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Necessary and Sufficient Conditions of the Wave Packet Frames in $L^2(R^n)$

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Abstract. The main goal of this paper is to consider the necessary and sufficient conditions of wave packet systems to be frames in higher dimensions. We establish the necessary and sufficient conditions for all kinds of wave packet frames of the different operator order in $L^2(R^n)$ with an arbitrary expanding matrix dilations, which include the corresponding results of wavelet analysis and Gabor theory as the special cases. Our way combines with some techniques in wavelet analysis and time-frequency analysis.

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1. Introduction

Frames were first introduced by Duffin and Schaeffer [12] in the context of nonharmonic Fourier series. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies, Grossmann, and Meyer [11]. Since then, the theory of frames began to be more widely studied. Traditionally, frames have been used in signal processing, image processing, data compression, and sampling theory. Recently, frames are also used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission [3,14], and to design high-rate constellation with full diversity in multiple-antenna code design [15]. We refer to the monograph of Daubechies [9] or the research-tutorial [4] for basic properties of frames. Recently, generalized frames were studied in papers [1] and [16].

An important example about frame is wavelet frame, which is obtained by translating and dilating a finite family of functions. Wavelets were introduced relatively recently, in the beginning of the 1980. They attracted considerable interest from the mathematical community and from members of many diverse disciplines in which wavelets had promising

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applications. Daubechies, Grossman and Meyer [11] combined the theory of the continuous wavelet transform with the theory of frames to define wavelet frames for $L^2(R)$. In 1990, Daubechies [10] obtained the first result on the necessary conditions for affine frames, and then in 1993, Chui and Shi [6] obtained an improved result. After about ten years, Casazza and Christensen [2] established a stronger condition which also works for wavelet frame. Recently, Shi *et al.* [20, 23, 25] obtained the necessary conditions and sufficient conditions of wavelet frames.

Another most important concrete realization of frame is Gabor frame. Gabor systems (Weyl-Heisenberg systems) were first introduced by Gabor [13]. They are generated by modulations and translations of a finite family of functions. In 2007, Shi and Chen [24] established some new necessary conditions for Gabor frames. These conditions are also sufficient for tight frames. In [21], Li, Wu and Zhang presented two new sufficient conditions for Gabor frame via Fourier transform. The conditions they proposed were stated in terms of the Fourier transforms of the Gabor system's generating functions, and the conditions were better than that of Daubechies. Furthermore, in [22], Li, Wu and Yang established a necessary condition and two sufficient conditions ensuring that the shift-invariant system is a frame for $L^2(R^n)$. As some applications, the results are used to obtain some known conclusions about wavelet frames and Gabor frames.

In [7], authors introduced wave packet systems by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. In [17], authors adopted the same expression to describe any collections of functions which are obtained by applying the same operations to a finite family of functions. In fact, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Wave packet systems have recently been successfully applied to some problems in harmonic analysis and operator theory [18, 19].

In [17], authors examined in detail both the continuous and discrete versions of wave packet systems by using a unified approach that the authors have developed in their previous work. They gave a classification of the wave packet system to be a Parseval frame. They constructed a very general example of wave packet frame. In [5], authors considered wave packet systems as special cases of generalized shift-invariant systems and presented a sufficient condition for a wave packet system to form a frame. They also presented certain natural conditions on the parameters in a wave packet system which exclude the frame property. Then, they gave a characterization of the wave packet system to be a Parseval frame. At last, they provided several examples which the dilations do not have to be expanding and the modulations do not have to be associated with a lattice. In paper [8], authors introduced analogues of the notion of Beurling density to describe completeness properties of wave packet systems via geometric properties of the sets of their parameters. In particular, they showed necessary conditions for the wave packet system to be a Bessel system. Also, they obtained the necessary conditions for existence of wave packet frames and provided large families of new, non-standard examples of wave packet frames with prescribed dimensions.

Except for above three systems mentioned, composite dilation wavelet systems and shearlet systems have widely studied recently. People can refer to the review in [26] for further knowledge about all reproducing systems generated by finite functions.

Since both Gabor systems and wavelet systems are some particular examples of wave packet systems, people ask naturally: how do we construct some examples of wave packet systems such that they possess simultaneously both Gabor systems and wavelet systems' advantages and, however, overcome their shortcomings? In need of applications, how do we develop the algorithm as classical multiresolution analysis in the setting of the wave packet systems?

So far as we know, few results are known about these problems. This impels people to make great efforts solve them.

The main goal of this paper is to consider the necessary conditions and sufficient conditions of wave packet frames in higher dimensions. We establish some necessary conditions and sufficient conditions for the wave packet frames of the different operator order in $L^2(R^n)$ with matrix dilations of the form $(Df)(x) = \sqrt{q}f(Ax)$, where A is an arbitrary expanding $n\times n$ matrix with integer coefficients and $q = |detA|$. At first, we give a necessary condition for the wave packet system to be a frame, which is a generalization of classical wavelet frame and Gabor frame. Of course, our way combines with some techniques in wavelet analysis and time-frequency analysis. In particular, we use some thoughts of C. K. Chui and X. L. Shi [6] in classifying the necessary condition for the Gabor frame. Also, we discuss necessary conditions for other wave packet frames with the different operator order. Secondly, we deduce a sufficient condition for the wave packet system to be a frame in $L^2(R^n)$. Also, we fuse some ways in wavelet analysis and Gabor theory and we mainly borrow some thoughts in classifying the sufficient conditions of the wavelet frame in papers [20–23, 25].

