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The Bounded Convergence Theorem for Riesz Space-Valued Choquet Integrals

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Abstract. The bounded convergence theorem on the Riesz space-valued Choquet integral is formalized for a sequence of measurable functions converging in measure and in distribution.

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1. Introduction

The Choquet integral is commonly used as the integral in non-additive measure theory, and has been already generalized to the framework of Riesz spaces; see Boccuto and Riečan [1], Duchoň *et al.* [3], and [7] among others. The fundamental three limit theorems that are used in applications of the integral are the monotone convergence, the dominated convergence, and the bounded convergence. In [7], the author has given a comprehensive discussion on the theory of Choquet integration for Riesz space-valued non-additive measures, and has formalized the monotone convergence theorem and the dominated convergence theorem under the smoothness condition on the involved Riesz space, called the monotone function continuity property. This paper is a continuation of [7], and this time, in exchange for the uniform essential boundedness of the integrands, we can obtain some convergence theorems for Choquet integrals with respect to Riesz space-valued non-additive measures (for short, Riesz space-valued Choquet integrals) under weaker conditions on the smoothness of the Riesz space and the modes of convergence of the integrands.

In Section 2, we give a definition and basic properties of the Riesz space-valued asymmetric Choquet integral by utilizing the existing theory of Riemann(-Stieltjes) integration in Riesz spaces. In Section 3, we formalize the bounded convergence theorem for Riesz space-valued Choquet integrals of a sequence of functions converging in measure. In Section 4, we formalize another form of the bounded convergence theorem for Riesz space-valued Choquet integrals of functions converging in distribution under the monotone function continuity property. The autocontinuity of non-additive measures plays a crucial role in these

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formalizations. The obtained results in Sections 3 and 4 contain the corresponding ones discussed in Denneberg [2] and Murofushi *et al.* [9] for real-valued Choquet integrals.

2. The Asymmetric Choquet integral

It is always assumed that V is a Riesz space, and the standard terminology of the theory of Riesz spaces [8] will be used. Denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all natural numbers, respectively.

In this section, we give a definition of the asymmetric Choquet integral for Riesz space-valued non-additive measures and study its basic properties. To this end, we utilize the existing theory of Riemann(-Stieltjes) integration in Riesz spaces [11]. We have already given its summary in [7, Appendix A] for readers' convenience and will use those results without mentioning explicitly.

From now on we assume that $(X,\mathscr{F})$ is a measurable space, that is, $\mathscr{F}$ is a σ-field of subsets of a non-empty set X. We also assume that V is a Dedekind complete and weakly σ-distributive Riesz space [13]. Denote by χ_A the characteristic function of a set A.

Definition 2.1. *A set function $\mu : \mathscr{F} \to V$ is called a non-additive measure if it satisfies the following two conditions:*

(i) $\mu(\emptyset) = 0$.
(ii) $\mu(A) \le \mu(B)$ *whenever $A,B \in \mathscr{F}$ and $A \subset B$ (monotonicity).*

See [2, 10, 12] for comprehensive information on real-valued non-additive measures.

Definition 2.2. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let $f : X \to \mathbb{R}$ be an $\mathscr{F}$-measurable function. The function $G_f : \mathbb{R} \to V$ defined by*

$$G_f(t) := \mu\left(\{x \in X : f(x) > t\}\right) \ (t \in \mathbb{R})$$

is called the decreasing distribution function of f with respect to μ.

Since the function G_f is decreasing, it follows from the theory of Riemann integration in Riesz spaces (see [7, Appendix A] for instance) that every $s > 0$, G_f is Riemann integrable on the bounded closed interval $[0,s]$, and the function $\varphi : [0,\infty) \to V$ defined by $\varphi(a) := \int_0^a G_f(t)dt$ for each $a \in [0,\infty)$ is increasing. In the same way, the function $\psi : (-\infty,0] \to V$ defined by $\psi(b) := \int_b^0 \{G_f(t) - \mu(X)\}dt$ for each $b \in (-\infty,0]$ is also increasing. Therefore, the following formalization is well-defined.

