

Lipschitz-quotients and the Kunen-Martin Theorem

YVES DUTRIEUX

Abstract. We show that there is a universal control on the Szlenk index of a Lipschitz-quotient of a Banach space with countable Szlenk index. It is in particular the case when two Banach spaces are Lipschitz-homeomorphic. This provides information on the Cantor index of scattered compact sets K and L such that $C(L)$ is a Lipschitz-quotient of $C(K)$ (that is the case in particular when these two spaces are Lipschitz-homeomorphic). The proof requires tools of descriptive set theory.

Keywords: Lipschitz equivalences, Szlenk index

Classification: 03E15, 46B20

In the non-linear classification of Banach spaces, it is an open problem to know whether two separable Lipschitz-homeomorphic Banach spaces are isomorphic. Several partial results appeared recently on the subject. We refer to [10] (especially Chapters 7 and 11) for an up-to-date account of the theory. In Theorem 3.18 of [2], it is shown that the class of Asplund spaces is stable under Lipschitz-quotient (this is false under uniform homeomorphism; see Theorem 1 in [12]). The aim of this paper is to precise this result: we show that there exists a universal control on the Szlenk index of a Lipschitz-quotient of a Banach space X , provided X has a countable Szlenk index. For that, we need to estimate the topological complexity of the relation of Lipschitz-quotient and apply the Kunen-Martin theorem.

1. Analyticity of the relation of Lipschitz quotient

The aim of this section is to prove that the relation of Lipschitz-quotient (see Definitions 3.1 and 3.2 in [2]) is analytic in a sense which will be made precise later. First, let us introduce some notation:

Notation. • E will denote the space $C(2^\omega)$ of all continuous functions on the Cantor set. Let us recall that E is universal for all separable Banach spaces.

- \mathcal{S} will denote the set of all closed subspaces of E . It is shown in Proposition 2.1 of [3] (see also pages 15 and 16) that the restriction of the Effros Borel structure on the closed subsets of E makes it into a standard Borel set.
- If X and Y are two Banach spaces, the fact that Y is a Lipschitz-quotient of X will be written $X \twoheadrightarrow_\ell Y$.

When we say that the relation of Lipschitz-quotient is analytic, we mean that the set $\{(X, Y) \in \mathcal{S}^2; X \twoheadrightarrow_\ell Y\}$ is analytic in the standard Borel structure of \mathcal{S} (see Definition 0.4, page 9 in [8]).

We will show the following crucial technical proposition:

Proposition 1. \twoheadrightarrow_ℓ is analytic.

Let us introduce some more notation:

Notation. • The sequence of the vectors x_n will be denoted by \mathbf{x} .

- When the sequence \mathbf{x} is dense in X , we write $X = \overline{\mathbf{x}}$.
- \mathbf{x} and \mathbf{y} being two sequences of vectors, we will write $\mathbf{x} \twoheadrightarrow_\ell \mathbf{y}$ to mean that there exist two constants L and C in ω such that

$$\forall k, l \in \omega, \quad \|y_k - y_l\| \leq L \|x_k - x_l\|$$

and such that, for any $n, p \in \omega$ and any $r \in \mathbb{Q}_+^*$ such that $\|y_p - y_n\| \leq r/C$, there exists a convergent subsequence $\mathbf{x}_\varphi = (x_{\varphi(m)})_{m \in \omega}$ verifying:

$$\mathbf{x}_\varphi \in B_X(x_n, r)^\omega \quad \text{and} \quad \mathbf{y}_\varphi \rightarrow y_p.$$

The link between \twoheadrightarrow_ℓ for spaces and \twoheadrightarrow_ℓ for sequences is given by the following lemma:

Lemma 2. *Let X and Y be two separable Banach spaces. Then $X \twoheadrightarrow_\ell Y$ if and only if there exist two sequences \mathbf{x} and \mathbf{y} such that $X = \overline{\mathbf{x}}$, $Y = \overline{\mathbf{y}}$ and $\mathbf{x} \twoheadrightarrow_\ell \mathbf{y}$.*

PROOF: If there exists a L -Lipschitz and C -co-Lipschitz map f from X to Y then, taking any dense sequence \mathbf{x} and defining \mathbf{y} as the image of \mathbf{x} by f , we clearly have

$$\forall k, l \in \omega, \quad \|y_k - y_l\| \leq L \|x_k - x_l\|.$$

Moreover, let $n, p \in \omega$ and $r \in \mathbb{Q}_+^*$ be such that $\|y_p - y_n\| \leq r/C$. Then, $y_p \in f(B_X(x_n, r))$. Since there is a preimage x of y_p in $B_X(x_n, r)$, there exists a subsequence \mathbf{x}_φ of \mathbf{x} in the open ball such that $\mathbf{x}_\varphi \rightarrow x$. Then $\mathbf{y}_\varphi \rightarrow f(x) = y_p$.