Let us now describe the organization of the material that follows. Section 2 is of a preliminary character: it contains various notations and some facts about the frame and the wave packet system. In Section 3, we establish some necessary conditions for all kinds of wave packet frames with the different operator order in $L^2(R^n)$. In Section 4, we give a sufficient condition for the wave packet system to be a frame in $L^2(R^n)$.

2. Preliminaries

Let us now establish some basic notations.

Throughout this paper, we use the following notations. R^n and Z^n denote the set of real numbers and the set of integers in n dimensions, respectively. $L^2(R^n)$ is the space of all square-integrable functions in n dimensions, and $\cdot$ and $\|\cdot\|$ denote the inner product and norm in $L^2(R^n)$, respectively, and $l^2(Z^n)$ denotes the space of all square-summable sequences.

For $x = (x_1, x_2, \cdots, x_n) \in R^n$, define $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. We denote by T^n the n-dimensional torus. By $L^p(T^n)$ we denote the space of all Z^n-periodic functions f (i.e., f is 1-periodic in each variable) such that $\int_{T^n} |f(x)|^p dx < +\infty$.

We use the Fourier transform in the form

$$\hat{f}(\omega) = \int_{R^n} f(x)e^{-2\pi i x\cdot\omega}dx, \tag{2.1}$$

where $\cdot$ denotes the standard inner product in R^n, and we often omit it when we can understand this from the background. Sometimes, $\hat{f}(\omega)$ is defined by $\mathscr{F}f$.

The Lebesgue measure of a set $S \subseteq R^n$ will be denoted by $|S|$. When measurable sets X and Y are equal up to a set of measure zero, we write $X \doteq Y$.

Let E_n denote the set of all expanding matrices. The expanding matrices mean that all eigenvalues have magnitude greater than 1. For $A \in E_n$, we denote by A^* the transpose of A. It is obvious that $A^* \in E_n$. Let $GL_n(R)$ denote the set of all $n \times n$ non-singular (or invertible) matrices with real entries. For $B \in GL_n(R)$ we denote by B^{-1} the invertible matrix of B. For the sake of simplicity, we denote $(A^*)^{-1}$ by $A^\sharp$.

Let us recall the definition of frame.

Definition 2.1. *Let H be a separable Hilbert space. A sequence $\{f_i\}_{i\in N}$ of elements of H is a frame for H if there exist constants $0 < C \le D < \infty$ such that for all $f \in H$,*

$$C\|f\|^2 \le \sum_{i=1}^{\infty} |< f, f_i >|^2 \le D\|f\|^2. \tag{2.2}$$

The numbers C, D are called lower and upper frame bounds, respectively (the largest C and the smallest D for which (2.2) holds are the optimal frame bounds). Those sequences which satisfy only the upper inequality in (2.2) are called Bessel sequences. A frame is tight if $C = D$. If $C = D = 1$, it is called a Parseval frame.

Let T_f denote the synthesis operator of $f = \{f_i\}_{i\in N}$, i.e., $T_f(c) = \sum_i c_i f_i$ for each sequence of scalars $c = (c_i)_{i\in N}$. Then the frame operator $Sh = T_f T_f^*(h)$ associated with $\{f_i\}_{i\in N}$ is a bounded, invertible, and positive operator mapping of H on itself. This provides the reconstruction formula

$$h = \sum_{i=1}^{\infty} < h, \tilde{f}_i > f_i = \sum_{i=1}^{\infty} < h, f_i > \tilde{f}_i, \forall\, h \in H. \tag{2.3}$$

where $\tilde{f}_i = S^{-1} f_i$. The family $\{\tilde{f}_i\}_{i\in N}$ is also a frame for H, called the canonical dual frame of $\{f_i\}_{i\in N}$. If $\{g_i\}_{i\in N}$ is any sequence in H which satisfies

$$h = \sum_{i=1}^{\infty} < h, g_i > f_i = \sum_{i=1}^{\infty} < h, f_i > g_i,\ \forall\, h \in H, \tag{2.4}$$

it is called an alternate dual frame of $\{f_i\}_{i\in N}$.

In this paper, we will work with three families of unitary operators on $L^2(R^n)$. Let $A \in E_n$ and $B, C \in GL_n(R)$. The first one consists of the dilation operator $D_A : L^2(R^n) \to L^2(R^n)$ defined by $(D_A f)(x) = q^{1/2} f(Ax)$ with $q = |detA|$. The second one consists of all translation operators $T_{Bk} : L^2(R^n) \to L^2(R^n)$, $k \in Z^n$, defined by $(T_{Bk} f)(x) = f(x - Bk)$. The third one consists of the modulation operator $E_{Cm} : L^2(R^n) \to L^2(R^n)$, $m \in Z^n$, defined by $(E_{Cm} f)(x) = e^{2i\pi Cm\cdot x} f(x)$.

Let $P \subset Z$ and $Q \subset R^n$. Let $S = P \times Q$. Then, we have $S \subset Z \times R^n$. Again, let $\{A_p : A_p \in P\} \subset E_n$ and $B \in GL_n(R)$. For the function $\psi \in L^2(R^n)$, we will consider the wave packet system Ψ defined by the following

$$\Psi = \left\{ \psi_{p,\ \nu,\ m}(x) \quad | \quad D_{A_p} E_\nu T_{Bm} \psi(x), m \in Z^n,\ (p, \nu) \in S \right\}. \tag{2.5}$$

Let $A_p = A^j (j \in Z), S = Z \times \{0\}$. Then, we obtain the wavelet systems. On the other side, we can get the Gabor systems when the set $\{A_p : A_p \in P\}$ only consists of the elementary matrix E. This simple observation already suggests that the wave packet systems provide greater flexibility than the wavelet systems or the Gabor systems.