Definition 2.3. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let $f : X \to \mathbb{R}$ be an $\mathscr{F}$-measurable function. We say that f is Choquet integrable with respect to μ if the following two conditions are satisfied:*

(i) *The set* $\left\{ \int_0^a G_f(t)dt : a > 0 \right\}$ *is bounded from above.*
(ii) *The set* $\left\{ \int_b^0 \{G_f(t) - \mu(X)\}dt : b < 0 \right\}$ *is bounded from below.*

In this case, the (asymmetric) Choquet integral of f with respect to μ is defined by

$$\int_X f d\mu := \sup_{a>0} \int_0^a G_f(t)dt + \inf_{b<0} \int_b^0 \{G_f(t) - \mu(X)\}dt.$$

The following proposition can be proved by the definition of the Choquet integral and the properties of the Riemann integral in Riesz spaces [7, Appendix A].

Proposition 2.1. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let $f, g, h : X \to \mathbb{R}$ be $\mathscr{F}$-measurable functions.*

(1) *Let $c \geq 0$. If f is Choquet integrable, then so is cf and it holds that $\int_X (cf) d\mu = c \int_X f d\mu$.*
(2) *Let $c \in \mathbb{R}$. If f is Choquet integrable, then so is $f + c$ and it holds that $\int_X (f + c) d\mu = \int_X f d\mu + c\mu(X)$.*
(3) *Assume that f and g are Choquet integrable and $f(x) \leq g(x)$ for all $x \in X$. Then it holds that $\int_X f d\mu \leq \int_X g d\mu$.*
(4) *Assume that f and g are Choquet integrable and there is $c > 0$ satisfying $|f(x) - g(x)| \leq c$ for all $x \in X$. Then it holds that $|\int_X f d\mu - \int_X g d\mu| \leq c\mu(X)$.*
(5) *Assume that $h(x) \leq f(x) \leq g(x)$ for all $x \in X$. If g and h are Choquet integrable, then so is f and it holds that $\int_X h d\mu \leq \int_X f d\mu \leq \int_X g d\mu$.*

Definition 2.4. [9, Definition 3.2] *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let $f : X \to \mathbb{R}$ be an $\mathscr{F}$-measurable function. Let Φ be a non-empty family of $\mathscr{F}$-measurable, real-valued functions on X.*

(1) *We say that f is essentially bounded if there is $r \in \mathbb{R}$ with $r > 0$ such that $G_f(r) = 0$ and $G_f(-r) = \mu(X)$.*
(2) *We say that Φ is uniformly essentially bounded if there is $r \in \mathbb{R}$ with $r > 0$ such that $G_f(r) = 0$ and $G_f(-r) = \mu(X)$ for all $f \in \Phi$.*

The following proposition shows that every essentially bounded, measurable function is Choquet integrable.

Proposition 2.2. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let $f : X \to \mathbb{R}$ be an $\mathscr{F}$-measurable function. If f is essentially bounded, then it is Choquet integrable. More precisely, if there is $r \in \mathbb{R}$ with $r > 0$ such that $G_f(r) = 0$ and $G_f(-r) = \mu(X)$, then it holds that*

$$\int_X f d\mu = \int_0^r G_f(t) dt + \int_{-r}^0 \left\{ G_f(t) - \mu(X) \right\} dt = \int_{-r}^r t\, d(-G_f).$$

Proof. Assume that there is $r \in \mathbb{R}$ with $r > 0$ such that $G_f(r) = 0$ and $G_f(-r) = \mu(X)$. Then $G_f(t) = 0$ for all $t \geq r$ and $G_f(t) = \mu(X)$ for all $t \leq -r$. Therefore, the set $\left\{ \int_0^a G_f(t) dt : a > 0 \right\}$ is bounded from above and

$$\sup_{a>0} \int_0^a G_f(t) dt = \int_0^r G_f(t) dt.$$

Similarly, the set $\left\{ \int_b^0 \left\{ G_f(t) - \mu(X) \right\} dt : b < 0 \right\}$ is bounded from below and

$$\inf_{b<0} \int_b^0 \left\{ G_f(t) - \mu(X) \right\} dt = \int_{-r}^0 \left\{ G_f(t) - \mu(X) \right\} dt.$$

Therefore, f is Choquet integrable and the first equality holds. The second equality follows from [7, Propositions A.3 and A.7]. ∎

3. The bounded convergence theorem

In this section, we formalize the bounded convergence theorem for Riesz space-valued Choquet integrals of a sequence of functions converging in measure.

