

A generalization of the Schauder fixed point theorem via multivalued contractions

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Abstract. We establish a fixed point theorem for a continuous function $f : X \rightarrow E$, where E is a Banach space and $X \subseteq E$. Our result, which involves multivalued contractions, contains the classical Schauder fixed point theorem as a special case. An application is presented.

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1. Introduction

The aim of this short note is to point out the following result.

Theorem 1. *Let E be a Banach space, X a nonempty closed convex subset of E , $f : X \rightarrow E$ a continuous function, $G : X \times X \rightarrow 2^E$ a multifunction with non-empty values. Moreover, assume that:*

- (i) $f(f(X) \cap X) \subseteq X$;
- (ii) $f(f^{-1}(X))$ is relatively compact;
- (iii) for every $x \in X$, one has $G(x, x) = \{0_E\}$ and the multifunction $G(x, \cdot)$ is upper semicontinuous and with convex graph;
- (iv) the multifunction $F_G : X \rightarrow 2^X$ defined by putting

$$F_G(x) = \left\{ y \in X : [G(x, y) + f(x)] \cap X \neq \emptyset \right\}$$

for all $x \in X$, is a multivalued contraction.

Then, f admits at least one fixed point.

The proof of Theorem 1 will be given in Section 2. When X is compact and $f(X) \subseteq X$, each assumption of Theorem 1 is satisfied. In particular, it suffices to take $G(x, y) \equiv \{0_E\}$. Hence, the classical Schauder fixed point theorem is a particular case of Theorem 1.

As an application of Theorem 1, in Section 2 we shall also prove the following result.

Theorem 2. *Let E be a Banach space, and let $X = B(x_0, R)$ be the closed ball centered at $x_0 \in E$ with radius $R > 0$. Let $f : X \rightarrow E$ be a continuous function satisfying conditions (i) and (ii) of Theorem 1. Moreover, assume that:*

- (iii)' $\alpha := \sup_{x \in X} \|x - f(x)\| < 2R$;
- (iv)' *the function $x \in X \rightarrow x - f(x)$ is a contraction with constant $L < \psi(\alpha)$, where*

$$\psi(t) := \begin{cases} \frac{1}{2} & \text{if } t \in [0, R] \\ 1 - \frac{t}{2R} & \text{if } t \in]R, 2R[. \end{cases}$$

Then, f admits at least one fixed point.

2. The proofs

This section is devoted to the proofs of Theorems 1 and 2. For the basic facts and definitions about multifunctions, we refer to [1], [4].

PROOF OF THEOREM 1: If we put $\text{Fix}(F_G) := \{x \in X : x \in F_G(x)\}$, by (iii) we have

$$\text{Fix}(F_G) = f^{-1}(X).$$

On the other hand, for every $x \in X$ we have

$$F_G(x) = \left\{ y \in X : G(x, y) \cap (X - f(x)) \neq \emptyset \right\}.$$

Hence, by (iii), it follows that the set $F_G(x)$ is closed and convex. Consequently, by (iv) and Theorem 1 of [5], the set $\text{Fix}(F_G)$, endowed with the relative norm topology, is a non-empty absolute extensor for paracompact spaces. Hence, in particular, it is an absolute retract (see [2, p. 92]). On the other hand, (i) is equivalent to the fact that $f(f^{-1}(X)) \subseteq f^{-1}(X)$. At this point, our conclusion follows from Theorem 10.8 at page 94 of [2]. □

If A and D are nonempty subsets of the Banach space E and $x \in E$, we put

$$d(x, D) := \inf_{v \in D} \|x - v\|, \quad d^*(A, D) := \sup_{u \in A} d(u, D).$$

