On Contractions and Invariants of Leibniz Algebras

Isamiddin S. Rakhimov and Kamel A. Mohd. Atan

Abstract. In this paper, contractions of complex Leibniz algebras are considered. A short summary of the history, relationships of different definitions and comparisons of them are given. We focus on the contractions of three-dimensional case of complex Leibniz algebras. A several contraction invariants that are useful in determining whether one algebra can be obtained as an contraction of another algebra are given.

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1. Introduction

In 1951 Segal [21] introduced the notion of contractions of Lie algebras on physical grounds: if two physical theories (like relativistic and classical mechanics) are related by a limiting process, then the associated invariance groups (like the Poincare and Galilean groups) should also be related by some limiting process. If the velocity of light is assumed to go to infinity, relativistic mechanics “transforms” into classical mechanics. This also induces a singular transition from the Poincare algebra to the Galilean one. Another example is a limiting process from quantum mechanics to classical mechanics under $\hbar \to 0$, that corresponds to the contraction of the Heisenberg algebras to the abelian ones of the same dimensions [5].

There are two approaches to the contraction problems of algebras. The first of them is based on physical considerations that is mainly oriented to applications of contractions. Contractions were used to establish connections between various kinematical groups and to shed a light on their physical meaning. In this way relationships between the conformal and Schrodinger groups was elucidated and various Lie algebras including a relativistic position operator were interrelated. Under dynamical group description of interacting systems, contractions corresponding to the coupling constant going to zero give noninteracting systems. Application of contractions allows to derive interesting results in the special function theory and on the variable separation method.
The second consideration is pure algebraical dealing with abstract algebraic structures.

Let $A$ be an $n$-dimensional algebra over a field $K$, (underlying vector space denoted $V$) with the binary operation $\lambda : V \times V \rightarrow V$. Consider a continuous function $g_t : (0, 1) \rightarrow GL(V)$. In other words, $g_t$ is a nonsingular linear operator on $V$ for all $t \in (0, 1]$. Define parameterized family of new isomorphic to $A$ algebra structures on $V$ via the old binary operation $\lambda$ as follows:

$$\lambda_t(x, y) = (g_t \ast \lambda)(x, y) = g_t^{-1}(\lambda(g_t(x), g_t(y)), x, y \in V.$$

**Definition 1.1.** If the limit $\lim_{t \to +0} \lambda_t = \lambda_0$ exists for all $x, y \in V$, then the algebraic structure $\lambda_0$ defined by this way on $V$ is said to be a contraction of the algebra $A$.

**Note 1.1.** Obviously, the contractions can be considered in basis level, i.e., let $\{e_1, e_2, \ldots, e_n\}$ be a basis of an $n$-dimensional algebra $A$. If the limit $\lim_{t \to +0} \lambda_t(e_i, e_j) = \lambda_0(e_i, e_j)$ exists, then the algebra $(V, \lambda_0)$ is a contraction of $A$.

**Definition 1.2.** A contraction from an algebra $A$ to algebra $A_0$ is said to be trivial if $A_0$ is abelian and improper if $A_0$ is isomorphic to $A$.

Note that both the trivial and the improper contractions always exist. Here is an example of the trivial and the improper contractions.

**Example 1.1.** Let $A = (V, \lambda)$ be an $n$-dimensional algebra. If we take $g_t = \text{diag}(t, t, \ldots, t)$, then $g_t \ast \lambda$ is abelian and at $g_t = \text{diag}(1, 1, \ldots, 1)$, we get $g_t \ast \lambda = A$.

In this paper we mainly focus on the algebraic aspects of the contractions. In current usage in algebra, the word degeneration also is equally used instead of contraction.