Definition 3.1. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let f be an $\mathscr{F}$-measurable, real-valued function on X and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of such functions.*

(1) *We say that $\{f_n\}_{n\in\mathbb{N}}$ converges to f almost everywhere and write $f_n \to f$ (a.e.) if there is $N \in \mathscr{F}$ with $\mu(N) = 0$ such that $f_n(x) \to f(x)$ for every $x \notin N$.*
(2) *We say that $\{f_n\}_{n\in\mathbb{N}}$ converges to f almost uniformly and write $f_n \to f$ (a.u.) if there is a decreasing net $\{E_\alpha\}_{\alpha\in\Gamma} \subset \mathscr{F}$ with $\mu(E_\alpha) \downarrow 0$ such that f_n converges to f uniformly on every set $X \setminus E_\alpha$.*
(3) *We say that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in measure and write $f_n \xrightarrow{\mu} f$ if $\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \to 0$ for every $\varepsilon > 0$.*

Definition 3.2. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure.*

(1) *μ is said to be autocontinuous from above if $\mu(A \cup B_n) \to \mu(A)$ for each $A \in \mathscr{F}$ and each $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ with $\mu(B_n) \to 0$.*
(2) *μ is said to be autocontinuous from below if $\mu(A \setminus B_n) \to \mu(A)$ for each $A \in \mathscr{F}$ and each $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ with $\mu(B_n) \to 0$.*
(3) *μ is said to be autocontinuous if it is autocontinuous from above and below.*
(4) *μ is said to be null-continuous if $\mu(\bigcup_{n=1}^\infty N_n) = 0$ whenever $N_n \in \mathscr{F}$ and $\mu(N_n) = 0$ for all $n \in \mathbb{N}$.*
(5) *μ is said to be continuous from above if $\mu(A_n) \downarrow \mu(A)$ whenever $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfy $A_n \downarrow A$.*

We give a typical example of Riesz space-valued non-additive measures satisfying the properties given above.

Example 3.1. Denote by $\mathscr{L}_0[0,1]$ the Dedekind complete Riesz space of all equivalence classes of Lebesgue measurable, real-valued functions on $[0,1]$. Let K be a Lebesgue integrable, real-valued function on $[0,1]^2$ with $K(s,t) \geq 0$ for almost all $(s,t) \in [0,1]^2$. Define the vector-valued set function by $\lambda(A)(s) := \int_A K(s,t)dt$ for every Borel subset A of $[0,1]$ and almost all $s \in [0,1]$. Then, λ is an $\mathscr{L}_0[0,1]$-valued *order countably additive* Borel measure on $[0,1]$, that is, it holds that $\sum_{k=1}^n \lambda(A_k) \to \lambda(A)$ whenever $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of mutually disjoint Borel subsets of $[0,1]$ with $A = \bigcup_{n=1}^\infty A_n$. Let $\mu(A) := \sqrt{\lambda(A)} + \lambda(A)^2$ for every Borel subset A of $[0,1]$. Then, μ is an $\mathscr{L}_0[0,1]$-valued non-additive measure which is autocontinuous, null-continuous, and continuous from above.

The following proposition shows that the convergence in measure follows from the almost convergence or the almost uniform convergence under the frequently used "quasi-additivity" conditions of non-additive measures given above. See [4, 5, 6] for the Egoroff theorem, the Lebesgue theorem, the Riesz theorem, and the Lusin theorem concerning the convergence of measurable functions.

