On the Dirichlet problem for functions of the first Baire class

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Abstract. Let $\mathcal{H}$ be a simplicial function space on a metric compact space X. Then the Choquet boundary $\operatorname{Ch} X$ of $\mathcal{H}$ is an F_σ-set if and only if given any bounded Baire-one function f on $\operatorname{Ch} X$ there is an $\mathcal{H}$-affine bounded Baire-one function h on X such that $h = f$ on $\operatorname{Ch} X$. This theorem yields an answer to a problem of F. Jellett from [8] in the case of a metrizable set X.

Keywords: weak Dirichlet problem, function space, Choquet simplexes, Baire-one functions

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1. Introduction

Let $\mathcal{H}$ be a *function space* on a compact metric space X. By this we mean a linear subspace of $\mathcal{C}(X)$ (the space of all real-valued continuous functions on X equipped with the sup-norm $||.||$) containing constant functions and separating points of X. Let $\mathcal{M}^1(X)$ denote the set of all probability Radon measures on X and ε_x the Dirac measure at $x \in X$. Let further $\mathcal{M}_x(\mathcal{H})$ be the set of all $\mathcal{H}$-*representing measures* for $x \in X$, i.e.

$$\mathcal{M}_x(\mathcal{H}) = \{\mu \in \mathcal{M}^1(X) : \mu(h) = h(x) \text{ for any } h \in \mathcal{H}\}.$$

A bounded Borel function f is called $\mathcal{H}$-*affine* if it satisfies $\mu(f) = f(x)$ for any $x \in X$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. The space of all $\mathcal{H}$-affine continuous functions will be denoted by $\mathcal{A}(\mathcal{H})$. *The Choquet boundary* $\operatorname{Ch} \mathrm{X}$ of $\mathcal{H}$ is defined as the set $\{x \in X : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$. The Choquet boundary is a G_δ-set and the Choquet representation theorem guarantees for any $x \in X$ the existence of a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $\mu(X \setminus \operatorname{Ch} \mathrm{X}) = 0$. We say that $(X, \mathcal{H})$ is *a simplicial space* if for any $x \in X$ there is a unique measure representing x carried by the Choquet boundary.

We introduce main examples of function spaces.

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Examples. 1. *Continuous functions.* Let X be a metric compact space. For $\mathcal{H} = \mathcal{C}(X)$ we have $\mathrm{Ch\,X} = X$ and $\mathcal{C}(X)$ is a simplicial space because there are no $\mathcal{H}$-representing measures except Dirac measures.

2. *Affine functions.* Let X be a metrizable convex compact subset of a Hausdorff locally convex space E and $\mathcal{H}$ the linear space $\mathcal{A}(X)$ of all continuous affine functions on X. In this case the Choquet boundary $\mathrm{Ch\,X}$ coincides with the set $\operatorname{ext} X$ of all extreme points of X. Then $(X, \mathcal{A}(X))$ is a simplicial space if and only if X is a Choquet simplex (for a definition of a Choquet simplex see e.g. [1] or [7]).

3. *Harmonic functions.* Let Ω be a bounded open subset of a Euclidean space $\mathbb{R}^n$, X the closure $\overline{\Omega}$ of Ω and $\mathcal{H}$ the linear space $H(\Omega)$ of all continuous functions on $\overline{\Omega}$ which are harmonic on Ω. We will study this example more deeply in Section 3.

A well-known theorem (cf. [11]) in the case of affine functions on a Choquet simplex X asserts that $\mathrm{Ch\,X}$ is closed if and only if any continuous function f on $\mathrm{Ch\,X}$ can be extended to an affine continuous function h on X. A similar result can be obtained for general function spaces. This paper answers the question (in the case of a metrizable space X) asked by F. Jellett in [8]. He posed a problem whether a similar assertion can be proved for F_σ-sets and functions of the first Baire class. In the sequel we prove a theorem which says that for a simplicial space $(X, \mathcal{H})$, the Choquet boundary is an F_σ-set if and only if any bounded function of the first Baire class on $\mathrm{Ch\,X}$ can be extended to a bounded $\mathcal{H}$-affine function h of the first Baire class on X.