Let $V$ be a vector space of dimension $n$ over an algebraically closed field $K$ (char$K$=0). The bilinear maps $V \times V \rightarrow V$ form a vector space $\text{Hom}(V \otimes V, V)$ of dimension $n^3$, which can be considered together with its natural structure of an affine algebraic variety over $K$ and denoted by $\text{Alg}_n(K)$. An $n$-dimensional algebra $A$ over $K$ may be considered as an element $\lambda(A)$ of $\text{Alg}_n(K)$ via the bilinear mapping $\lambda : A \otimes A \rightarrow A$ defining an binary algebraic operation on $A$. The linear reductive group $GL_n(K)$ acts on $\text{Alg}_n(K)$ by $(g \ast \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y))$ ("transport of structure"). Two algebras $\lambda_1$ and $\lambda_2$ are isomorphic if and only if they belong to the same orbit under this action. For given two algebras $\lambda$ and $\mu$, we say that $\lambda$ degenerates to $\mu$, if $\mu$ lies in the Zariski closure of the orbit $\lambda$. We denote this by $\lambda \rightarrow \mu$.

**Definition 1.3.** An algebra $L$ over a field $K$ is called a Leibniz algebra if its binary operation $\lambda$ satisfies the following Leibniz identity:

$$\lambda((x, \lambda(y, z)) = \lambda(\lambda(x, y), z) - \lambda(\lambda(x, z), y).$$

The set of all Leibniz algebra structures on an $n$-dimensional vector space $V$ over a field $K$ is denoted by $\text{LB}_n(K)$. The set $\text{LB}_n(K)$ can be included in the above mentioned $n^3$-dimensional affine space as follows: let $\{e_1, e_2, \ldots, e_n\}$ be a basis of the vector space $V$. Then the table of multiplication of $L$ is represented by point $(\gamma^L_{ij})$ of this affine space as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^n \gamma^L_{ij} e_k.$$
Thus, the algebra $L$ corresponds to the point $(\gamma_{ij}^k)$. $\gamma_{ij}^k$ are called structure constants of $L$. The Leibniz identity gives polynomial relations among $\gamma_{ij}^k$. Hence we regard $LB_n$ as a subvariety of $K^{n^3}$.

**Definition 1.4.** A Leibniz algebra $\lambda$ is said to degenerate to a Leibniz algebra $\mu$, if $\mu$ is represented by a structure which lies in the Zariski closure of the $GL_n(K)$-orbit of the structure which represents $\lambda$.

In this case entire orbit $\text{Orb}(\mu)$ lies in the closure of $\text{Orb}(\lambda)$. We denote this, as has been mentioned above, by $\lambda \to \mu$, i.e., $\mu \in \text{Orb}(\lambda)$.

**Note 1.2.** Degeneration is transitive, that is, if $\lambda \to \mu$ and $\mu \to \nu$, then $\lambda \to \nu$.

**Note 1.3.** The algebras whose orbits are open in $LB_n(K)$ are called rigid. There are irreducible components of $GL_n$-variety $LB_n$ generated by the rigid laws, i.e. the orbit’s closure of the law is an irreducible component. But there exist components of $LB_n$ given by nonrigid laws. One can consider some “rigid” families parameterized by one, two or several parameters. In this cases, the union of the orbits of these families also give irreducible components. Thus there are two types of irreducible components of a variety $LB_n$ with the action of the algebraic group $GL_n$ (“transport of structure”) on it. The first type is irreducible component generated by a rigid law. The second type is irreducible component generated by a family of nonrigid laws. In the last case the representatives of these laws are not degenerated by other laws.

From now on all the algebras considered are supposed to be over the field of complex numbers $\mathbb{C}$. We make use of a few useful facts from the algebraic groups theory, concerning the degenerations. The first of them is on constructive subsets of algebraic varieties over $\mathbb{C}$, the closures of which relative to Euclidean and Zariski topologies coincide. Since $GL_n(\mathbb{C})$-orbits are constructive sets, the usual Euclidean topology on $\mathbb{C}^{n^3}$ leads to the same degenerations as the Zariski topology does. Now we may express the concept of degeneration in a slightly different way, that is, the following condition will imply that $\lambda \to \mu$:

$$\exists g_t \in GL_n(\mathbb{C}(t)) \text{ such that } \lim_{t \to 0} g_t * \lambda = \mu,$$

where $\mathbb{C}(t)$ is the field of fractions of the polynomial ring $\mathbb{C}[t]$.

