0}\omega^{a_0}).$$

Remark that the normally flat semiparallel M^m in $N^n(c)$ with $s \neq 0$ are investigated in [DN], where most of the statements are made by referring to the papers published previously, without explicit deductions. The results obtained in this subsection 2.3 specify the exposition in [DN], but their proper task here is to be basic for the characterization of inner geometry of the normally flat semiparallel submanifolds M^m in $N^n(c)$ as the isometrically immersed semisymmetric Riemannian manifolds.

3. Characterization of inner geometry

Let M^m be a normally flat semiparallel submanifold in a space form $N^n(c)$. If $c = 0$, i.e. $N^n(c)$ is E^n, it is suitable to join the scopes of indices $a, b, \dots$ and $i_0, j_0, \dots$ and introduce indices $u, v, \dots$ which run over $\{m^*+1, \dots, m\}$. Then the last two groups of relations in (2.21) can be joined into

$$(3.1)\qquad \omega^u_{i_\rho} = \lambda_{(\rho)u}\omega^{i_\rho}.$$

If $c \neq 0$ then the scope of indices $i_0, j_0, \dots$ is empty and (3.1) coincide with the corresponding relations in (2.25) and (2.26).

If we use exterior differentiation, then (3.1) give, due to the first group of relations in (2.21), that

$$(3.2)\quad \sum_u \lambda_{(\rho)u}\lambda_{(\tau)u} = 0,\ \ (\rho \neq \tau),\quad d\lambda_{(\rho)u} = \sum_v (\lambda_{(\rho)v}\omega^v_u + \lambda_{(\rho)u}\lambda_{(\rho)v}\omega^v).$$

Fix a value ρ and consider the distribution determined by the differential system $\omega^{i_\tau} = 0$ for all values of $\tau \neq \rho$ and i_τ. Here

$$d\omega^{i_\tau} = \omega^{j_\tau} \wedge \omega^{i_\tau}_{j_\tau} + \omega^u \wedge (-\lambda_{(\tau)u}\omega^{i_\tau}),$$

therefore this system is totally integrable and thus the considered distribution is a foliation. Similarly, due to $d\omega^u = \omega^v \wedge \omega^u_v$, the distribution determined by the system $\omega^u = 0$ for all values of u is a foliation. Hence the intersection of both these foliations is also a foliation.

Consider its leaves. For each of them, due to (2.9)

$$(3.3)\quad dx = \sum_{i_\rho} e_{i_\rho}\omega^{i_\rho} \;\; (\rho- \text{ fixed}), \quad de_{i_\rho} = \sum_{j_\rho} e_{j_\rho}\omega^{j_\rho}_{i_\rho} + \omega^{i_\rho}(l_{(\rho)} + k_{(\rho)}) - xc\omega^{i_\rho},$$

where $l_{(\rho)} = \sum_u e_u\lambda_{(\rho)u}$. It is seen that each leaf is totally umbilical with the mean curvature vector $H_{(\rho)} = l_{(\rho)} + k_{(\rho)}$.

Now the following assertion can be used.

Proposition 3.1 (see [L6, Proposition 5.4])**.** *A totally umbilic submanifold with the mean curvature vector H in $N^n(c)$, if complete, coincides with an $N^m(c^*)$, embedded into $N^n(c)$, where $c^* = H^2 + c$ and $dH = -H^2dx$, thus $dH^2 = 0$.*

If $c \geq 0$ then for the leaf considered above there holds $c^*_{(\rho)} = H^2_{(\rho)} + c \geq 0$ and $= 0$ is impossible here because this would lead to $c = 0$ and $H_{(\rho)} = 0$, thus to a totally geodesic submanifold in some Euclidean space, so to $k_{(\rho)} = 0$, which is excluded, because all $k_{(\rho)}$ are here nonsimple non-zero.

Therefore $c^*_{(\rho)} > 0$ and thus this leaf is a sphere $S^{m_\rho}(c^*_{(\rho)})$ in $N^n(c)$ (i.e. in E^n if $c = 0$, or $S^n(c) \subset E^{n+1}$ if $c > 0$) with the radius $r_\rho = (c^*_{(\rho)})^{-2}$ and normal vector $H^*_{(\rho)} = l_{(\rho)} + k^*_{(\rho)}$. The $(m_\rho{+}1)$-plane of this sphere is spanned by its arbitrary point x, its tangent vectors e_{i_ρ} and normal vector $H^*_{(\rho)}$ at this point.