2. Results

Let X be a metric space. We write $B^b(X)$ for the space of all bounded real-valued Borel functions on X. Let f be a real-valued function on X. Then the function f is of *the first Baire class* or *a Baire-one function* (written $f \in B_1(X)$) if f is a pointwise limit of a sequence $\{f_n\}$ of continuous functions on X. Let us denote the set of all bounded functions of the first Baire class on X by $B_1^b(X)$. Due to [10, Theorem 2.12], a function f is of the first Baire class on a compact metric space X if and only if for every nonempty closed set F and every couple $a < b$, the sets $\{x \in F : f(x) < a\}$ and $\{x \in F : f(x) > b\}$ are not simultaneously dense in F (*the* [D–P] *condition*). A set B is called *ambivalent* if it is both an F_σ and G_δ-set, or equivalently, if the characteristic function χ_B of the set B is in $B_1(X)$. Due to the [D–P] condition, a subset B of a metric compact space is ambivalent if and only if for every nonempty closed set F, the sets $F \cap B$ and $F \setminus B$ are not simultaneously dense in F (*the* [A] *condition*).

A metric space X is said to be *a Baire space* if and only if the intersection of each countable family of dense open sets in X is dense. A set $A \subset X$ is *residual* if its complement $X \setminus A$ is a set of the first category, i.e. $X \setminus A = \bigcup_{n=1}^{\infty} A_n$ where

A_n is a nowhere dense subset of X for every integer n. We will employ the fact that a G_δ-subspace F of a complete metric space X is a Baire space. Note also that a residual subset of a Baire space is dense. A suitable reference for details on Baire spaces is [6].

For a set B in a metric space X let us denote by $\operatorname{der}(B)$ the set of all accumulation points of B.

Theorem. *Let $(X, \mathcal{H})$ be a simplicial space. Then the following assertions are equivalent:*

(i) $\operatorname{Ch} \mathrm{X}$ *is an F_σ-set,*
(ii) *given $f \in B_1^b(\operatorname{Ch} \mathrm{X})$ there exists an $\mathcal{H}$-affine function $h \in B_1^b(X)$ such that $h = f$ on $\operatorname{Ch} \mathrm{X}$.*

In what follows we assume that $(X, \mathcal{H})$ is a simplicial space. Let us denote by μ_x the unique probability measure on X representing a point x supported by $\operatorname{Ch} \mathrm{X}$. We will consider the operator $T \colon B^b(X) \to B^b(X)$ defined by $Tf(x) = \int_X f \, d\mu_x$ for $f \in B^b(X)$. According to [11, Proposition 9.10], T maps $\mathcal{C}(X)$ into $B_1^b(X)$. Thus T maps a bounded Borel function f on X onto a bounded Borel function Tf. Let us notice that $Tf(x) = f(x)$ for $x \in \operatorname{Ch} \mathrm{X}$.

Let B be a Borel set, $\operatorname{Ch} \mathrm{X} \subset B \subset X$ (in particular $B = \operatorname{Ch} \mathrm{X}$). Given a bounded Borel function g on B, define Tg as Tf, where a bounded Borel function f on X is defined by $f = g$ on B and $f = 0$ elsewhere. Since any measure μ_x is carried by the Choquet boundary we see that $Tg(x) = Tf(x) = \mu_x(f) = \mu_x(g)$ for every point $x \in X$.

Lemma 1. *Let $f \in B^b(X)$. Then Tf is an $\mathcal{H}$-affine function on X.*

Proof: Given $y \in X$ and $\lambda \in \mathcal{M}_y(\mathcal{H})$, define a linear functional μ on $\mathcal{C}(X)$ by the formula $\mu(g) = \int_X Tg \, d\lambda$, $g \in \mathcal{C}(X)$. Then μ is obviously a probability measure representing the point y. The equality

$$\mu(\operatorname{Ch} \mathrm{X}) = \int_X \mu_x(\operatorname{Ch} \mathrm{X}) \, d\lambda = \int_X 1 \, d\lambda = 1$$

now implies that μ is supported by $\operatorname{Ch} \mathrm{X}$. Therefore $\mu = \mu_y$ because $(X, \mathcal{H})$ is a simplicial space. Thus we obtain

$$\lambda(Tf) = \int_X \mu_x(f) \, d\lambda = \mu(f) = \mu_y(f) = Tf(y)$$

and the proof is complete. □

Lemma 2. *Suppose that $f \in B^b(\mathrm{Ch\,X})$ and $F \in B^b(X)$ is an $\mathcal{H}$-affine function such that $F = f$ on $\mathrm{Ch\,X}$. Then $F = Tf$.*