Proposition 3.1. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let f be an $\mathscr{F}$-measurable, real-valued function and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of such functions.*

(1) *If $\{f_n\}_{n\in\mathbb{N}}$ converges to f almost uniformly, then it converges in measure.*
(2) *Assume that μ is continuous from above. If $\{f_n\}_{n\in\mathbb{N}}$ converges to f almost everywhere, then it converges in measure.*
(3) *Assume that μ is null-continuous. If $\{f_n\}_{n\in\mathbb{N}}$ converges to f almost uniformly, then it converges almost everywhere.*

Proof. (1) Let $\{E_\alpha\}_{\alpha\in\Gamma}\subset\mathscr{F}$ be a decreasing net with $\mu(E_\alpha)\downarrow 0$ such that f_n converges to f uniformly on each set $X\setminus E_\alpha$. Fix $\varepsilon>0$. Then we have

$$\limsup_{n\to\infty}\mu(\{|f_n-f|\geq\varepsilon\})\leq\mu(E_\alpha)$$

for every $\alpha\in\Gamma$, so that $\mu(\{|f_n-f|\geq\varepsilon\})\to 0$.

(2) By assumption, there is a set $N\in\mathscr{F}$ with $\mu(N)=0$ such that $f_n(x)\to f(x)$ for every $x\notin N$. Fix $\varepsilon>0$. Then $\mu(\bigcap_{n=1}^\infty\bigcup_{k=n}^\infty\{|f_k-f|\geq\varepsilon\})=0$. Thus, by the continuity of μ from above, we have $\inf_{n\in\mathbb{N}}\mu(\bigcup_{k=n}^\infty\{|f_k-f|\geq\varepsilon\})=0$, so that $\mu(\{|f_n-f|\geq\varepsilon)\to 0$.

(3) It follows from assumption that $\mu(\bigcap_{n=1}^\infty\bigcup_{k=n}^\infty\{|f_k-f|\geq\varepsilon\})=0$ for every $\varepsilon>0$. For each $i\in\mathbb{N}$, let $N_i:=\bigcap_{n=1}^\infty\bigcup_{k=n}^\infty\{|f_k-f|\geq 1/i\}$ and $N:=\bigcup_{i=1}^\infty N_i$. Then $\mu(N_i)=0$ for each $i\in\mathbb{N}$, so that $\mu(N)=0$ since μ is null-continuous. Further, $f_n(x)\to f(x)$ for every $x\notin N$. ∎

Lemma 3.1. *Let $\mu:\mathscr{F}\to V$ be a non-additive measure. Assume that μ is autocontinuous. Let f be an $\mathscr{F}$-measurable, real-valued function on X and $\{f_n\}_{n\in\mathbb{N}}$ a uniformly essentially bounded sequence of such functions. If $\{f_n\}_{n\in\mathbb{N}}$ converges in measure to f, then $\{f_n,f\}_{n\in\mathbb{N}}$ is also uniformly essentially bounded.*

Proof. We have only to show that f is essentially bounded. By assumption, there is $r\in\mathbb{R}$ with $r>0$ such that $G_{f_n}(r)=0$ and $G_{f_n}(-r)=\mu(X)$ for all $n\in\mathbb{N}$. Let $A:=\{f>r+1\}$ and $B_n:=\{|f_n-f|>1\}$ for each $n\in\mathbb{N}$. Since $f_n\xrightarrow{\mu}f$, we have $\mu(B_n)\to 0$, so that the autocontinuity of μ implies $\mu(A\setminus B_n)\to\mu(A)$. Since $A\setminus B_n\subset\{f_n>r\}$ for all $n\in\mathbb{N}$, we have

$$0\leq G_f(r+1)=\liminf_{n\to\infty}\mu(A\setminus B_n)\leq\liminf_{n\to\infty}G_{f_n}(r)=0.$$

Thus, we have $G_f(r+1)=0$. In a similar way, we have $G_f(-r-1)=\mu(X)$. ∎

Theorem 3.1. *Let $\mu:\mathscr{F}\to V$ be a non-additive measure. The following conditions are equivalent:*

