**BULLETIN** *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

*Bull. Malays. Math. Sci. Soc. (2)* **34***(3) (2011), 445–454*

# Some Jensen's Type Inequalities for Log-Convex Functions of Selfadjoint Operators in Hilbert Spaces

[1,2]S. S. DRAGOMIR
[1]Mathematics, School of Engineering & Science, Victoria University,
PO Box 14428 Melbourne City, MC 8001, Australia
[2]School of Computational and Applied Mathematics
University of the Witwatersrand,
Private Bag 3, Wits 2050, Johannesburg, South Africa
sever.dragomir@vu.edu.au

**Abstract.** Some Jensen's type inequalities for Log-Convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

2010 Mathematics Subject Classification: 47A63, 47A99

Keywords and phrases: Selfadjoint operators, positive operators, Jensen's inequality, convex functions, functions of selfadjoint operators.

## 1. Introduction

Let $A$ be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism $\Phi$ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of $A$, denoted $Sp(A)$, and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$ as follows (see for instance [6, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator $A$.

*Communicated by* Mohammad Sal Moslehian.
*Received:* August 4, 2010; *Revised:* October 13, 2010.

If $A$ is a selfadjoint operator and $f$ is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on $H$. Moreover, if both $f$ and $g$ are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{1.1}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [6] and the references therein. For other results, see [7, 9, 11, 12]. For recent results, see [1, 3, 4].

## 2. Some Jensen's type inequalities for log-convex functions

The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [10] (see also [6, p. 5]).

**Theorem 2.1.** [10] *Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{2.1}$$

*for each $x \in H$ with $\|x\| = 1$.*

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of *log-convex functions*, namely functions $f : I \to (0, \infty)$ for which $\ln f$ is convex.

We observe that such functions satisfy the elementary inequality

$$f((1-t)a + tb) \leq [f(a)]^{1-t} [f(b)]^t$$

for any $a, b \in I$ and $t \in [0, 1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex function. However, obviously, there are functions that are convex but not log-convex.

As an imediate consequence of the Mond-Pečarić inequality above we can provide the following result.

**Theorem 2.2.** *Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$g(\langle Ax, x\rangle) \leq \exp \langle \ln g(A)x, x\rangle \leq \langle g(A)x, x\rangle \tag{2.2}$$

*for each $x \in H$ with $\|x\| = 1$.*

*Proof.* Consider the function $f := \ln g$, which is convex on $[m, M]$. Writing (2.1) for $f$ we get $\ln[g(\langle Ax, x\rangle)] \leq \langle \ln g(A)x, x\rangle$, for each $x \in H$ with $\|x\| = 1$, which, by taking the exponential, produces the first inequality in (2.2).

If we also use (2.1) for the exponential function, we get

$$\exp \langle \ln g(A)x, x\rangle \leq \langle \exp[\ln g(A)]x, x\rangle = \langle g(A)x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$ and the proof is complete. ∎

It is also important to observe that, as a special case of Theorem 2.1 we have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality.

**Theorem 2.3.** [8] *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. Then*

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r > 1$ *and* $x \in H$ *with* $\|x\| = 1$;
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for all* $0 < r < 1$ *and* $x \in H$ *with* $\|x\| = 1$;
(iii) *If $A$ is invertible, then* $\langle A^{-r} x, x\rangle \geq \langle Ax, x\rangle^{-r}$ *for all* $r > 0$ *and* $x \in H$ *with* $\|x\| = 1$.

Since the function $g(t) = t^{-r}$ for $r > 0$ is log-convex, we can improve the Hölder-McCarthy inequality as follows.

**Proposition 2.1.** *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible, then*

$$\langle Ax, x\rangle^{-r} \leq \exp\left\langle \ln\left(A^{-r}\right) x, x\right\rangle \leq \left\langle A^{-r} x, x\right\rangle \tag{2.3}$$

*for all* $r > 0$ *and* $x \in H$ *with* $\|x\| = 1$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [6, p. 57].

**Theorem 2.4.** *Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a convex function on $[m, M]$, then*

$$\langle f(A) x, x\rangle \leq \frac{M - \langle Ax, x\rangle}{M - m} \cdot f(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot f(M) \tag{2.4}$$

*for each* $x \in H$ *with* $\|x\| = 1$.

This result can be improved for log-convex functions as follows.

**Theorem 2.5.** *Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$\begin{aligned} \langle g(A) x, x\rangle &\leq \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A - m1_H}{M-m}}\right] x, x\right\rangle \\ &\leq \frac{M - \langle Ax, x\rangle}{M - m} \cdot g(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot g(M) \end{aligned} \tag{2.5}$$

*and*

$$\begin{aligned} g(\langle Ax, x\rangle) &\leq [g(m)]^{\frac{M - \langle Ax, x\rangle}{M-m}} [g(M)]^{\frac{\langle Ax, x\rangle - m}{M-m}} \\ &\leq \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A - m1_H}{M-m}}\right] x, x\right\rangle \end{aligned} \tag{2.6}$$

*for each* $x \in H$ *with* $\|x\| = 1$.