The second fact concerning the closure of $GL_n(\mathbb{C})$-orbits states that the boundary of each orbit is a union of finitely many orbits with dimensions strictly less than dimension of the given orbit. It follows that each irreducible component of the variety, on which algebraic group acts, contains only one open orbit that has a maximal dimension. It is obvious that in the content of variety of algebras, the representatives of this kind orbits are rigid.

It is an interesting but difficult problem to determine the number of irreducible components of an algebraic variety. In this note we study the variety of 3-dimensional Leibniz algebras. As for other classes of algebras the known cases are as follows: for associative algebras $\text{alg}_n(\mathbb{C})$: $\text{alg}_4(\mathbb{C})$ has been studied in [9], $\text{alg}_5(\mathbb{C})$ has been treated in [18] and [12]; for nilpotent associative algebras case (see [18]); for nilpotent Lie algebras $NL_n(\mathbb{C})$: at $n \leq 5$, it can be found in [13, 3] and $NL_6$ was described by Seeley [20], $NL_7$ and $NL_8$ were investigated by Goze, Anchochea Bermudez [10] and Goze, Khakimdjanov [11]; the variety of filiform Lie Algebras were investigated by Goze, Khakimdjanov [11]; the variety
of low-dimensional Jordan algebras has been treated by Kashuba in [15]; for nilpotent Leibniz algebras in dimension less than 5, the geometric classification can be found in [1]. A slightly different approach to the geometric classification problem of algebras can be found in [6], [7] and [8].

2. Invariance Arguments

For a given Leibniz algebra \( L \), we define:

- \( \mathcal{R}(L) = \{ x \in L | \lambda(L, x) = 0 \} \) — the right annihilator of \( L \);
- \( \mathcal{Z}(L) = \{ x \in L | \lambda(x, L) = 0 \} \) — the left annihilator of \( L \);
- \( Z(L) = \{ x \in L \lambda(x, L) = \lambda(L, x) = 0 \} \) — the center of \( L \);
- \( \text{Aut}(L) \) — the group of automorphisms of \( L \);
- \( L^k = \lambda(L^{k-1}, L) \) — the \( k \)-th degree of \( L \), where \( k \in \mathbb{N} \);
- \( \text{SA}(L) \) — the maximal abelian subalgebra of \( L \);
- \( \text{Com}(L) \) — the maximal commutative subalgebra of \( L \);
- \( \text{SLie}(L) \) — the maximal Lie subalgebra of \( L \);
- \( \text{HL}_i(L, L) \) — the \( i \)-th Leibniz cohomology group.

**Invariance Argument 1.** Here is a result from [1] on Zariski closed subsets of \( LB_n \).

**Theorem 2.1.** For any \( m, r \in \mathbb{N} \), the following subsets of \( LB_n \) are closed relative to the Zariski topology:

1. \( \{ L \in LB_n | \dim L^m \leq r \} \)
2. \( \{ L \in LB_n | \dim \mathcal{R}(L) \geq m \} \)
3. \( \{ L \in LB_n | \dim \mathcal{Z}(L) \geq m \} \)
4. \( \{ L \in LB_n | \dim \text{Aut}(L) > m \} \)
5. \( \{ L \in LB_n | \dim \text{Com}(L) \geq m \} \)
6. \( \{ L \in LB_n | \dim \text{SLie}(L) \geq m \} \)
7. \( \{ L \in LB_n | \dim \text{HL}_1(L, L) \geq m \} \)

The proof is an easy consequence of the following fact from algebraic group theory. Let \( G \) be a complex reductive algebraic group acting rationally on an algebraic set \( X \). Let \( B \) be a Borel subgroup of \( G \). Then \( \overline{G} = G * \overline{B} \) [13].

**Corollary 2.1.** An algebra \( L \) does not degenerate to algebra \( L' \) if at least one of the following conditions is valid:

1. \( \dim L^m < \dim L'^m \) for some \( m \), \( \dim \mathcal{R}(L) > \dim \mathcal{R}(L') \),
2. \( \dim \mathcal{Z}(L) > \dim \mathcal{Z}(L') \),
3. \( \dim \text{Aut}(L) \geq \dim \text{Aut}(L') \),
4. \( \dim \text{Com}(L) > \dim \text{Com}(L') \),
5. \( \dim \text{SLie}(L) > \dim \text{SLie}(L') \).