Conversely, let us suppose that $X = \overline{\mathbf{x}}$, $Y = \overline{\mathbf{y}}$ and $\mathbf{x} \twoheadrightarrow_\ell \mathbf{y}$ with constants L and C . We can define $f : X \rightarrow Y$ by $f(x_n) = y_n$ for all $n \in \omega$ and f is L -Lipschitz. Moreover f clearly satisfies:

$$(1) \quad \forall n, p \in \omega, \forall r \in \mathbb{Q}_+^*, \quad \|y_p - y_n\| \leq \frac{r}{C}, \quad \exists x \in B_X(x_n, r), \quad f(x) = y_p.$$

Let us state and prove some facts:

Fact 1. For every $x \in X$, $p \in \omega$, $r \in \mathbb{Q}_+^*$ and $C' > C$ such that the inequality $\|y_p - f(x)\| \leq r/C'$ holds, there exists $z \in B_X(x, r)$ such that $f(z) = y_p$.

Let \mathbf{x}_φ be a subsequence of \mathbf{x} converging to x and verifying, for all $n \in \omega$, $\|x - x_{\varphi(n)}\| \leq k$, $k > 0$ being chosen such that $Lk + r/C' \leq r/C''$, with $C'' > C$ and $C''/C \in \mathbb{Q}$. Then we have $\|y_p - f(x_{\varphi(n)})\| \leq r/C''$. By (1), there exists $z_n \in B_X(x_{\varphi(n)}, Cr/C'')$ such that $f(z_n) = y_p$. Since $\mathbf{x}_\varphi \rightarrow x$, for n large enough, $z_n \in B(x, r)$. Taking $z = z_n$ for such an n gives the result.

Fact 2. f is surjective.

Let $y \in Y$ and let \mathbf{y}_φ be a subsequence such that $\|y - y_{\varphi(n)}\| \leq 2^{-n-1}/C'$ ($C' > C$), for all $n \in \omega$. Applying Fact 1, one can define by induction a sequence \mathbf{z} such that $z_0 = x_{\varphi(0)}$, $\|z_{k+1} - z_k\| \leq 2^{-k}$ and $f(z_k) = y_{\varphi(k)}$ for all $k \in \omega$. The limit z of \mathbf{z} satisfies $f(z) = y$.

Fact 3. For every $C' > C$, f is C' -co-Lipschitz.

The proof is similar to the proof of Fact 2 and will be omitted.

Finally, f is a Lipschitz-quotient map from X to Y and $X \rightarrow_\ell Y$. □

We now give a characterization of the condition $\mathbf{x} \rightarrow_\ell \mathbf{y}$ which is useful for our purpose. We denote by G the set of all infinite subsets of ω . As a G_δ set of a compact set, it is a Polish space. Let us also define

$$\mathcal{G} = G^{\omega \times \omega \times \mathbb{Q}_+^*}.$$

Lemma 3. Let \mathbf{x} and \mathbf{y} be two sequences of vectors. The condition $\mathbf{x} \rightarrow_\ell \mathbf{y}$ is equivalent to the existence of $P \in \mathcal{G}$ such that the conjunction of the following two conditions holds:

1. There exists $L \in \omega$ such that

$$\forall k, l \in \omega, \quad \|y_k - y_l\| \leq L \|x_k - x_l\|.$$

This first condition will be denoted by $L(\mathbf{x}, \mathbf{y})$.

2. There exists $C \in \omega$ that satisfies: for any $n, p \in \omega$ and $r \in \mathbb{Q}_+^*$ such that $\|y_p - y_n\| \leq r/C$, we have $\|x_m - x_n\| \leq r$ for all $m \in P_{n,p,r}$ and

$$\forall q \in \omega, \exists Q \in 2^{<\omega}; \forall m', m \in P_{n,p,r} \setminus Q, \quad \|x_{m'} - x_m\| + \|y_m - y_p\| \leq 1/q.$$

This second condition will be denoted by $C(\mathbf{x}, \mathbf{y}, P)$.