By changing the order of the operators, we can also define the following one-to-one function systems from $S \times Z^n$ into $L^2(R^n)$:

$$\Psi^1 = \left\{ \psi_{p,\ \nu,\ m}(x) \quad | \quad D_{A_p} T_{Bm} E_\nu \psi(x), m \in Z^n,\ (p, \nu) \in S \right\},$$

$$(2.6)\qquad \begin{aligned} \Psi^2 &= \{\psi_{p,\,v,\,m}(x) \mid E_v D_{A_p} T_{Bm}\psi(x), m\in Z^n,\ (p,v)\in S\},\\ \Psi^3 &= \{\psi_{p,\,v,\,m}(x) \mid E_v T_{Bm} D_{A_p}\psi(x), m\in Z^n,\ (p,v)\in S\},\\ \Psi^4 &= \{\psi_{p,\,v,\,m}(x) \mid T_{Bm} D_{A_p} E_v\psi(x), m\in Z^n,\ (p,v)\in S\},\\ \Psi^5 &= \{\psi_{p,\,v,\,m}(x) \mid T_{Bm} E_v D_{A_p}\psi(x), m\in Z^n,\ (p,v)\in S\}. \end{aligned}$$

Then, we will give the definitions of the wave packet frame and the frame wave packet.

Definition 2.2. *We say that the wave packet system Ψ defined by (2.5) is a wave packet frame if it is a frame for $L^2(R^n)$. Then, the function ψ is called a frame wave functions.*

For other wave packet systems $\Psi^i\,(1\le i\le 5)$ defined by (2.6), we can define the corresponding wave packet frames and the frame wave packets like Definition 2.2.

In order to prove theorems to be presented in next section, we need the following lemmas.

Lemma 2.1. *Suppose that $\{f_k\}_{k=1}^{+\infty}$ is a family of elements in a Hilbert space H such that there exist constants $0<C\le D<+\infty$ satisfying (2.2) for all f belonging to a dense subset D of H. Then, the same inequalities (2.2) are true for all $f\in H$; that is, $\{f_k\}_{k=1}^{+\infty}$ is a frame for H.*

For proof of Lemma 2.1, people can refer to the book [9].

Therefore, we will consider the following set of functions:

$$(2.7)\qquad D=\left\{f\in L^2(R^n):\ \hat{f}\in L^\infty(R^n)\text{ and }\hat{f}\text{ has compact support in }R^n\setminus\{0\}\right\}.$$

The following result is well known, we can find it in [9].

Lemma 2.2. *D is a dense subset of $L^2(R^n)$.*

The following useful facts can be found in paper [5, Lemma 2.2].

Lemma 2.3. *Let $A\in GL_n(R)$, $y,z\in R^n$ and $f\in L^2(R^n)$. Then the following holds:*

(1) $(T_y f)^\wedge = E_y\hat{f}$, $(E_z f)^\wedge = T_z\hat{f}$, $(D_A f)^\wedge = D_{A^\sharp}\hat{f}$;
(2) $T_y E_z f = e^{-2\pi i z\cdot y}E_z T_y f$, $D_A E_y f = E_{A^* y} D_A f$, $D_A T_y f = T_{A^{-1}y} D_A f$;
(3) $(T_y E_z f)^\wedge = e^{-2\pi i z\cdot y}T_z E_{-y}\hat{f}$;
(4) $(D_A T_y f)^\wedge(\xi) = E_{-A^\sharp y} D_{A^\sharp}\hat{f}(\xi) = |detA|^{-\frac{1}{2}}\hat{f}(A^\sharp\xi)e^{-2\pi i A^{-1}y\cdot\xi}$.

3. Necessary conditions of wave packet frames

We firstly give some existing results of wavelet frame and Gabor frame in real line R. Let a and b be the real numbers with $a>1, b>0, \psi\in L^2(R)$, and the system $\psi_{j,k}(x) := \{a^{\frac{j}{2}}\psi(a^j x-kb)\}_{j,k\in Z}$ be a wavelet system. In 1990, Daubechies[10] proved that if the system $\psi_{j,k}(x)$ forms a wavelet frame in $L^2(R)$ with bounds C and D, then

$$bC\ln a\le\int_0^{+\infty}\frac{|\hat{\psi}(\omega)|^2}{\omega}d\omega\le bD\ln a$$

and

$$(3.1)\qquad bC\ln a\le\int_{-\infty}^{0}\frac{|\hat{\psi}(\omega)|^2}{|\omega|}d\omega\le Db\ln a.$$

In 1993, C. K. Chui and X. L. Shi [6] established the following improvement if $\psi(x)$ is a frame wavelet:

$$bC \leq \sum_{j\in Z} |\hat{\psi}(a^j\omega)|^2 \leq bD,\ a.e.\ \omega. \tag{3.2}$$

Let a and b be the real numbers with $a>1, b>0, \psi \in L^2(R)$, and the system $G_{m,k}(x) := \{e^{2i\pi amx}\psi(x-kb)\}_{m,k\in Z}$ be a Gabor system. O. Christensen [4] introduced that if the system $G_{m,k}(x)$ forms a Gabor frame in $L^2(R)$ with bounds C and D, then

$$bC \leq \sum_{m\in Z} |\hat{\psi}(\omega - am)|^2 \leq bD,\ a.e.\ \omega. \tag{3.3}$$

Motivating by the fundament works in (3.2) and (3.3), we will give a necessary condition of wave packet frame Ψ defined by (2.5) for higher dimension with an arbitrary expansive matrix dilation in the following.