The Invariance Arguments below are stated in general sitting and Leibniz algebras case is deduced from these as a special case.

**Invariance Argument 2.** Let \( A \) be an \( n \)-dimensional algebra over a field \( K \) and \( e_1, e_2, ..., e_n \) be a basis on it. Then the element \( x = x_1 \otimes e_1 + x_2 \otimes e_2 + ... + x_n \otimes e_n \in K[x_1, x_2, ..., x_n] \otimes_K A \), where \( x_1, x_2, ..., x_n \) are independent variables, is called the generic element of \( A \). Denote by \( f_A(R_x) \) a Cayley-Hamilton polynomial of the right-multiplication operator to the generic element \( x \) in the algebra \( A = K[x_1, x_2, ..., x_n] \otimes_K K \). It is known that \( f_A(R_x) \) doesn’t depend on choosing of the basis used.
Proposition 2.1. If an algebra A degenerates to algebra B, then \( f_A(R_x) = 0 \) in B.

Invariance Argument 3. Let \( \{e_1, e_2, \ldots, e_n\} \) be a basis of A and \( \text{tr}(R_{e_i}) = 0 \) for all i. If there exists a basis \( \{f_1, f_2, \ldots, f_n\} \) of B such that \( \text{tr}(R_{f_i}) \neq 0 \) for some i, then A does not degenerate to B.

Invariance Argument 4. Let A be given by the structure constants \( \gamma_1, \gamma_2, \ldots, \gamma_r \) and \( (i, j) \) be pair of positive integers such that

\[
c_{ij} = \frac{\text{tr}(R_x)^i \text{tr}(R_y)^j}{\text{tr}((R_x)^i \circ (R_y)^j)}.
\]

Then \( c_{ij} \) is a polynomial of \( \gamma_1, \gamma_2, \ldots, \gamma_r \) and it does not depend on the elements \( x, y \) of A. If neither of these polynomials is zero, we call \( c_{ij} \) an \( (i, j) \)-invariant of A. Suppose that A has an \( (i, j) \)-invariant \( c_{ij} \). Then all \( B \in \text{Orb}(A) \) must have the same \( (i, j) \)-invariant.

Invariance Argument 5. Let assume that in the previous invariance argument either \( \text{tr}(R_x)^i \text{tr}(R_y)^j = 0 \) or \( \text{tr}((R_x)^i \circ (R_y)^j) = 0 \) for all \( x, y \in A \) and some pair \( (i, j) \). Then the same holds for all \( B \in \text{Orb}(A) \).

3. Variety of 3-dimensional complex Leibniz algebras

In two dimensional Leibniz algebras case one has the following table.

| Table 1. Isomorphism classes of two-dimensional Leibniz algebras |
|-----------------|-----------------|-----------------|
| L₁       | \( e₁e₂ = e₁, \ e₂e₁ = e₁ \) | Solvable Leibniz algebra |
| L₂       | \( e₂e₁ = e₁ \) | Nilpotent Leibniz algebra |
| L₃       | \( e₁e₂ = e₁, \ e₂e₁ = e₂ \) | Solvable Lie algebra |
| L₄       | - | Abelian |

It is easy to see here that the algebras \( L₁ \) and \( L₃ \) are rigid. Hence, \( \text{LB}_2(\mathbb{C}) \) has two irreducible components generated by \( L₁ \) and \( L₃ \), respectively.

Theorem 3.1. Up to isomorphism, there exist four one parametric families and thirteen explicit representatives of complex Leibniz algebras of dimension three.

Proof. The proof can be obtained by combining algebraic classification of Lie (see [14]) and Leibniz algebras (see [2]) in dimension three. The updated list of non Lie complex three-dimensional Leibniz algebras is available from [19] and [4].