It is easy to see that these vectors are mutually orthogonal to the same vectors for a subindex τ, different from ρ. Indeed, all e_i are orthogonal to e_u and $k^*_{(\tau)}$, thus to $l_{(\tau)}$ and $H^*_{(\tau)}$. Moreover, $\langle e_{i_\rho}, e_{j_\tau}\rangle = 0$ and $\langle H^*_{(\rho)}, H^*_{(\tau)}\rangle = 0$ for $\rho \neq \tau$, the latter due to (2.16) and (3.2), which implies $\langle l_{(\rho)}, l_{(\tau)}\rangle = 0$.

Hence the $(m_\rho{+}1)$- and $(m_\tau{+}1)$-planes of $S^{m_\rho}(c^*_{(\rho)})$ and $S^{m_\tau}(c^*_{(\tau)})$ are orthogonal for $\rho \neq \tau$. This gives that the leaves of the foliation determined by $\omega^u = 0$ are product submanifolds $S^{m_1}(c^*_{(1)}) \times \cdots \times S^{m_p}(c^*_{(p)})$.

For the radius $r_\rho = (l^2_{(\rho)} + k^2_{(\rho)} + c)^{-2}$ of one of these spheres from (3.2) and (2.19) or (2.23) it follows that $dr_\rho = r_{\rho u}\omega^u$, where $r_{\rho u} = -r_\rho\lambda_{(\rho)u}$. The same (3.2) gives now

$$(3.4)\qquad dr_{\rho u} = r_{\rho v}\omega^v_u.$$

The orthogonal complement distribution of the foliation above with leaves $S^{m_1}(c^*_{(1)}) \times \cdots \times S^{m_p}(c^*_{(p)})$ is determined by $\omega^{i_\rho} = 0$ with arbitrary ρ and i_ρ.

Due to (2.21) or (2.25) this distribution is a foliation and for each of its leaves there hold

$$\Omega_u^v = \sum_{\rho, i_\rho} \omega_u^{i_\rho} \wedge \omega_{i_\rho}^v = -\sum_{\rho, i_\rho} \lambda_{(\rho)u} \omega^{i_\rho} \wedge \lambda_{(\rho)v} \omega^{i_\rho} = 0,$$

therefore all these leaves are locally Euclidean and on each of them there exists a local orthonormal frame field so that the integral lines of the basic vector fields are geodesic coordinate lines of some affine (actually Descartes') coordinates $x^{m^*+1}, \dots, x^m$. This frame field is parallel with respect to ∇ and accordingly for it $\omega_u^v = 0$, if we use the same notation as before. Moreover, $\omega^u = dx^u$.

Now from (3.4) it follows that $dr_{\rho u} = 0$, thus $r_{\rho u} = c_{\rho u} = \text{const}$, and hence $r_\rho = \sum_u c_{\rho u} x^u + c_\rho$.

For every two different values of ρ and τ there is $\langle l_{(\rho)}, l_{(\tau)} \rangle = 0$ (see above). Among p mutually orthogonal vectors $l_{(1)}, \dots, l_{(p)}$ in an $(m - m^*)$-dimensional vector subspace there can be some $\bar{p}$ non-zero vectors, where $\bar{p} \leq \min\{m - m^*, p\}$. By renumbering, if necessary, they can be made the first $\bar{p}$ vectors $l_{(1)}, \dots, l_{(\bar{p})}$, so that $l_{(\bar{\rho})} \neq 0$ for $\bar{\rho}$ running over $\{1, \dots, \bar{p}\}$; the other $l_{(\rho')}$ are then zero vectors, where ρ' runs over the remaining values $\bar{p}+1, \dots, p$. For the latter $c_{\rho' u} = 0$ so that $r_{\rho'} = c_{\rho'}$ are some constants.