Theorem 3.1. *Suppose that wave packet system $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{m\in Z^n,(p,\nu)\in S}$ defined by (2.5) is a frame with frame bounds A_1 and A_2, then we have*

$$bA_1 \leq \sum_{(p,\nu)\in S} |\hat{\psi}(A_p^\sharp\omega - \nu)|^2 \leq bA_2,\ a.e.\ \omega, \tag{3.4}$$

where $b = |detB|$.

Proof. Because wave packet system $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{m\in Z^n,(p,\nu)\in S}$ is a frame with frame bounds A_1 and A_2, for all $f \in L^2(R^n)$,we have

$$A_1\|f\|^2 \leq \sum_{(p,\nu)\in S}\sum_{m\in Z^n} |< f, D_{A_p}E_\nu T_{Bm}\psi >|^2 \leq A_2\|f\|^2. \tag{3.5}$$

Let $\hat{f} \in C_c(R)$ and $\hat{f}$ have compact support.

Let $q_p = |detA_p|$. According to Lemma 2.3 and Plancheral theorem, we have

$$\begin{aligned}
&\sum_{(p,\nu)\in S}\sum_{m\in Z^n} |< f, D_{A_p}E_\nu T_{Bm}\psi >|^2 \\
&= \sum_{(p,\nu)\in S}\sum_{m\in Z^n} |< \mathscr{F}f, \mathscr{F}D_{A_p}E_\nu T_{Bm}\psi >|^2 \\
&= \sum_{(p,\nu)\in S}\sum_{m\in Z^n} |< \hat{f}, D_{A_p^\sharp}T_\nu E_{-Bm}\hat{\psi} >|^2 \\
&= \sum_{p\in P} q_p^{-1} \sum_{\nu\in Q}\sum_{m\in Z^n} |\int_{R^n} \hat{f}(\omega)\overline{\hat{\psi}(A_p^\sharp\omega-\nu)}e^{2\pi iBm(A_p^\sharp\omega-\nu)}d\omega|^2 \\
&= \sum_{p\in P} q_p \sum_{\nu\in Q}\sum_{m\in Z^n} |\int_{R^n} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega|^2
\end{aligned} \tag{3.6}$$

where we change variables by $\omega' = A_p^\sharp\omega - \nu$ in the last equality.

We assert:

$$\begin{aligned}
&\sum_{p\in P} q_p \sum_{\nu\in Q}\sum_{m\in Z^n} |\int_{R^n} \hat{f}(A_p^{*j}(\omega+\nu))\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega|^2 \\
&= \sum_{(p,\nu)\in S} \frac{q_p}{b}\int_{B^\sharp([0,1]^n)} |\sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s)|^2 d\omega.
\end{aligned} \tag{3.7}$$

For fixed $(p, \nu) \in S$, we have

$$\begin{aligned}
&\int_{B^\sharp([0,1]^n)} \sum_{s\in Z^n} |\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s)|d\omega \\
&= \sum_{s\in Z^n} \int_{B^\sharp([0,1]^n)} |\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s)|d\omega \\
&= \sum_{s\in Z^n} \int_{B^\sharp s+B^\sharp([0,1]^n)} |\hat{f}(A_p^*(\omega+\nu))\bar{\hat{\psi}}(\omega)|d\omega \\
&= \int_{R^n} |\hat{f}(A_p^*(\omega+\nu))\bar{\hat{\psi}}(\omega)|d\omega \\
&\leq (\int_{R^n} |\hat{f}(A_p^*(\omega+\nu))|^2 d\omega)^{\frac{1}{2}} (\int_{R^n} |\bar{\hat{\psi}}(\omega)|^2 d\omega)^{\frac{1}{2}} \\
&< \infty,
\end{aligned} \tag{3.8}$$

where the fourth inequality is obtained by using Cauchy-Schwarz's inequality.

Thus we can define a function $F_p : R \to C$ by

$$F_p(\omega) = \sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s), \, a.e. \ \omega. \tag{3.9}$$

$F_p(\omega)$ is $B^\sharp T^n$-periodic, and the above argument gives that $F_p(\omega) \in L^1(B^\sharp[0,1]^n)$. In fact, we even have $F_p(\omega) \in L^2(B^\sharp[0,1]^n)$. To see this, we first see that

$$|F_p(\omega)|^2 \leq \sum_{s\in Z^n} |\hat{f}(A_p^*(\omega+B^\sharp s+\nu))|^2 \sum_{s\in Z^n} |\hat{\psi}(\omega+B^\sharp s)|^2. \tag{3.10}$$

Since $\hat{f} \in C_c(R)$, the function $\omega \to \sum_{s\in Z^n} |\hat{f}(A_p^*(\omega+B^\sharp s+\nu))|^2$ is bounded. According to above argument, we easily get $F_p(x) \in L^2(B^\sharp[0,1]^n)$. Then, according to the definition of $F_p(\omega)$, we have

$$\begin{aligned}
&\int_{R^n} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}(\omega)} e^{2\pi iBm\omega} d\omega \\
&= \sum_{s\in Z^n} \int_{B^\sharp s+B^\sharp([0,1]^n)} \hat{f}(A_p^*(\omega+\nu))\bar{\hat{\psi}}(\omega) e^{2\pi iBm\omega} d\omega \\
&= \sum_{s\in Z^n} \int_{B^\sharp([0,1]^n)} \hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s) e^{2\pi iBm\omega} d\omega \\
&= \int_{B^\sharp([0,1]^n)} \left(\sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s) \right) e^{2\pi iBm\omega} d\omega \\
&= \int_{B^\sharp([0,1]^n)} F_p(\omega) e^{2\pi iBm\omega} d\omega.
\end{aligned} \tag{3.11}$$

Parseval's equality shows that

$$\sum_{m\in Z^n} |\int_{B^\sharp([0,1]^n)} F_p(\omega) e^{2\pi iBm\omega} d\omega|^2 = \frac{1}{b} \int_{B^\sharp([0,1]^n)} |F_p(\omega)|^2 d\omega; \tag{3.12}$$

Combining (3.11),(3.12) and the definition of $F_p(\omega)$, we obtain that

$$\begin{aligned}(3.13)\qquad &\sum_{m\in Z^n}|\int_{R^n}\hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega|^2\\ =\ &\frac{1}{b}\int_{B^\sharp([0,1]^n)}|\sum_{s\in Z^n}\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s)|^2d\omega.\end{aligned}$$

So, we obtain (3.7). Thus, we complete the assertion.