PROOF: It is an easy reformulation of the condition $\mathbf{x} \rightarrow_\ell \mathbf{y}$: for a given (n, p, r) , $P_{n,p,r}$ is the set $\{\varphi(m); m \in \omega\}$ where \mathbf{x}_φ is the subsequence of the definition of $\mathbf{x} \rightarrow_\ell \mathbf{y}$. □

Lemma 4. *Let \mathcal{A} be the set*

$$\left\{ (X, Y, \mathbf{x}, \mathbf{y}, P) \in \mathcal{S}^2 \times (E^\omega)^2 \times \mathcal{G}; X = \overline{\mathbf{x}}, Y = \overline{\mathbf{y}}, L(\mathbf{x}, \mathbf{y}), C(\mathbf{x}, \mathbf{y}, P) \right\}.$$

Then \mathcal{A} is a Borel set.

PROOF: It is enough to see that the sets

$$\begin{aligned} \mathcal{B} = \{ (X, \mathbf{x}) \in E \times E^\omega; X = \overline{\mathbf{x}} \}, \quad \mathcal{C} = \{ (\mathbf{x}, \mathbf{y}) \in (E^\omega)^2; L(\mathbf{x}, \mathbf{y}) \} \\ \text{and} \quad \mathcal{D} = \{ (\mathbf{x}, \mathbf{y}, P) \in (E^\omega)^2 \times \mathcal{G}; C(\mathbf{x}, \mathbf{y}, P) \} \end{aligned}$$

are Borel sets.

It is easy to check that \mathcal{C} is an F_σ .

Let us define \mathcal{O} a countable basis of the topology of E . Recall that the Effros Borel structure on the closed subsets of E is generated by the basis:

$$\left(\{ F \subseteq E; O \cap F \neq \emptyset \} \right)_{O \in \mathcal{O}}.$$

$X = \overline{\mathbf{x}}$ is equivalent to the two conditions:

- (i) $x_n \in O$ implies $O \cap X \neq \emptyset$, for all $n \in \omega$ and all $O \in \mathcal{O}$.
- (ii) For all $O \in \mathcal{O}$, $O \cap X \neq \emptyset$ implies that there exists $n \in \omega$ such that $x_n \in O$.

Then, it is easy to see that \mathcal{B} is a Borel set.

\mathcal{D} is the union over \mathcal{C} of the intersection over n, p, r of:

$$\begin{aligned} \{ \|y_n - y_p\| > r/C \} \cup \left[\bigcap_{m \in \omega} (\{ m \notin P_{n,p,r} \} \cup \{ \|x_m - x_n\| \leq r \}) \cap \right. \\ \left. \left(\bigcap_{q \in \omega} \bigcup_{Q \in 2^{< \omega}} \bigcap_{m, m' \in \omega} [\{ m \notin P_{n,p,r} \text{ or } m' \notin P_{n,p,r} \} \cup \right. \right. \\ \left. \left. \{ \|x_{m'} - x_m\| + \|y_m - y_p\| \leq 1/q \} \right) \right]. \end{aligned}$$

Therefore, \mathcal{D} is a Borel set. □

The set $\{(X, Y); X \rightarrow_\ell Y\}$ being the projection on the first two coordinates of the set \mathcal{A} , it is analytic. This concludes the proof of our technical proposition. □

Before investigating the consequences of Proposition 1, let us add some more details on the Lipschitz-homeomorphisms between Banach spaces. In Theorem 2.4 of [3], Benoît Bossard proved that the linear isomorphism relation is analytic and non Borel. It is therefore natural to ask whether the Lipschitz-homeomorphism relation is also non Borel.

Notation. Let X and Y be two subspaces of E . When X and Y are Lipschitz-homeomorphic, we write $X \sim_\ell Y$.

Proposition 5. *The relation \sim_ℓ is analytic and non Borel.*

PROOF: The proof of the analyticity of \sim_ℓ is similar to (and technically simpler than) the proof of the analyticity of \rightarrow_ℓ . It will thus be omitted.

Let us show that \sim_ℓ is non Borel. Let us introduce $\mathcal{C} = \omega^{<\omega}$ and the group $G = 2^{\mathcal{C}}$. G is isomorphic to the Cantor group. Let p be a real number greater than 1 and different from 2. It suffices for our purpose to show that the set $\mathcal{L} = \{X \in \mathcal{S}; X \sim_\ell L_p(G)\}$ is non Borel.