(i) *μ is autocontinuous;*

(ii) *The bounded convergence theorem holds for μ, that is, if a uniformly essentially bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathscr{F}$-measurable, real-valued functions on X converges in measure to an $\mathscr{F}$-measurable, real-valued function f on X, then it holds that $\int_X f_n d\mu\to\int_X f d\mu$.*

Proof. (i) ⇒ (ii): By Lemma 3.1, we may assume that there is a real number $r>0$ such that $G_{f_n}(r)=G_f(r)=0$ and $G_{f_n}(-r)=G_f(-r)=\mu(X)$ for all $n\in\mathbb{N}$, so that by Proposition 2.2, f_n and f are all Choquet integrable.

Let $g:=\{f\vee(-r)\}\wedge r$ and $g_n:=\{f_n\vee(-r)\}\wedge r$ $(n=1,2,\ldots)$. Then $G_f(t)=G_g(t)$ and $G_{f_n}(t)=G_{g_n}(t)$ for all $t\in\mathbb{R}$ and $n\in\mathbb{N}$. Thus, $\int_X f d\mu=\int_X g d\mu$ and $\int_X f_n d\mu=\int_X g_n d\mu$ for all $n\in\mathbb{N}$.

Fix $\varepsilon>0$. Since $|g(x)|\leq r$ for all $x\in X$, one can find an $\mathscr{F}$-measurable, simple function $h:X\to\mathbb{R}$ such that $|h(x)|\leq r$ and $|h(x)-g(x)|<\varepsilon$ for all $x\in X$. Then by Proposition 2.1, it holds that

$$\left|\int_X h d\mu-\int_X g d\mu\right|\leq\varepsilon\mu(X). \tag{3.1}$$

We first prove that there is $\{p_n\}_{n\in\mathbb{N}} \subset V$ with $p_n \downarrow 0$ such that for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$, it holds that

$$\mu(\{h > t\}) \leq \mu(\{g_n > t - 2\varepsilon\}) + p_n. \tag{3.2}$$

To prove this, let $B_n := \{|g_n - h| > 2\varepsilon\}$ for all $n \in \mathbb{N}$. Since $|g_n(x) - g(x)| \leq |f_n(x) - f(x)|$ for all $x \in X$, it follows that $g_n \xrightarrow{\mu} g$. Further, since $|h(x) - g(x)| < \varepsilon$ for all $x \in X$, we have $\{|g_n - h| > 2\varepsilon\} \subset \{|g_n - g| > \varepsilon\}$ for all $n \in \mathbb{N}$. Thus, it holds that $\mu(\{|g_n - h| > 2\varepsilon\}) \to 0$, so that $\mu(B_n) \to 0$. Since the family $\{\{h > t\} : t \in \mathbb{R}\}$ consists of finitely many sets, say $A_1, A_2, \ldots, A_m$, the autocontinuity of μ from below shows that for each $k = 1, 2, \ldots, m$, there is $\{p_n^{(k)}\}_{n\in\mathbb{N}} \subset V$ with $p_n^{(k)} \downarrow 0$ such that $\mu(A_k) \leq \mu(A_k \setminus B_n) + p_n^{(k)}$ for every $n \in \mathbb{N}$. Let $p_n := \sup_{1\leq k\leq m} p_n^{(k)}$ for each $n \in \mathbb{N}$. Then $p_n \downarrow 0$, and for every $k = 1, \ldots, m$ and $n \in \mathbb{N}$, it holds that

$$\mu(A_k) \leq \mu(A_k \setminus B_n) + p_n. \tag{3.3}$$

Take $n \in \mathbb{N}$ and $t \in \mathbb{R}$ arbitrarily. Since $\{h > t\} = A_{k_0}$ for some k_0 $(1 \leq k_0 \leq m)$ and $\{h > t\} \setminus B_n \subset \{g_n > t - 2\varepsilon\}$, the desired inequality (3.2) follows from (3.3).