In the following two tables, we give all isomorphic types of 3-dimensional complex Leibniz algebras and their volumes of invariants (the names in column 1 correspond to the increasing of the automorphisms group’s dimension). Note that the table contains Lie and Leibniz algebras low-dimensional cases as well.

| Table 2. Isomorphism classes of three-dimensional Leibniz algebras |
|-----------------|-----------------|-----------------|
| \( L₁(\alpha) \), \( \alpha \neq 0, \ \alpha \in \mathbb{C} \) | \( e₁e₃ = \alpha e₁, \ e₂e₃ = e₁ + e₂, \ e₃e₁ = e₁ \) | Solvable Leibniz algebra |
| L₂       | \( e₃e₁ = e₁, \ e₂e₁ = e₁ + e₂ \) | Solvable Leibniz algebra |
| L₃       | \( e₁e₂ = e₃, \ e₁e₁ = 2e₁, \ e₂e₁ = -e₃, \ e₂e₂ = -2e₂, \ e₃e₁ = -2e₁, \ e₃e₂ = 2e₂ \) | Simple Lie algebra: \( sl₂ \) |
Table 3. Contraction Invariants of three-dimensional Leibniz algebras

<table>
<thead>
<tr>
<th>$L$</th>
<th>$dL_2$</th>
<th>$d\mathfrak{R}(L)$</th>
<th>$d\mathfrak{S}(L)$</th>
<th>$dZ(L)$</th>
<th>$d\text{Aut}(L)$</th>
<th>$d\text{SA}(L)$</th>
<th>$d\text{Com}(L)$</th>
<th>$d\text{Lie}(L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_4(\alpha)$, $\alpha \in \mathbb{C}$</td>
<td>$e_1e_3 = \alpha e_1$, $e_2e_3 = -e_2$, $e_3e_2 = e_1$</td>
<td>Solvable Leibniz algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_5$</td>
<td>$e_1e_3 = e_1$, $e_2e_3 = e_1$, $e_3e_2 = e_1$</td>
<td>Solvable Leibniz algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_6$</td>
<td>$e_1e_3 = e_2$, $e_3e_2 = e_1$</td>
<td>Nilpotent Leibniz algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_7$</td>
<td>$e_1e_3 = e_2$, $e_2e_1 = e_2$</td>
<td>Solvable Leibniz algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{8}(\alpha)$, $\alpha \neq 0;</td>
<td>\alpha</td>
<td>&lt; 1$</td>
<td>$e_1e_2 = e_2$, $e_1e_3 = \alpha e_3$, $e_2e_1 = -e_2$, $e_3e_1 = -\alpha e_3$</td>
<td>Solvable Lie algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_9$</td>
<td>$e_1e_2 = e_2$, $e_2e_1 = -e_2$</td>
<td>Solvable Lie algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{10}$</td>
<td>$e_1e_2 = e_2$, $e_1e_3 = e_2 + e_3$, $e_2e_1 = -e_2$, $e_3e_1 = -e_2 - e_3$</td>
<td>Solvable Lie algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{11}(\alpha)$, $\alpha \in \mathbb{C}$</td>
<td>$e_2e_2 = e_1$, $e_2e_3 = e_1$, $e_3e_2 = \alpha e_1$</td>
<td>Nilpotent Leibniz algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{12}$</td>
<td>$e_2e_2 = e_1$, $e_2e_3 = e_1$, $e_3e_2 = e_1$</td>
<td>Associative, commutative, nilpotent Leibniz algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{13}$</td>
<td>$e_1e_3 = e_1$, $e_2e_3 = e_2$, $e_3e_3 = e_1$</td>
<td>Solvable Leibniz algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{14}$</td>
<td>$e_1e_1 = e_2$</td>
<td>Associative, commutative, nilpotent Leibniz algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{15}$</td>
<td>$e_1e_2 = e_2$, $e_1e_3 = e_3$, $e_2e_1 = -e_2$, $e_3e_2 = -e_3$</td>
<td>Solvable Lie algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{16}$</td>
<td>$e_1e_2 = e_3$, $e_2e_1 = -e_3$</td>
<td>Nilpotent Lie algebra, Heisenberg algebra $\mathcal{H}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{17}$</td>
<td>-</td>
<td>Abelian</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
In the table below $R_x$ and $L_x$ stand for the right and the left multiplication operators, respectively and $I$ stands for the identity operator.