PROOF: Pick $y \in X$. Since F is $\mathcal{H}$-affine, we have

$$F(y) = \int_{\mathrm{Ch\,X}} F(x)\, d\mu_y(x) = \int_{\mathrm{Ch\,X}} f(x)\, d\mu_y(x) = Tf(y).$$

□

Lemma 3. *Let $\mathrm{Ch\,X}$ be an F_σ-set and $f \in B_1^b(\mathrm{Ch\,X})$. Then Tf is an $\mathcal{H}$-affine function of the first Baire class.*

PROOF: Due to the assumption we write $\mathrm{Ch\,X} = \bigcup_{n=1}^\infty F_n$ where F_n are compact sets such that $F_1 \subset F_2 \subset \cdots \subset \mathrm{Ch\,X}$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of continuous functions on $\mathrm{Ch\,X}$ converging pointwise to f. We may assume that $\|f\|$, $\|f_n\|$ are bounded by a positive number M. Since $(X, \mathcal{H})$ is a simplicial space, according to [3, Corollary 3.6], there exist $\mathcal{H}$-affine continuous functions h_n on X such that $h_n = f_n$ on $\mathrm{Ch\,X}$ and $\|h_n\| = \|f_n\|$.

The proof will be completed by showing that $h_n(x) \to Tf(x)$ for all $x \in X$. For fixed $x \in X$ and ε positive choose an integer n_0 such that $\int_X |f - f_n|\, d\mu_x < \varepsilon$ and $\mu_x(F_n) > 1 - \varepsilon$ for all $n \geq n_0$. For such n we have

$$\begin{aligned} |Tf(x) - h_n(x)| &= \left| \int_X (f - h_n)\, d\mu_x \right| \\ &\leq \int_X |f - f_n|\, d\mu_x + \int_X |f_n - h_n|\, d\mu_x \\ &\leq \varepsilon + \int_{\mathrm{Ch\,X} \setminus F_{n_0}} 2M\, d\mu_x \leq \varepsilon(1 + 2M), \end{aligned}$$

which proves the lemma. □

We start the main part of the proof of the Theorem with the following lemma.

Lemma 4. *Let F be a metric compact space and G be a subset of F such that $\overline{G} = F = \overline{F \setminus G}$. Let $K \subset G$ be a closed subset of F. Then K is nowhere dense in G.*

PROOF: Since K is a closed set in F, it is a closed subset of G as well. Suppose that K is not nowhere dense in G. Find a nonempty open set $U \subset F$ such that $U \cap G \neq \emptyset$ and $U \cap G \subset K$. Since $F \setminus G$ is dense in F, we may find a point $x \in U \cap (F \setminus G)$. Due to density of G in F, there is a sequence $\{x_n\}$ of points of G such that $x = \lim_{n\to\infty} x_n$. Since $x \in U$ and U is open in F, we may assume that $x_n \in U \cap G$ for each integer n. Since $U \cap G \subset K$ and K is a closed set, $x \in K \subset G$. This contradiction concludes the proof. □

Lemma 5. *If* $\mathrm{Ch}\,\mathrm{X}$ *is not an* F_σ*-set, then there exists a function* $f \in B_1^b(X)$ *such that* $Tf \notin B_1^b(X)$.

Proof: Suppose that the Choquet boundary $\mathrm{Ch}\,\mathrm{X}$ of $\mathcal{H}$ is not an F_σ-set. Thus it is not an ambivalent set and according to condition [A] we can find a nonempty closed set F satisfying $F = \overline{F \cap \mathrm{Ch}\,\mathrm{X}} = \overline{F \setminus \mathrm{Ch}\,\mathrm{X}}$. Let B denote the set $\{x \in F \setminus \mathrm{Ch}\,\mathrm{X} : \mu_x(F) \geq \frac{1}{2}\}$. Suppose that B is not dense in F. Then there exists an open set $U \subset X$ satisfying $U \cap F \neq \emptyset$ and $U \cap F \cap B = \emptyset$. The function $f = \chi_F$ is of the first Baire class. Since

$$Tf(x)\begin{cases} = 1 & \text{for } \ x \in F \cap \mathrm{Ch}\,\mathrm{X} \cap U, \\ \leq \frac{1}{2} & \text{for } \ x \in (F \setminus \mathrm{Ch}\,\mathrm{X}) \cap U, \end{cases}$$

we see that Tf is not in $B_1^b(X)$ due to condition [D–P] applied to the set $\overline{U \cap F}$. Thus we may suppose that B is dense in F.