In a similar way, there is $\{q_n\}_{n\in\mathbb{N}} \subset V$ with $q_n \downarrow 0$ such that for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$, it holds that

$$\mu(\{g_n > t + 2\varepsilon\}) \leq \mu(\{h > t\}) + q_n. \tag{3.4}$$

Let $u_n := p_n \vee q_n$ for each $n \in \mathbb{N}$. Then $u_n \downarrow 0$. Fix $n \in \mathbb{N}$. Since $|h(x)| \leq r$ for all $x \in X$, by Proposition 2.2, h is Choquet integrable and

$$\int_X h\,d\mu = \int_0^r \mu(\{h > t\})dt + \int_{-r}^0 \big\{\mu(\{h > t\}) - \mu(X)\big\}dt. \tag{3.5}$$

Then, by (3.2) we have

$$\int_X h\,d\mu \leq \int_0^r G_{g_n}(t - 2\varepsilon)dt + \int_{-r}^0 \big\{G_{g_n}(t - 2\varepsilon) - \mu(X)\big\}\,dt + 2ru_n. \tag{3.6}$$

On the other hand, it holds that

$$\begin{aligned} \int_X (g_n + 2\varepsilon)d\mu &= \int_0^{r+2\varepsilon} G_{g_n+2\varepsilon}(t)dt + \int_{-r+2\varepsilon}^0 \big\{G_{g_n+2\varepsilon}(t) - \mu(X)\big\}\,dt \\ &= \int_0^{r+2\varepsilon} G_{g_n}(t - 2\varepsilon)dt + \int_{-r+2\varepsilon}^0 \big\{G_{g_n}(t - 2\varepsilon) - \mu(X)\big\}\,dt \\ &\geq \int_0^r G_{g_n}(t - 2\varepsilon)dt + \int_{-r}^0 \big\{G_{g_n}(t - 2\varepsilon) - \mu(X)\big\}\,dt. \end{aligned} \tag{3.7}$$

Since $\int_X (g_n + 2\varepsilon)d\mu = \int_X g_n d\mu + 2\varepsilon\mu(X)$ by Proposition 2.1, it follows from (3.6) and (3.7) that

$$\int_X h\,d\mu \leq \int_X g_n d\mu + 2\varepsilon\mu(X) + 2ru_n. \tag{3.8}$$

In a similar way, by (3.4) and (3.5), it holds that

$$\int_X h\,d\mu \geq \int_X g_n d\mu - 2\varepsilon\mu(X) - 2ru_n. \tag{3.9}$$

As shown in the second paragraph of this proof, f and g, as well as f_n and g_n, have the same Choquet integrals. Thus by (3.1) and (3.9), we have

$$\left| \int_X f_n d\mu - \int_X f d\mu \right| = \left| \int_X g_n d\mu - \int_X g d\mu \right| \leq 3\varepsilon\mu(X) + 2ru_n,$$

so that $\limsup_{n\to\infty} |\int_X f_n d\mu - \int_X f d\mu| \leq 3\varepsilon\mu(X)$. Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$, we have $\int_X f_n d\mu \to \int_X f d\mu$.

(ii) $\Rightarrow$ (i): Let $A, B_n \in \mathscr{F}$ $(n = 1, 2, \ldots)$ and assume that $\mu(B_n) \to 0$. Then the sequence $\{\chi_{A\cup B_n}\}_{n\in\mathbb{N}}$ is uniformly essentially bounded and $\chi_{A\cup B_n} \xrightarrow{\mu} \chi_A$. Thus, by assumption, we have $\mu(A \cup B_n) = \int_X \chi_{A\cup B_n} d\mu \to \int_X \chi_A d\mu = \mu(A)$, so that μ is autocontinuous from above. The autocontinuity of μ from below can be proved in a similar way. ∎

Remark 3.1. (1) Theorem 3.1 extends a part of [9, Theorem 3.3] to Riesz space-valued Choquet integrals.