The dual of G is the group \widehat{G} of all finite subsets of \mathcal{C} where we identify b , a finite subset of \mathcal{C} , and its Walsh function w_b . For any tree T on ω , let us define the set $FB(T)$ of all finite branches of T . The space L_p^T is the closed (for the L_p norm) linear span of the set $\{w_b; b \in FB(T)\}$. Theorem 4.34 in [7] shows that all the spaces L_p^T are complemented subspaces of $L_p(G)$. According to Theorem 4.35 in [7], $L_p(G)$ does not embed in L_p^T if T is well-founded (that we write $T \in WF$). Conversely, if T has an infinite branch, then obviously $L_p(G)$ is isomorphic to a complemented subspace of L_p^T . Pełczyński's decomposition method then implies that $L_p(G)$ is isomorphic to L_p^T if and only if $T \notin WF$. Now we need the following fact:

Fact 4. *The map θ defined on the set \mathcal{T} of all trees on ω by $\theta(T) = L_p^T$ is Borel.*

Let O be an open set of E . It is enough to show that the set $\Omega = \{T \in \mathcal{T}; \theta(T) \cap O \neq \emptyset\}$ is Borel. Since $\theta(T) = \overline{\text{span}}\{w_b; b \in FB(T)\}$, we have, defining $\Lambda = \{(\lambda_b) \in \mathbb{Q}^{FB(\mathcal{C})}; \sum_b \lambda_b w_b \in O\}$:

$$\Omega = \bigcup_{(\lambda_b) \in \Lambda} \bigcap \{T; b \subseteq T\}.$$

It is now clear that Ω is a Borel set, which ends the proof of Fact 4.

According to Corollary 2.9 in [6], $\mathcal{L} = \{X \in \mathcal{S}; X \text{ isomorphic to } L_p(G)\}$. Thus, $\mathcal{L} = \theta(\mathcal{T} \setminus WF)$ is non Borel. Indeed, if it was Borel then, since $\mathcal{T} \setminus WF = \theta^{-1}(\mathcal{L})$ and θ is Borel, $\mathcal{T} \setminus WF$ would be Borel which is absurd. \square

It would come as a very big surprise for us if the relation of Lipschitz-quotient is actually Borel.

2. Control on the Szlenk index of a Lipschitz quotient

Our main result is a consequence of Proposition 1:

Theorem 6. *There exists a universal function $\psi_1 : \omega_1 \rightarrow \omega_1$ such that, if X is a Banach space with countable Szlenk index and Y a Lipschitz-quotient of X , then $\text{Sz}(Y) \leq \psi_1(\text{Sz}(X))$.*

PROOF: Let us recall that, for separable Banach spaces, having a countable Szlenk index is equivalent to having a separable dual (see Proposition 4.12 of [3] for example). Thus, we will show that the general case boils down to the separable case and then use Theorem 3.18 of [2] concerning Asplund spaces.

According to Corollary 3.17 in [2], if f is a Lipschitz-quotient from a Banach space X onto another Banach space Y , then for any separable subspaces X_0 and Y_0 in X and Y respectively, there exist X_1 and Y_1 , separable subspaces of X and Y respectively such that $X_0 \subseteq X_1$, $Y_0 \subseteq Y_1$ and the restriction of f to X_1 is a Lipschitz quotient mapping from X_1 onto Y_1 . Moreover, the Szlenk index of a Banach space, when countable, is the supremum of the Szlenk indices of its separable subspaces (Proposition 3.1 in [4]). Thus, it is enough to deal with separable Banach spaces in our proof. Since the Szlenk index is invariant under linear isomorphism and since E is universal for separable Banach spaces, we can restrict our study to subspaces of E . It is shown in Lemma 3.5 and Theorem 4.13 of [3] that the set of all separable Asplund subspaces of E is a co-analytic set and that the Szlenk index is a Π_1^1 -rank on it (see page 140 of [5] for a definition of Π_1^1 -rank). For any ordinal ξ , let us call S_ξ the set of all subspaces of E whose Szlenk index is less than or equal to ξ and P_ξ the set of all subspaces of E Lipschitz homeomorphic to some element of S_ξ . With this notation, S_{ω_1} is the co-analytic set of all Asplund subspaces of E . Let ξ be a countable ordinal. The set S_ξ is Borel. According to Proposition 1, the set $H = \{(X, Y); X \in S_\xi \text{ and } X \twoheadrightarrow_\ell Y\}$ is analytic. Since P_ξ is the projection of H on the second coordinate, it is also analytic. Theorem 3.18 in [2] shows that P_ξ is included in S_{ω_1} . Kunen-Martin's theorem (see Theorem 7 p. 148 in [5] for instance) then proves that P_ξ is included in S_ζ for some countable ordinal ζ . We can define ψ_1 by $\psi_1(\xi) = \zeta$. \square

In the special case of Lipschitz-homeomorphisms, we obtain the following result:

Corollary 7. *There exists a universal function $\psi_2 : \omega_1 \rightarrow \omega_1$ such that, if X is a Banach space with a countable Szlenk index and Y is a Banach space which is Lipschitz-homeomorphic to X , then $\text{Sz}(Y) \leq \psi_2(\text{Sz}(X))$.*

Theorem 5.5 in [1] proves that, if X and Y are uniformly homeomorphic, then $\text{Sz}(X) \leq \omega$ if and only if $\text{Sz}(Y) \leq \omega$. Thus, if we consider the minimal choices for ψ_1 and ψ_2 , we have $\psi_2(\omega) = \omega$. It is not clear to us whether $\psi_2(\omega^2)$ equals ω^2 . We do not know either the value of $\psi_1(\omega)$. More generally, it could be possible that, in fact, ψ_1 and ψ_2 are simply the identity.

As a corollary of Theorem 6, we get the following theorem about the Cantor index of scattered compact sets:

Corollary 8. *There exists a universal function $\lambda : \omega_1 \rightarrow \omega_1$ such that, if K is a scattered compact with a countable derivative empty and if $C(L)$ is a Lipschitz-quotient of $C(K)$, then the Cantor index $i(L)$ of L is less than or equal to $\lambda(i(K))$.*

PROOF: This corollary is a straightforward consequence of Theorem 6 and of Theorem 5.1 in [4]. \square

Example 4.9 from [11] shows that there exist two non metrizable scattered compact sets K and L with a countable derivative empty such that $C(K)$ and $C(L)$ are Lipschitz-isomorphic but not isomorphic. Thus Corollaries 7 and 8 deal with a situation which is known not to be linear. In the linear case, the Bessaga-Pelczyński result (see Theorem 3 in [13]) gives a necessary and sufficient condition for two countable compact sets K and L to be such that $C(K)$ is isomorphic to $C(L)$ (namely, that $i(K) < i(L) \cdot \omega$ and conversely). Thus, in the context of countable compact sets, it would be natural to compare λ and the function $\xi \mapsto \xi \cdot \omega$.

REFERENCES

- [1] Godefroy N.J., Kalton G., Lancien G., *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. (2000), to appear.
- [2] Bates W., Johnson J., Lindenstrauss D., Preiss G., Schechtman S., *Affine approximation of Lipschitz functions and nonlinear quotients*, Geom. Funct. Anal. **6** (1999), 1092–1127.
- [3] Bossard B., *Théorie descriptive des ensembles en géométrie des espaces de Banach*, Thèse de doctorat de l'Université Paris VI, (in French), 1994.
- [4] Lancien G., *On the Szlenk index and the weak*-dentability index*, Quart. J. Math. Oxford (2) **47** (1996), 59–71.
- [5] Kechris A., Louveau A., *Descriptive set theory and the structure of sets of uniqueness*, London Math. Soc. Lecture Note Series **128** (1987).
- [6] Heinrich P., Mankiewicz S., *Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces*, Studia Math. **73** (1982), 225–251.
- [7] Bourgain J., *New classes of L^p -spaces*, Lecture Notes in Mathematics 889, Springer Verlag, 1981.
- [8] Christensen J., *Topology and Borel structure; descriptive topology and set theory with applications to functional analysis and measure theory*, North-Holland Math. Stud. **10** (1974).
- [9] Bossard B., *Codages des espaces de Banach séparables; familles analytiques et coanalytiques d'espaces de Banach*, C.R. Acad. Sci. Paris, Série I, **316** (1993), 1005–1010.
- [10] Benyamini J., Lindenstrauss Y., *Geometric nonlinear functional analysis vol. I*, AMS Colloquium Publications **48** (2000).
- [11] Godefroy N., Kalton G., Lancien G., *Subspaces of $c_0(\mathbb{N})$ and Lipschitz isomorphisms*, Geom. Funct. Anal. (2000), to appear.

- [12] Ribe M, *Existence of separable uniformly homeomorphic non isomorphic Banach spaces*, Israel J. Math. **48** (1984), 139–147.
- [13] Bessaga A., Pełczyński C., *Spaces of continuous functions (IV)*, Studia Math. **19** (1960), 53–62.

UNIVERSITÉ PIERRE ET MARIE CURIE, EQUIPE D'ANALYSE FONCTIONNELLE, BOÎTE 186,
4, PLACE JUSSIEU, 75005 PARIS, FRANCE

E-mail: dutrieux@ccr.jussieu.fr

(Received January 12, 2001, revised July 6, 2001)