For the first ones the level hypersurfaces of the function $r_{\bar{\rho}}$ in a considered locally Euclidean leaf are some hyperplanes $r_{\bar{\rho}} = \text{const}$, whose normal directions are determined by $-r_{\bar{\rho}} l_{(\bar{\rho})} = \sum_u c_{\bar{\rho} u} \frac{\partial}{\partial x^u}$. Now the Descartes' coordinates $\bar{x}^{m^*+1}, \dots, \bar{x}^m$ can be introduced on each leaf so that the first $\bar{p}$ of the $\bar{x}^{\bar{\rho}}$-axis have these normal directions of $l_{(\bar{\rho})} \neq 0$. Then every $r_{\bar{\rho}}$ is a linear function of $\bar{x}^{\bar{\rho}}$. This, together with the fact that $(m_{\bar{\rho}}+1)$-planes of spheres $S^{m_{\bar{\rho}}}(c^*_{\bar{\rho}})$ are parallel, shows that these spheres along the $\bar{x}^{\bar{\rho}}$-axis generate in the inner geometry of M^m a Riemannian cone $C^{m_{\bar{\rho}}+1}$, which is a warped product $\mathbf{R} \times_{r_{\bar{\rho}}} S^{m_{\bar{\rho}}}(1)$ (see [BO], [DN]).

The different cones $C^{m_{\bar{\rho}}+1}$ and $C^{m_{\bar{\tau}}+1}$ lie in totally orthogonal $(m_{\bar{\rho}}+2)$- and $(m_{\bar{\tau}}+2)$-dimensional submanifolds of M^m, which are totally orthogonal to $(m_{\rho'}+1)$-dimensional parallel mutually orthogonal submanifolds of the spheres $S^{m_{\rho'}}(c^*_{\rho'})$. All this shows that M^m is intrinsically a Riemannian direct product of Riemannian cones $C^{m_1+1}, \dots, C^{m_{\bar{p}}+1}$, spheres $S^{m_{\bar{p}+1}}(c^*_{\bar{p}+1}), \dots, S^{m_p}(c^*_p)$, and perhaps a totally geodesic $(m - m^* - \bar{p})$-dimensional submanifold (i.e. a $(m - m^* - \bar{p})$-plane in inner geometry). For all $m_{\bar{\rho}} > 2$ the cones $C^{m_{\bar{\rho}}+1}$ are elliptic cones (according to Szabó; see Theorem 1.1), for $m_{\bar{\rho}} = 2$ they are three-dimensional Riemannian manifolds of conullity two, which are of planar type as shows comparison of (3.1) with (1.4), where a_u is now $-\lambda_{(\bar{\rho})u}$; note that $i_{\bar{\rho}}$ takes here only two values.

As a result the following theorem can be formulated.

Theorem 3.2. *A normally flat semiparallel submanifold M^m in $N^n(c)$ with $c \geq 0$ (i.e. in E^n or $S^n(c)$) is intrinsically a semisymmetric Riemannian manifold which is, in general, a direct product of symmetric spaces (namely some spheres or their open parts and a locally Euclidean space), and some Riemannian cones, which are either elliptic cones (in the sense of Szabó* [Sz]*; see Theorem* 1.1 *above), or three-dimensional Riemannian manifolds of conullity two of the planar type (in the sense of* [BKV]*).*

Here, of course, in special cases only some types of factors will occur in the direct product.

If $N^n(c)$ is a hyperbolic space $H^n(c)$, i.e. if $c < 0$, the situation is the same as in Theorem 3.2 with the only difference that either instead of exactly one elliptic cone there is a hyperbolic or Euclidean cone, or instead of exactly one sphere there is a hyperbolic or Euclidean space.

This can be verified as above, but, because the ambient space is now E_1^{n+1}, among mutually orthogonal non-zero vectors $H^*_\rho = l_{(\rho)} + k^*_{(\rho)}$ $(\rho \in \{1, \dots, p\})$ there is exactly one with negative or zero scalar square; let it be for $\rho = 1$.