Moreover, we denote by $d_H(A, D)$ the Hausdorff distance between A and D , namely we put

$$d_H(A, D) := \max \{d^*(A, D), d^*(D, A)\}.$$

PROOF OF THEOREM 2: We want to apply Theorem 1 by taking $G(x, y) = \{y - x\}$. Of course, condition (iii) of Theorem 1 is satisfied. We now prove that assumption (iv) is also satisfied. To this aim, we first observe that for each $x \in X$ one has

$$(1) \quad F_G(x) = X \cap B(x - f(x) + x_0, R).$$

Now we claim that, for each $z \in X$ and each $v \in E$, with $\|v\| < 2R$, one has

$$(2) \quad d(z, X \cap B(v + x_0, R)) \leq \psi(\|v\|)^{-1} d(z, B(v + x_0, R)).$$

To prove (2), fix z and v as above. We distinguish two cases.

(a) $\|v\| \leq R$. Since $x_0 \in X \cap B(v + x_0, R)$, we have

$$d^*(X \cap B(v + x_0, R), E \setminus X) \geq d(x_0, E \setminus X) = R.$$

Consequently, by Lemma 1 of [3] we have that the inequality

$$d(z, X \cap B(v + x_0, R)) \leq \frac{2R}{\rho} d(z, B(v + x_0, R))$$

holds for all $\rho \in]0, R[$. Of course, this implies

$$d(z, X \cap B(v + x_0, R)) \leq 2 d(z, B(v + x_0, R)),$$

as desired.

(b) $\|v\| > R$. Since

$$u := x_0 + v \left(1 - \frac{R}{\|v\|}\right) \in X \cap B(v + x_0, R)$$

and $B(u, 2R - \|v\|) \subseteq X$, we get

$$d^*(X \cap B(v + x_0, R), E \setminus X) \geq d(u, E \setminus X) \geq 2R - \|v\|.$$

Again by Lemma 1 of [3], the inequality

$$d(z, X \cap B(v + x_0, R)) \leq \frac{2R}{\rho} d(z, B(v + x_0, R))$$

holds for all $\rho \in]0, 2R - \|v\|[$. This implies

$$d(z, X \cap B(v + x_0, R)) \leq \frac{2R}{2R - \|v\|} d(z, B(v + x_0, R)),$$

as desired. Hence, (2) holds.

At this point, fix $x, y \in X$. By (1) and (2) we have

$$\begin{aligned} d^*(F_G(y), F_G(x)) &\leq \psi(\|x - f(x)\|)^{-1} d^*(F_G(y), B(x - f(x) + x_0, R)) \\ &\leq \psi(\|x - f(x)\|)^{-1} d_H(B(y - f(y) + x_0, R), B(x - f(x) + x_0, R)). \end{aligned}$$

Since ψ decreases in $[0, \alpha]$, by assumption (iv)' and by the previous inequality we get

$$d_H(F_G(y), F_G(x)) \leq \frac{L}{\psi(\alpha)} \|x - y\|,$$

hence F_G is a multivalued contraction. By Theorem 1 our claim follows. □

Remark. When E is an Hilbert space, the more precise estimation given in Lemma 1 of [3] allows us to take the function $\psi(t)$ in the statement of Theorem 2 in the following better way:

$$\psi(t) := \begin{cases} 1 & \text{if } t \in [0, R] \\ \frac{2R}{t} - 1 & \text{if } t \in]R, 2R[. \end{cases}$$

REFERENCES

- [1] Aubin J.P., Frankowska H., *Set-Valued Analysis*, Birkhäuser, Boston, 1990.
- [2] Dugundji J., Granas A., *Fixed Point Theory*, Vol. I, PWN - Polish Scientific Publishers, Warsaw, 1982.
- [3] Hoffmann A., *The distance to the intersection of two convex sets expressed by the distances to each of them*, Math. Nachr. **157** (1992), 81–98.
- [4] Klein E., Thompson A.C., *Theory of Correspondences*, John Wiley and Sons, New York, 1984.
- [5] Ricceri B., *Une propriété topologique de l'ensemble des points fixes d'une contraction multivoque à valeurs convexes*, Atti Acc. Lincei Rend. Fis. **81** (8) (1987), 283–286.

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