Choose a countable set $S_1 \subset B$ dense in B, $S_1 = \{x_n\}_{n=1}^{\infty}$. Denote $\mu_n = \mu_{x_n}$. Fix an integer n. Since

$$\mu_n(F) \geq \frac{1}{2} \quad \text{ and } \quad \mu_n(F \setminus \mathrm{Ch}\,\mathrm{X}) = 0,$$

inner reqularity of Radon measures allows us to find a compact subset K_n of X such that $K_n \subset F \cap \mathrm{Ch}\,\mathrm{X}$ and $\mu_n(K_n) \geq \frac{1}{4}$.

Set $Y = F \cap \mathrm{Ch}\,\mathrm{X}$ and $K = \bigcup_{n=1}^{\infty} K_n$. Due to Lemma 4 the set K is a countable union of closed nowhere dense subsets of Y. Hence K is of the first Baire category in Y. Since Y is a G_δ-subset of a compact metric space, it is a Baire space. Since the set $Y \setminus K$ is residual in Y, it is dense in Y. Due to density of Y in F we obtain that $Y \setminus K$ is dense in F. Find a countable set $S_2 \subset Y \setminus K$ such that S_2 is dense in F.

Thus we have two countable sets $S_1,\ S_2$ such that

$$\begin{aligned} &S_1 \subset F \setminus \mathrm{Ch}\,\mathrm{X}, \\ &S_2 \subset F \cap (\mathrm{Ch}\,\mathrm{X} \setminus K), \end{aligned}$$

and both of them are dense in F. Let us denote $F_0 = \{x_1\}$. We will construct by induction nonempty sets $\{F_n\}_{n=1}^{\infty}$ and nonempty open sets $\{V_n\}_{n=1}^{\infty}$, $\{U_n\}_{n=1}^{\infty}$ such that for every integer n

(i) $\bigcup_{k=0}^{n} F_k$ is closed,
(ii) $\bigcup_{k=0}^{n} F_k \subset \bigcap_{k=1}^{n} U_k$,
(iii) $K_n \subset V_n$,
(iv) $U_n \cap V_n = \emptyset$,
(v) $\mathrm{der}(F_n) \cap S_1 = F_{n-1}$ and $\mathrm{der}(F_n) \cap S_2 = F_{n-1}$,
(vi) $F_n \subset S_1 \cup S_2$.

First, let us find disjoint open sets U_1, V_1 such that $x_1 \in U_1$ and $K_1 \subset V_1$. Since S_1 and S_2 are dense in F, there exists a set $F_1 \subset S_1 \cup S_2$ with $F_1 \subset U_1$, $\operatorname{der}(F_1 \cap S_1) = \{x_1\}$ and $\operatorname{der}(F_1 \cap S_2) = \{x_1\}$. Then all the required conditions are clearly satisfied.

Suppose that F_j, V_j, U_j with desired properties have been constructed for $j \leq n$. Since $S_1 \cup S_2$ is disjoint from K, condition (vi) implies that K_{n+1} is disjoint from $\bigcup_{k=0}^{n} F_k$. Find two disjoint open sets U_{n+1}, V_{n+1} satisfying $\bigcup_{k=0}^{n} F_k \subset U_{n+1}$ and $K_{n+1} \subset V_{n+1}$. Let us construct $F_{n+1} \subset S_1 \cup S_2$ such that $F_{n+1} \subset \bigcap_{k=1}^{n+1} U_k$ and $\operatorname{der}(F_{n+1} \cap S_1) = F_n$, $\operatorname{der}(F_{n+1} \cap S_2) = F_n$. Then all the required conditions are satisfied.