**Table 4. Characteristic Polynomials of three-dimensional Leibniz algebras**

<table>
<thead>
<tr>
<th>$L$</th>
<th>$dL^2$</th>
<th>$dR(L)$</th>
<th>$dS(L)$</th>
<th>$dZ(L)$</th>
<th>$dAut(L)$</th>
<th>$dSA(L)$</th>
<th>$dCom(L)$</th>
<th>$dLie(L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{11} (\alpha \neq 0)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$L_{12}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$L_{13}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$L_{14}$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$L_{15}$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$L_{16}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$L_{17}$</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

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Cont. Table 3

<table>
<thead>
<tr>
<th>$L$</th>
<th>The characteristic polynomial of $R_x$ in $L$</th>
<th>The characteristic polynomial of $L_x$ in $L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1 (\alpha \neq 0, -1, \alpha \in \mathbb{C})$</td>
<td>$R_x \left( R_x^2 - (trR_x)R_x + \frac{\alpha}{(\alpha + 1)^2}(trR_x)^2 I \right)$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_1 (\alpha = -1)$</td>
<td>$R_x(R_x^2 - \frac{1}{3}trR_x^2 I)$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$R_x^2(R_x - trR_x I)$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_3$</td>
<td>$R_x^2(R_x - trR_x I)$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_4 (\alpha \neq 1)$</td>
<td>$R_x \left( R_x^2 - (trR_x)R_x - \frac{\alpha}{(\alpha + 1)^2}(trR_x)^2 I \right)$</td>
<td>$L_x^3(L_x - trL_x I)$</td>
</tr>
<tr>
<td>$L_4 (\alpha = 1)$</td>
<td>$R_x(R_x^2 - \frac{1}{3}trR_x^2 I)$</td>
<td>$L_x^3(L_x - trL_x I)$</td>
</tr>
<tr>
<td>$L_5$</td>
<td>$R_x^2(R_x - trR_x I)$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_6$</td>
<td>$R_x^2$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_7$</td>
<td>$R_x^2(R_x - trR_x I)$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_8 (\alpha \neq 0;</td>
<td>\alpha</td>
<td>&lt; 1)$</td>
</tr>
<tr>
<td>$L_8 (\alpha = -1)$</td>
<td>$R_x(R_x^2 - \frac{1}{3}trR_x^2 I)$</td>
<td>$L_x \left( L_x^2 - \frac{1}{3}trL_x^2 I \right)$</td>
</tr>
<tr>
<td>$L_9$</td>
<td>$R_x^2(R_x - trR_x I)$</td>
<td>$L_x \left( L_x^2 - trL_x I \right)$</td>
</tr>
<tr>
<td>$L_{10}$</td>
<td>$R_x^2(R_x - \frac{1}{3}trR_x^2 I)^2$</td>
<td>$L_x \left( L_x^2 - \frac{1}{3}trL_x I \right)^2$</td>
</tr>
<tr>
<td>$L_{11} (\alpha \in \mathbb{C})$</td>
<td>$R_x^3$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_{12}$</td>
<td>$R_x^3$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_{13}$</td>
<td>$R_x(R_x - \frac{1}{3}trR_x I)^2$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_{14}$</td>
<td>$R_x^3$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_{15}$</td>
<td>$R_x(R_x - \frac{1}{3}trR_x I)^2$</td>
<td>$L_x \left( L_x^2 - \frac{1}{3}trL_x I \right)^2$</td>
</tr>
<tr>
<td>$L_{16}$</td>
<td>$R_x^3$</td>
<td>$L_x^3$</td>
</tr>
<tr>
<td>$L_{17}$</td>
<td>$R_x^3$</td>
<td>$L_x^3$</td>
</tr>
</tbody>
</table>
By using the Invariance Arguments, we find all possible degenerations of 3-dimensional complex Leibniz algebras.