Choose $\omega_0\in R$ to be Lebesgue point of the function $\sum_{(p,\nu)\in S}|\hat{\psi}(A_p^\sharp\omega-\nu)|^2$. Letting $B(\varepsilon)$ denote the ball of radius $\varepsilon>0$ about the origin and ε be sufficiently small, define f_ε by

$$\hat{f}_\varepsilon(\omega)=\frac{1}{\sqrt{|B(\varepsilon)|}}\chi_{B(\varepsilon)}(\omega-\omega_0). \tag{3.14}$$

Therefore, we obtain

$$\|f_\varepsilon\|^2=\|\hat{f}_\varepsilon\|^2=1. \tag{3.15}$$

Thus, we have

$$\sum_{(p,\nu)\in S}|\hat{\psi}(A_p^\sharp\omega_0-\nu)|^2=\lim_{\varepsilon\to 0}\int_{|\omega-\omega_0|<\varepsilon}\frac{1}{|B(\varepsilon)|}\sum_{(p,\nu)\in S}|\hat{\psi}(A_p^\sharp\omega-\nu)|^2d\omega. \tag{3.16}$$

From the definition of f, (3.5),(3.6) and (3.7), we have

$$\begin{aligned}(3.17)\qquad &\int_{|\omega-\omega_0|<\varepsilon}\frac{1}{|B(\varepsilon)|}\sum_{(p,\nu)\in S}|\hat{\psi}(A_p^\sharp\omega-\nu)|^2d\omega\\ &=\sum_{(p,\nu)\in S}\int_{B^\sharp([0,1]^n)}|\hat{f}_\varepsilon(\omega)|^2|\hat{\psi}(A_p^\sharp\omega-\nu)|^2d\omega\\ &=\sum_{(p,\nu)\in S}q_p\int_{B^\sharp([0,1]^n)}|\sum_{s\in Z^n}\hat{f}_\varepsilon(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s)|^2d\omega\\ &=b\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<f_\varepsilon,D_{A_p}E_\nu T_{Bm}\psi>|^2\\ &\le bA_2,\end{aligned}$$

where the third equality is obtained by changing variables $\omega'=A_p^*(\omega+\nu)$.

Let $\varepsilon\to 0$, using the definition of Lebesgue point, we get

$$\sum_{(p,\nu)\in S}|\hat{\psi}(A_p^\sharp\omega_0-\nu)|^2\le bA_2. \tag{3.18}$$

According to the definition of Lebesgue point, by the similar technique of Chui and Shi [6], we obtain

$$\sum_{(p,\nu)\in S}|\hat{\psi}(A_p^\sharp\omega_0-\nu)|^2\ge bA_1. \tag{3.19}$$

We leave the assertion to readers.

Comparing with (3.17) and (3.18), by changing variables by $\omega=\omega_0$, we have (3.4).

Therefore, we have completed the proof of Theorem 3.1. ∎

Remark 3.1. In particular, let A the elementary matrix E in the Theorem 3.1, then, we obtain the necessary condition of the Gabor frames as the following, which is a generalization of the known result [4] in higher dimensions.

Corollary 3.1. *Let $B,C \in GL_n(R)$. Suppose that the Gabor system $\{E_{Ck}T_{Bm}\psi(x)\}_{k,m\in Z^n}$ is a frame with frame bounds A_1 and A_2, then*

$$bA_1 \le \sum_{k\in Z^n} |\hat{\psi}(\omega - Ck)|^2 \le bA_2, \ a.e.\, \omega, \tag{3.20}$$

where $b = |detB|$.

On the other side, let $P = \{A^j : j \in Z, A \in GL_n(R)\}$ and $Q = \{0\}$ in the Theorem 3.1, then, we obtain the necessary condition of the wavelet frames as the following, which is a generalization of Chui and Shi [6] in higher dimensions.

Corollary 3.2. *Let $A \in E_n, B \in GL_n(R)$. Suppose that wavelet system $\{D_A^j T_{Bm}\psi(x)\}_{j\in Z, m\in Z^n}$ is a frame with frame bounds A_1 and A_2, then*

$$bA_1 \le \sum_{j\in Z} |\hat{\psi}(A^{*j}\omega)|^2 \le bA_2, \ a.e.\, \omega, \tag{3.21}$$

where $b = |detB|$.

In the following, we will discuss necessary conditions for other wave packet frames $\Psi^i\,(1 \le i \le 5)$ defined by (2.6) with the different operator order.

For wave packet systems Ψ^1, from Lemma 2.3, we have

$$D_{A_p}T_{Bm}E_\nu\psi(x) = e^{-2\pi iBm\cdot\nu}D_{A_p}E_\nu T_{Bm}\psi(x). \tag{3.22}$$

If wave packet system $\{D_{A_p}T_{Bm}E_\nu\psi(x)\}_{m\in Z^n,(p,\nu)\in S}$ defined by (2.6) is a frame with frame bounds A_1 and A_2, then, from Theorem 3.1 and (3.22), the inequality (3.4) holds.