(2) In Theorem 3.1, we do not need to assume the monotone function continuity property of the Riesz space V and the pointwise convergence of the integrands $\{f_n\}_{n\in\mathbb{N}}$; see [7, Theorem 4.15].

4. Another form of the bounded convergence theorem

In this section, we formalize another form of the bounded convergence theorem for Riesz space-valued Choquet integrals of functions converging in distribution. To this end, we need a notion of continuity of Riesz space-valued functions. Recall that a double sequence $\{u_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ of elements of V is called a *regulator* in V if it is order bounded and $u_{i,j} \downarrow 0$ for each $i \in \mathbb{N}$, that is, $u_{i,j} \geq u_{i,j+1}$ for each $i, j \in \mathbb{N}$ and $\inf_{j\in\mathbb{N}} u_{i,j} = 0$ for each $i \in \mathbb{N}$. Denote by Θ the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$.

Definition 4.1. [7, Definition 4.1] *Let $g : \mathbb{R} \to V$ be a function and $t_0 \in \mathbb{R}$. We say that g is continuous at t_0 if there is a regulator $\{u_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ in V with the property that for every $\theta \in \Theta$, one can find $\delta > 0$ such that for each $t \in \mathbb{R}$ with $|t - t_0| < \delta$, it holds that $|g(t) - g(t_0)| \leq \sup_{i\in\mathbb{N}} u_{i,\theta(i)}$. We say that g is continuous on $\mathbb{R}$ if it is continuous at every point of $\mathbb{R}$.*

Definition 4.2. [2] *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let f be an $\mathscr{F}$-measurable, real-valued function on X and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of such functions. We say that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in distribution and write $f_n \xrightarrow{G} f$ if $G_{f_n}(t) \to G_f(t)$ for every continuity point t of G_f.*

Remark 4.1. Since G_f is decreasing, it is easy to see that $\inf_{\varepsilon>0} G_f(t_0 - \varepsilon) = \sup_{\varepsilon>0} G_f(t_0 + \varepsilon) = G_f(t_0)$ holds for every continuity point t_0 of G_f.

Proposition 4.1. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. The following conditions are equivalent:*

(i) *μ is autocontinuous.*

(ii) *If a sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathscr{F}$-measurable, real-valued functions on X converges in measure to an $\mathscr{F}$-measurable, real-valued function f on X, then it converges in distribution to the same limit function f.*

Proof. (i) $\Rightarrow$ (ii): Assume that $f_n \xrightarrow{\mu} f$. Let $t_0 \in \mathbb{R}$ be a continuity point of G_f. Fix $\varepsilon > 0$. We first prove

$$\limsup_{n\to\infty} G_{f_n}(t_0) \le G_f(t_0 - \varepsilon). \tag{4.1}$$

Let $A := \{f > t_0 - \varepsilon\}$ and let $B_n := \{|f_n - f| > \varepsilon\}$ for all $n \in \mathbb{N}$. Since $f_n \xrightarrow{\mu} f$, we have $\mu(B_n) \to 0$, so that the autocontinuity of μ implies $\mu(A \cup B_n) \to \mu(A)$. Since $\{f_n > t_0\} \subset A \cup B_n$ for all $n \in \mathbb{N}$, it follows that $\limsup_{n\to\infty} G_{f_n}(t_0) \le \limsup_{n\to\infty} \mu(A \cup B_n) = \mu(A) = G_f(t_0 - \varepsilon)$.

In a similar way, it can be shown that

$$G_f(t_0 + \varepsilon) \le \liminf_{n\to\infty} G_{f_n}(t_0). \tag{4.2}$$

Since t_0 is a continuity point of G_f and $\varepsilon > 0$ is arbitrary, by (4.1), (4.2), and Remark 4.1, $G_{f_n}(t_0) \to G_f(t_0)$, which implies that $f_n \xrightarrow{G} f$.