If in (3.3), with $k^*_{(\rho)}$ instead of $k_{(\rho)}$, the vector $l_{(1)} + k^*_{(1)} = H_{(1)} - xc = H^*_{(1)}$ has negative scalar square $c^*_{(1)} = H^2_{(1)} + c < 0$, then the leaf determined by $\omega^{i_\rho} = 0$, $\omega^a = 0$, where i_ρ and a run over all their values for $\rho \neq 1$, is a hyperbolic space $H^{m_1}(c^*_{(1)})$. Here $c^*_{(1)}$ can be constant on M^m, but if not, then $r_1 = (c^*_{(1)})^{-2}$ is a linear function of $\bar{x}^1$ (see above) and these leaves along the $\bar{x}^1$-axis generate in the inner geometry of M^m a Riemannian cone, a warped product $\mathbf{R} \times_{r_1} H^{m_1}(-1)$, which is a hyperbolic cone, in the sense of Szabó.

If in (3.3) for $\rho = 1$ the vector $l_{(1)} + k^*_{(1)}$ has zero scalar square $c^*_{(1)} = H^2_{(1)} + c = (l_{(1)})^2 + (k^*_{(1)})^2 = 0$, then the leaf determined by $\omega^{i_\rho} = 0$, $\omega^a = 0$ $(\rho \neq 1)$ is intrinsically a Euclidean space E^{m_1}.

From (2.14) it follows that on the leaf determined by $\omega^{i_\rho} = 0$ $(\rho \neq 1)$ there holds $dk^*_{(1)} = -(k^*_{(1)})^2 \sum_{j_1} e_{j_1} \omega^{j_1}$, because $k_{(1)}$ is nonsimple and therefore $K_{(1)} = 0$. It follows that $d(k^*_{(1)})^2 = 0$ and thus $(k^*_{(1)})^2$ =const on this leaf. But this gives that $(l_{(1)})^2 = -(k^*_{(1)})^2$ is also a constant on this leaf.

Consider the vector field $l_{(1)}$ on M^m and let t be the canonical parameter on the integral lines of this field. If we take the m_1-dimensional Euclidean leaves above along these lines then we obtain a Riemannian cone. Let x describe one of theses leaves. Then $x' = x + tl_{(1)}$ by $t =$ const describes another leaf. Here $dx' = d(x + tl_{(1)}) = (1 - t(l_{(1)})^2) \sum_{i_1} e_{i_1} \omega^{i_1}$. Thus, for the inner metric of the other leaf there holds $(dx')^2 = (1 - t(l_{(1)})^2)(dx)^2$. The result shows that, if

$m_1 > 2$, the Riemannian cone is a Euclidean cone, in the sense of Szabó (see [Sz], also [BKV]), but if $m_1 = 2$ it is a three-dimensional Riemannian manifold of conullity two, which is of planar type as follows if we compare (2.26) with (1.4).

All this can be summarized as follows.

Theorem 3.3. *A normally flat semiparallel submanifold M^m in a hyperbolic space $H^n(c)$ is intrinsically a semisymmetric Riemannian manifold, which is, in general, a direct product of symmetric spaces, some Riemannian cones, which are: the elliptic cones, one hyperbolic or Euclidean cone (in the sense of Szabó; see Theorem 1.1 above), or three-dimensional Riemannian manifolds of conullity two of the planar type (in the sense of [BKV]).*

In special cases only some types of factors will occur in the direct product.

Note that the statements of Theorems 3.2 and 3.3, in the parts concerning the elliptic, Euclidean or hyperbolic cones, are anticipated in [DN], Remark 3.3, but without the assertion about the uniqueness of hyperbolic or Euclidean cone and without any explicit deduction.

Note also that a Euclidean cone is generated by the horospheres intersecting orthogonally the parallel lines of a given pencil in $H^n(c)$ (parallel in the Lobachevski-Bolyai geometry; see e.g. [N], [MC]).

Let us conclude with the following remark. The investigations in [L7] and [L8] led to a

Conjecture. *If a semiparallel submanifold M^m in the Euclidean space E^n is intrinsically a Riemannian manifold of conullity two, a particular case of a semisymmetric Riemannian manifold, then this manifold can be only of planar type (in the sense of [K], [BKV]).*

This conjecture is confirmed up to now for $m = 3$ and arbitrary n (see [L7]), and for M^m generated by $(m-2)$-dimensional plane leaves in E^n (see [L8]). The last parts of Theorems 3.2 and 3.3 show that this conjecture can be extended to submanifolds in space forms and then confirmed for normally flat semiparallel submanifolds.

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