For wave packet systems Ψ^2, from (2) of Lemma 2.3, we have

$$E_\nu D_{A_p}T_{Bm}\psi(x) = D_{A_p}E_{A^\sharp\nu}T_{Bm}\psi(x). \tag{3.23}$$

If wave packet system $\{D_{A_p}T_{Bm}E_\nu\psi(x)\}_{m\in Z^n,(p,\nu)\in S}$ defined by (2.6) is a frame with frame bounds A_1 and A_2, then, in the same way, the inequality (3.4) holds.

Then, from Theorem 3.1 and (3.23), we have

Corollary 3.3. *Suppose that wave packet system $\{E_\nu D_{A_p}T_{Bm}\psi(x)\}_{m\in Z^n,(p,\nu)\in S}$ defined by (2.6) is a frame with frame bounds A_1 and A_2, then we have*

$$bA_1 \le \sum_{(p,\nu)\in S} |\hat{\psi}(A_p^\sharp(\omega-\nu))|^2 \le bA_2, \ a.e.\, \omega, \tag{3.24}$$

where $b = |detB|$.

For wave packet systems $\Psi^i\,(1 \le 3 \le 5)$, according to the same reason, we have

$$E_\nu T_{Bm}D_{A_p}\psi(x) = D_{A_p}E_{A^\sharp\nu}T_{A_pBm}\psi(x). \tag{3.25}$$

$$T_{Bm}E_\nu D_{A_p}\psi(x) = e^{-2\pi iA_pBm\cdot\nu}D_{A_p}E_{A^\sharp\nu}T_{A_pBm}\psi(x). \tag{3.26}$$

$$T_{Bm}D_{A_p}E_\nu\psi(x) = e^{-2\pi iA_pBm\cdot\nu}D_{A_p}E_{A\nu}T_{A_pBm}\psi(x). \tag{3.27}$$

The problems turn into being more complicated because all of three equalities are involved in the operator T_{A_pBm}. We can not obtain directly the results from Theorem 3.1. We will discuss them in the future.

4. Sufficient condition of wave packet frames

Not all choices for ψ, A_p, ν and B lead to the wave packet system $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{(p,\nu)\in S,\ m\in Z^n}$ to be a wave packet frame, even if ψ satisfies (3.4).

In this section, we will derive a sufficient condition for the wave packet system to be a frame in $L^2(R^n)$.

Theorem 4.1. *Suppose that wave packet system $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{m\in Z^n,(p,\nu)\in S}$ is defined by (2.5). Define the constants C_1, C_2 as the following*

(4.1)

$$C_1 := \frac{1}{b}\left\{\inf_{\omega\in R^n}\left[\sum_{(p,\nu)\in S}|\hat{\psi}(A_p^*\omega-\nu)|^2 - \sum_{(p,\nu)\in S}\sum_{s\in Z^n\setminus\{0\}}|\hat{\psi}(A_p^\sharp\omega-\nu)||\bar{\hat{\psi}}(A_p^\sharp\omega-\nu+B^\sharp s)|\right]\right\} > 0,$$

$$C_2 := \frac{1}{b}\left\{\sup_{\omega\in R^n}\left[\sum_{(p,\nu)\in S}\sum_{s\in Z^n}|\hat{\psi}(A_p^\sharp\omega-\nu)||\bar{\hat{\psi}}(A_p^\sharp\omega-\nu+B^\sharp s)|\right]\right\} < \infty, \tag{4.2}$$

where $b=|detB|$. Then, the wave packet system $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{m\in Z^n,(p,\nu)\in S}$ is a frame with frame bounds C_1, C_2.

Proof. By Lemma 2.1 and Lemma 2.2, it suffices to show that Theorem 4.1 holds for all $f\in D$.

To do this, we need to estimate the series

$$\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<f, D_{A_p}E_\nu T_{Bm}\psi>|^2. \tag{4.3}$$

Because $f\in D$, the number of k is finite, so (3.6), (3.7) and the Fourier transform inversion formula imply that

$$\begin{aligned}
&\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<f, D_{A_p}E_\nu T_{Bm}\psi>|^2\\
&= \sum_{(p,\nu)\in S}\frac{q_p}{b}\int_{B^\sharp([0,1]^n)}|\sum_{s\in Z^n}\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s)|^2 d\omega\\
&= \sum_{(p,\nu)\in S}\frac{q_p}{b}\int_{B^\sharp([0,1]^n)}\sum_{s\in Z^n}\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s) \qquad (4.4)\\
&\sum_{m\in Z^n}\bar{\hat{f}}(A_p^*(\omega+B^\sharp m+\nu))\hat{\psi}(\omega+B^\sharp m)d\omega\\
&= \sum_{(p,\nu)\in S}\frac{q_p}{b}\int_{R^n}\overline{\hat{f}(A_p^*(\omega+\nu))}\hat{\psi}(\omega)\left[\sum_{s\in Z^n}\hat{f}(A_p^*(\omega+B^\sharp s+\nu))\bar{\hat{\psi}}(\omega+B^\sharp s)\right]d\omega.
\end{aligned}$$