Put $H = \bigcup_{n=0}^{\infty} F_n$. Conditions (ii) and (iv) imply that $H \cap \bigcup_{n=1}^{\infty} V_n = \emptyset$. Thus the set $\overline{H}$ is a closed set disjoint with K. Moreover, by (v) both sets $H \cap S_1$ and $H \cap S_2$ are dense in H. Thus $\overline{H \cap S_1} = \overline{H} = \overline{H \cap S_2}$. Set $f = \chi_{\overline{H}}$. Then f is a function of the first Baire class on X. If x is in $H \cap S_1$ then

$$\mu_n(\overline{H}) \leq \mu_n(X \setminus K) \leq \mu_n(X \setminus K_n) \leq \frac{3}{4},$$

which implies

$$Tf(x) \begin{cases} = 1, & x \in H \cap S_2, \\ \leq \frac{3}{4}, & x \in H \cap S_1. \end{cases}$$

By applying condition [D–P] to the set $\overline{H}$, we get that Tf is not a function of the first Baire class and the proof is complete. □

Proof of the Theorem: The implication (i)⇒(ii) is a consequence of Lemma 1 and Lemma 3. For the converse, suppose that $\operatorname{Ch} X$ is not an F_σ-set. Due to Lemma 5 there exists a function $f \in B_1^b(X)$ such that Tf is not in $B_1^b(X)$. Then $g = f|_{\operatorname{Ch} X}$ is clearly a Baire-one function on $\operatorname{Ch} X$. If F is an $\mathcal{H}$-affine Borel function equal to g on $\operatorname{Ch} X$ then Lemma 2 yields $F = Tg = Tf$. But Tf is not a function of the first Baire class and this proves the Theorem. □

3. An application in potential theory

Let Ω be an open bounded subset of $\mathbb{R}^n$ and let the function space $\mathcal{H}$ consist of all functions continuous on $\overline{\Omega}$ harmonic on Ω. For a real-valued function f defined on the boundary $\partial\Omega$ we denote by Hf the PWB-solution of the Dirichlet problem on Ω with the boundary condition f provided it exists. Given $x \in \Omega$, we have $Hf(x) = \lambda_x(f)$ where λ_x is a harmonic measure representing the point x. In this case the Choquet boundary of $\mathcal{H}$ coincides with the set $\partial_{\text{reg}}\Omega$ of all regular points of Ω. According to a deep result of J. Bliedtner and W. Hansen [4] the function space $(\overline{\Omega}, \mathcal{H})$ is simplicial. Moreover, $\mathcal{H} = \mathcal{A}(\mathcal{H})$ and for any $x \in \Omega$ the measure μ_x equals λ_x.

If we reformulate the general results into the language of potential theory we get the following assertions.

Proposition 1. *The set of regular points $\partial_{\text{reg}}\Omega$ is closed if and only if for any continuous function f defined on $\partial_{\text{reg}}\Omega$ there exists a function h continuous on $\overline{\Omega}$ and harmonic on Ω such that $h = f$ on $\partial_{\text{reg}}\Omega$.*

PROOF: Follows by [1, Theorem II.4.3]. □

Proposition 2. *The set of all regular points $\partial_{\text{reg}}\Omega$ is an F_σ-set if and only if for any bounded function f of the first Baire class defined on $\partial_{\text{reg}}\Omega$ there exists a bounded $H(\Omega)$-affine function h of the first Baire class on $\overline{\Omega}$ such that $h = f$ on $\partial_{\text{reg}}\Omega$.*

PROOF: The proof is a direct consequence of the Theorem. □

4. Final remarks and open problems

1. It seems to be an open problem whether or not the Theorem is valid if we omit the condition of metrizability of the space X. If X is a compact Hausdorff space only then the Choquet boundary Ch X need not be a Borel set and the situation is much more complicated.

2. The first implication of the Theorem has been known since sixties. The proof can be found e.g. in [5] and [9].

3. Consider again the function space of Example 3 (harmonic functions). Following a definition of H. Bauer [2], the set Ω is termed *semiregular* if the PWB-solution Hf can be continuously extended to the closure $\overline{\Omega}$ of Ω for any continuous function f on $\partial\Omega$. Proposition 1 tells us that Ω is semiregular if and only if the set $\partial_{\text{reg}}\Omega$ is closed.

4. Let X be a compact convex subset of a locally convex space E. If X is a Choquet simplex and the set of all extreme point $\operatorname{ext} X$ is closed we call X *a Bauer simplex*. Alfsen [1] is a suitable reference for further details on Bauer simplexes.

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