(ii) $\Rightarrow$ (i): Let $A, B_n \in \mathscr{F}$ $(n = 1, 2, \ldots)$ and assume that $\mu(B_n) \to 0$. Since $\chi_{A\cup B_n} \xrightarrow{\mu} \chi_A$ and G_{χ_A} is continuous at $1/2$, it follows that $G_{\chi_{A\cup B_n}}(1/2) \to G_{\chi_A}(1/2)$. Thus, $\mu(A \cup B_n) \to \mu(A)$, so that μ is autocontinuous from above. In a similar way, the autocontinuity of μ from below can be proved. Thus the proof is complete. ∎

Remark 4.2. Proposition 4.1 extends [9, Theorem 3.1] to Riesz space-valued non-additive measures.

In [7], we introduced and imposed a new property of Riesz spaces concerning the cardinality of the set of points of discontinuity of a Riesz space-valued monotone function to obtain analogues of the monotone convergence theorem and the dominated convergence theorem of Riesz space-valued Choquet integrals. Recall that the set of points of discontinuity of a monotone real-valued function is at most countable. We have an example showing that this is not the case of a Riesz space-valued monotone function.

Example 4.1. [7, Example 4.4] For each $t \in [0, 1]$, define the element $h_t \in \mathbb{R}^{[0,1]}$ by

$$h_t(\xi) := \begin{cases} 1 & \text{if } 0 \le \xi \le t, \\ 0 & \text{if } \xi > t \end{cases}$$

for all $\xi \in [0, 1]$. Let $g(t)(\xi) := h_t(\xi)$ for all $t, \xi \in [0, 1]$. It is readily seen that $g : [0, 1] \to \mathbb{R}^{[0,1]}$ is increasing and discontinuous at every point of $(0, 1]$. Thus the set of points of discontinuity of g is uncountable.

Owing to Example 4.1, the following property defines a new class of Riesz spaces.

Definition 4.3. *We say that a Riesz space V has the monotone function continuity property if every V-valued monotone function defined on any closed, finite interval on $\mathbb{R}$ has at most countably many points of discontinuity.*

Many important function spaces and sequence spaces enjoy this property; see [7, Example 4.9].

Theorem 4.1. *Let $\mu : \mathscr{F} \to V$ be a non-additive measure. Let f be an essentially bounded, $\mathscr{F}$-measurable, real-valued function on X and $\{f_n\}_{n\in\mathbb{N}}$ a uniformly essentially bounded sequence of such functions. Assume that V has the monotone function continuity property. If $f_n \xrightarrow{G} f$, then it holds that $\int_X f_n d\mu \to \int_X f d\mu$.*

Proof. By assumption, we may assume that there is $r \in \mathbb{R}$ with $r > 0$ such that $G_{f_n}(r) = G_f(r) = 0$ and $G_{f_n}(-r) = G_f(-r) = \mu(X)$ for all $n \in \mathbb{N}$. Then, by Proposition 2.2, f_n and f are all Choquet integrable.

Since the functions G_{f_n} and G_f are decreasing, they are functions of bounded variation on $[-r, r]$. It is easy to see that $\{G_{f_n}\}_{n\in\mathbb{N}}$ satisfies conditions (i) and (ii) of [7, Theorem A.10]. Since V has the monotone function continuity property, G_f has at most countably many points of discontinuity, so that the convergence $f_n \xrightarrow{G} f$ implies condition (iii) of [7, Theorem A.10]. Thus we have $\int_0^r G_{f_n}(t)dt \to \int_0^r G_f(t)dt$. Similarly, we also have $\int_{-r}^0 \{G_{f_n}(t) - \mu(X)\}dt \to \int_{-r}^0 \{G_f(t) - \mu(X)\}dt$. Therefore $\int_X f_n d\mu \to \int_X f d\mu$ and the proof is complete. ∎

Remark 4.3. (1) Theorem 4.1 almost extends [2, Theorem 8.9] and [9, Theorem 3.2] to Riesz space-valued Choquet integrals.

(2) In Theorem 4.1, we do not need to assume the continuity of the measure μ and the pointwise convergence of the integrands $\{f_n\}_{n\in\mathbb{N}}$; see [7, Theorem 4.15].

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