Then, by (4.4) and changing variables $\omega' = A_p^*\omega$, we can write

$$\begin{aligned}
&\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<f, D_{A_p}E_\nu T_{Bm}\psi>|^2\\
&= \frac{1}{b}\sum_{(p,\nu)\in S}\sum_{s\in Z^n}\int_{R^n}\overline{\hat{f}(\omega+A_p^*\nu)}\hat{\psi}(A_p^\sharp\omega)\hat{f}(\omega+A_p^*B^\sharp s+A_p^*\nu)\bar{\hat{\psi}}(A_p^\sharp\omega+B^\sharp s)d\omega \qquad (4.5)\\
&= Q_1+Q_2,
\end{aligned}$$

where,

$$Q_1 = \frac{1}{b} \sum_{(p,\nu)\in S} \int_{R^n} |\hat{f}(\omega + A_p^* \nu)\hat{\psi}(A_p^\sharp \omega)|^2 d\omega \tag{4.6}$$

and

$$Q_2 = \frac{1}{b} \sum_{(p,\nu)\in S} \sum_{s\in Z^n\setminus\{0\}} \int_{R^n} \overline{\hat{f}(\omega + A_p^* \nu)}\hat{\psi}(A_p^\sharp \omega)\hat{f}(\omega + A_p^* B^\sharp s + A_p^* \nu)\bar{\hat{\psi}}(A_p^\sharp \omega + B^\sharp s) d\omega. \tag{4.7}$$

Thus, we can rearrange the series Q_2 as

$$\begin{aligned} Q_2 &= \frac{1}{b} \sum_{\nu\in Q} \sum_{p\in P} \sum_{s\in Z^n\setminus\{0\}} \int_{R^n} \overline{\hat{f}(\omega + A_p^* \nu)}\hat{f}(\omega + A_p^* \nu + A_p^* B^\sharp s) \left(\hat{\psi}(A_p^\sharp \omega)\bar{\hat{\psi}}(A_p^\sharp \omega + B^\sharp s)\right) d\omega \\ &= \frac{1}{b} \sum_{p\in P} \sum_{s\in Z^n\setminus\{0\}} \int_{R^n} \overline{\hat{f}(\omega)}\hat{f}(\omega + A_p^* B^\sharp s) \left(\sum_{\nu\in Q} \hat{\psi}(A_p^\sharp \omega - \nu)\bar{\hat{\psi}}(A_p^\sharp \omega - \nu + B^\sharp s)\right) d\omega, \end{aligned} \tag{4.8}$$

where the second equality is obtained by changing variables $\omega' = \omega + A_p^* \nu$.

Let

$$\Delta_s(\omega) = \sum_{\nu\in Q} \hat{\psi}(A_p^\sharp \omega - \nu)\bar{\hat{\psi}}(A_p^\sharp \omega - \nu + B^\sharp s). \tag{4.9}$$

According to Hölder's inequality and (4.8), we have

$$\begin{aligned} |Q_2| &\le \frac{1}{b} \sum_{p\in P} \sum_{s\in Z^n\setminus\{0\}} \int_{R^n} \left(|\bar{\hat{f}}(\omega)|\sqrt{|\Delta_s(\omega)|}\right)\left(|\hat{f}(\omega + A_p^* B^\sharp s)|\sqrt{|\Delta_s(\omega)|}\right) d\omega \\ &\le \frac{1}{b} \sum_{p\in P} \sum_{s\in Z^n\setminus\{0\}} \left[\int_{R^n} |\bar{\hat{f}}(\omega)|^2 |\Delta_s(\omega)| d\omega \int_{R^n} |\hat{f}(\omega + A_p^* B^\sharp s)|^2 |\Delta_s(\omega)| d\omega\right]^{\frac{1}{2}}. \end{aligned} \tag{4.10}$$

Therefore, by (4.10) and Cauchy-Schwarz's inequality,

$$\begin{aligned} |Q_2| \quad &\le \frac{1}{b} \sqrt{\sum_{p\in P} \int_{R^n} |\hat{f}(\omega)|^2 \sum_{s\in Z^n\setminus\{0\}} |\Delta_s(\omega)| d\omega} \\ &\times \sqrt{\sum_{p\in P} \int_{R^n} |\hat{f}(\omega + A_p^* B^\sharp s)|^2 \sum_{s\in Z^n\setminus\{0\}} |\Delta_s(\omega)| d\omega}. \end{aligned} \tag{4.11}$$

Combining with (4.11), we get:

$$\int_{R^n} |\hat{f}(\omega + A_p^* B^\sharp s)|^2 |\Delta_s(\omega)| d\omega = \int_{R^n} |\hat{f}(\omega)|^2 |\Delta_s(\omega - A_p^* B^\sharp s)| d\omega \tag{4.12}$$

and

$$\begin{aligned} \Delta_s(\omega - A_p^* B^\sharp s) &= \sum_{\nu\in Q} \hat{\psi}(A_p^\sharp \omega - \nu - B^\sharp s)\bar{\hat{\psi}}(A_p^\sharp \omega - \nu) \\ &= \overline{\sum_{\nu\in Q} \hat{\psi}(A_p^\sharp \omega - \nu)\bar{\hat{\psi}}(A_p^\sharp \omega - \nu - B^\sharp s)} \\ &= \overline{\Delta_{-s}(\omega)}. \end{aligned} \tag{4.13}$$

By changing variables $s' = -s$, we can obtain we obtain

$$\sum_{s\in Z^n\setminus\{0\}} |\Delta_{-s}(\omega)| = \sum_{s\in Z^n\setminus\{0\}} |\Delta_s(\omega)|. \tag{4.14}$$

Thus, we have

$$\begin{aligned} |Q_2| &\leq \frac{1}{b}\sum_{p\in P}\int_{R^n} |\hat{f}(\omega)|^2 \sum_{s\in Z^n\setminus\{0\}} |\Delta_s(\omega)| d\omega \\ &\leq \frac{1}{b}\|f\|^2 \sup_{\omega\in R}\sum_{p\in P}\sum_{s\in Z^n\setminus\{0\}} |\Delta_s(\omega)|. \end{aligned} \tag{4.15}$$