\[ L_1 \to L_2, \ L_5, \ L_6, \ L_7, L_{11}(\alpha = 0), L_{13}, L_{14}, L_{17}; \]
\[ L_2 \to L_7, L_{11}(\alpha = 0), L_{13}, L_{14}, L_{17}; \]
\[ L_3 \to L_9, L_{16}, L_{17}; \]
\[ L_4(\alpha = 0) \to L_7, L_9, L_{11}(\alpha = 0), L_{12}, L_{13}, L_{14}, L_{16}, L_{17}; \]
\[ L_4(\alpha \neq 0) \to L_4(\alpha = 0), L_5, L_6, L_7, L_9, L_{11}(\alpha = 0), L_{12}, L_{13}, L_{14}, L_{16}, L_{17}; \]
\[ L_5 \to L_7, L_9, L_{11}(\alpha = 0), L_{14}, L_{17}; \]
\[ L_6 \to L_7, L_{11}(\alpha = 0), L_{13}, L_{14}, L_{17}; \]
\[ L_7 \to L_7, L_{11}(\alpha = 0), L_{14}, L_{17}; \]
\[ L_8 \to L_9, L_{10}, L_{15}, L_{16}, L_{17}; \]
\[ L_9 \to L_{16}, L_{17}; \]
\[ L_{10} \to L_{15}, L_{16}, L_{17}; \]
\[ L_{11}(\alpha = 0) \to L_{14}, L_{17}; \]
\[ L_{11}(\alpha \neq 0) \to L_{11}(\alpha = 0), L_{14}, L_{17}; \]
\[ L_{12} \to L_{14}, L_{16}, L_{17}; \]
\[ L_{13} \to L_{16}, L_{17}; \]
\[ L_{14} \to L_{17}; \]
\[ L_{15} \to L_{16}, L_{17}; \]
\[ L_{16} \to L_{17}; \]
\[ L_{17} \to L_{17}; \]

The algebra \( L_3 \) and the parametric family of algebras \( L_1(\alpha), L_4(\alpha), L_8(\alpha), L_{11}(\alpha) \), do not appear on the right hand side of this list after the arrows, this means that \( L_3 \) is rigid and the group of algebras \( L_1(\alpha), L_4(\alpha), L_8(\alpha), L_{11}(\alpha) \), form rigid families of algebras, i.e., they are not degeneration of other Leibniz algebra structures in dimension three. The rigidity of the algebra \( L_3 \) follows from the following fact as well. A Lie algebra \( L \) is Leibniz rigid if and only if \( L \) is a Lie-rigid and \( H^2(L, L) = HL^2(L, L) \).

For the Leibniz algebras that cannot be excluded from the rigidity class by these invariance arguments, we apply the following additional arguments:

1. A Leibniz algebra can not be degenerated by a Lie algebra.
2. Use existing 3-dimensional Lie algebras degenerations ([13], [3], [20]).
3. Use existing 3-dimensional nilpotent Leibniz algebras degenerations ([1]).
4. Use Associative algebras degenerations ([18]).

The final result can now be written as follows.

**Theorem 3.2.** The algebra \( L_3 \) and the continues family of algebras

\[ L_1(\alpha), L_4(\alpha), L_8(\alpha), (|\alpha| < 1, \alpha \neq 0), L_{11}(\alpha), \]

generate the rigid irreducible components of \( LB_3(\mathbb{C}) \) with the dimensions:

\[ \mathcal{C}_1 = \overline{\text{Orb}(L_3)}, \quad \dim \mathcal{C}_1 = 6, \]
\[ \mathcal{C}_2 = \bigcup_{\alpha} \overline{\text{Orb}(L_1(\alpha))}, \quad \dim \mathcal{C}_2 = 7, \]
\[ \mathcal{C}_3 = \bigcup_{\alpha} \overline{\text{Orb}(L_4(\alpha))}, \quad \dim \mathcal{C}_3 = 6, \]
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\[ \mathcal{C}_4 = \bigcup_{|\alpha| < 1, \alpha \neq 0} \text{Orb}(L_8(\alpha)), \quad \dim \mathcal{C}_3 = 5, \]

\[ \mathcal{C}_5 = \bigcup_{\alpha \neq 0} \text{Orb}(L_{11}(\alpha)), \quad \dim \mathcal{C}_5 = 5. \]

Therefore

\[ \dim \text{LB}_3(\mathbb{C}) = \max\{\dim \mathcal{C}_1, \dim \mathcal{C}_2, \dim \mathcal{C}_3, \dim \mathcal{C}_4, \dim \mathcal{C}_5\} = 7. \]

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References