According to the definition of $\Delta_s(\omega)$ and (4.15), we get

$$\begin{aligned} |Q_2| &\leq \frac{1}{b}\|f\|^2 \sup_{\omega\in R}\sum_{p\in P}\sum_{s\in Z^n\setminus\{0\}} |\sum_{\nu\in Q} \hat{\psi}(A_p^\sharp\omega-\nu)\bar{\hat{\psi}}(A_p^\sharp\omega-\nu+B^\sharp s)| \\ &\leq \frac{1}{b}\|f\|^2 \sup_{\omega\in R}\sum_{p\in P}\sum_{s\in Z^n\setminus\{0\}}\sum_{p\in P} |\sum_{\nu\in Q} \hat{\psi}(A_p^\sharp\omega-\nu)||\bar{\hat{\psi}}(A_p^\sharp\omega-\nu+B^\sharp s)|. \end{aligned} \tag{4.16}$$

From (4.6), we easily obtain

$$|Q_1| \leq \frac{1}{b}\|f\|^2 \sup_{\omega\in R}\sum_{(p,\nu)\in S} |\hat{\psi}(A_p^*\omega-\nu)|^2. \tag{4.17}$$

Combining with (4.5), (4.16) and (4.17), we have

$$\begin{aligned} &\sum_{(p,\nu)\in S}\sum_{m\in Z^n} |< f, D_{A_p}E_\nu T_{Bm}\psi >|^2 \\ &\leq \frac{1}{b}\|f\|^2 \left\{ \sup_{\omega\in R^n}\left[\sum_{(p,\nu)\in S}\sum_{s\in Z^n} |\hat{\psi}(A_p^\sharp\omega-\nu)||\bar{\hat{\psi}}(A_p^\sharp\omega-\nu+B^\sharp s)|\right]\right\}. \end{aligned} \tag{4.18}$$

In the similar way, we can get

$$\begin{aligned} &\sum_{(p,\nu)\in S}\sum_{m\in Z^n} |< f, D_{A_p}E_\nu T_{Bm}\psi >|^2 \\ &\geq \frac{1}{b}\|f\|^2 \left\{ \inf_{\omega\in R^n}\left[\sum_{(p,\nu)\in S} |\hat{\psi}(A_p^*\omega-\nu)|^2 \right.\right. \\ &\quad \left.\left. -\sum_{(p,\nu)\in S}\sum_{s\in Z^n\setminus\{0\}} |\hat{\psi}(A_p^\sharp\omega-\nu)||\bar{\hat{\psi}}(A_p^\sharp\omega-\nu+B^\sharp s)|\right]\right\}. \end{aligned} \tag{4.19}$$

That is to say, if the constants C_1, C_2 are defined by (4.1) and (4.2), the wave packet system $\{D_{A_p}E_\nu T_{Bm}\psi(x)\}_{m\in Z^n,(p,\nu)\in S}$ is a frame with frame bounds C_1, C_2.
Therefore, we have completed the proof of Theorem 4.1. ∎

In particular, let A the elementary matrix E in the Theorem 4.1, then, we obtain the sufficient condition of the Gabor frames as the following, which is a special case of Corollary 6.3 in [22].

Corollary 4.1. *Let $B, C \in GL_n(R)$, $g(x) \in L^2(R^n)$. Define the constants A_1, A_2 as the following*

$$A_1 = \inf_{\xi} \sum_{m \in Z^n} \left(|\hat{g}(\xi - Bm)|^2 - \sum_{k \neq 0} |\hat{g}(\xi - Bm)||\hat{g}(\xi - Bm + C^* k)| \right) > 0,$$

$$A_2 = \sup_{\xi} \left(\sum_{m \in Z^n} \sum_{k \in Z^n} |\hat{g}(\xi - Bm)||\hat{g}(\xi - Bm + C^* k)| \right) < +\infty,$$

then, Gabor system $\{E_{Ck} T_{Bm} g(x)\}_{k,m \in Z^n}$ is a frame for $L^2(R^n)$ with frame bounds A_1 and A_2.

On the other side, let $P = \{A^j : j \in Z, A \in E_n,\}$ $B = E$ and $Q = \{0\}$ in the Theorem 4.1, then, we obtain the sufficient condition of the wavelet frames as the following, which is the case of a single generator of Corollary 5.3 in paper [22].

Corollary 4.2. *Let $\psi(x) \in L^2(R^n), A \in E_n$. Suppose that the constants C, D satisfy*

$$C = \inf_{\xi} \sum_{j \in Z} \left(|\hat{\psi}\left((A^\sharp)^j \xi\right)|^2 - \sum_{m \neq 0} |\hat{\psi}((A^\sharp)^j \xi)||\hat{\psi}((A^\sharp)^j (\xi + m))| \right) > 0,$$

$$D = \sup_{\xi} \left(\sum_{m \in Z^n} \sum_{j \in Z} |\hat{\psi}((A^\sharp)^j \xi)||\hat{\psi}((A^\sharp)^j (\xi + m))| \right) < +\infty,$$

then wavelet system $\{D_A^j T_k \psi(x) \mid j \in Z,\ k \in Z^n\}$ is a frame for $L^2(R^n)$ with bounds C and D.

Remark 4.1. Note that O. Christensen and A. Rahimi [5] presented a sufficient condition for a wave packet system Ψ^1 defined by (2.6) to form a frame by making use of the theory of generalized shift-invariant systems. In this paper, we devoted to classifying the wave packet system Ψ defined by (2.5), which includes the corresponding results of wavelet analysis and Gabor theory as the special cases.

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