*Proof.* Observe that, by the log-convexity of $g$, we have

$$g(t) = g\left(\frac{M - t}{M - m} \cdot m + \frac{t - m}{M - m} \cdot M\right) \leq [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \tag{2.7}$$

for any $t \in [m, M]$.

Applying the property (1.1) for the operator $A$, we have that

$$\langle g(A) x, x \rangle \leq \langle \Psi(A) x, x \rangle$$

for each $x \in H$ with $\|x\| = 1$, where $\Psi(t) := [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}}, \quad t \in [m, M]$. This proves the first inequality in (2.5).

Now, observe that, by the weighted arithmetic mean-geometric mean inequality we have

$$[g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \leq \frac{M-t}{M-m} \cdot g(m) + \frac{t-m}{M-m} \cdot g(M)$$

for any $t \in [m, M]$.

Applying the property (1.1) for the operator $A$ we deduce the second inequality in (2.5).

Further on, if we use the inequality (2.7) for $t = \langle Ax, x \rangle \in [m, M]$ then we deduce the first part of (2.6).

Now, observe that the function $\Psi$ introduced above can be rearranged to read as

$$\Psi(t) = g(m) \left[ \frac{g(M)}{g(m)} \right]^{\frac{t-m}{M-m}}, \quad t \in [m, M]$$

showing that $\Psi$ is a convex function on $[m, M]$.

Applying Mond-Pečarić's inequality for $\Psi$ we deduce the second part of (2.6) and the proof is complete. ∎

The above result from Theorem 2.5 can be utilized to produce the following reverse inequality for negative powers of operators.

**Proposition 2.2.** *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $Sp(A) \subseteq [m, M]$ $(0 < m < M)$, then*

$$\langle A^{-r} x, x \rangle \leq \left\langle \left[ m^{\frac{M 1_H - A}{M-m}} M^{\frac{A - m 1_H}{M-m}} \right]^{-r} x, x \right\rangle \leq \frac{M - \langle Ax, x \rangle}{M-m} \cdot m^{-r} + \frac{\langle Ax, x \rangle - m}{M-m} \cdot M^{-r} \tag{2.8}$$

*and*

$$\langle Ax, x \rangle^{-r} \leq \left[ g(m)^{\frac{M - \langle Ax, x \rangle}{M-m}} g(M)^{\frac{\langle Ax, x \rangle - m}{M-m}} \right]^{-r} \leq \left\langle \left[ m^{\frac{M 1_H - A}{M-m}} M^{\frac{A - m 1_H}{M-m}} \right]^{-r} x, x \right\rangle \tag{2.9}$$

*for all $r > 0$ and $x \in H$ with $\|x\| = 1$.*

## 3. Jensen's inequality for differentiable log-convex functions

The following result provides a reverse for the Jensen type inequality (2.1).

**Theorem 3.1.** [5] *Let $J$ be an interval and $f : J \to \mathbb{R}$ be a convex and differentiable function on $\mathring{J}$ (the interior of $J$) whose derivative $f'$ is continuous on $\mathring{J}$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$(0 \leq) \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \leq \langle f'(A) Ax, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A) x, x \rangle \tag{3.1}$$

*for any* $x \in H$ *with* $\|x\| = 1$.

The following result may be stated.

**Proposition 3.1.** *Let* $J$ *be an interval and* $g : J \to \mathbb{R}$ *be a differentiable log-convex function on* $\mathring{J}$ *whose derivative* $g'$ *is continuous on* $\mathring{J}$. *If* $A$ *is a selfadjoint operator on the Hilbert space* $H$ *with* $Sp(A) \subseteq [m, M] \subset \mathring{J}$, *then*

$$(3.2) \qquad (1 \leq) \frac{\exp \langle \ln g(A) x, x \rangle}{g(\langle Ax, x \rangle)} \leq \exp \left[ \left\langle g'(A) [g(A)]^{-1} Ax, x \right\rangle - \langle Ax, x \rangle \cdot \left\langle g'(A) [g(A)]^{-1} x, x \right\rangle \right]$$

*for each* $x \in H$ *with* $\|x\| = 1$.

*Proof.* It follows by the inequality (3.1) written for the convex function $f = \ln g$ that

$$\langle \ln g(A) x, x \rangle \leq \ln g(\langle Ax, x \rangle) + \left\langle g'(A) [g(A)]^{-1} Ax, x \right\rangle - \langle Ax, x \rangle \cdot \left\langle g'(A) [g(A)]^{-1} x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

Now, taking the exponential and dividing by $g(\langle Ax, x \rangle) > 0$ for each $x \in H$ with $\|x\| = 1$, we deduce the desired result (3.2). ∎

**Remark 3.1.** Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible, then

$$(3.3) \qquad (1 \leq) \langle Ax, x \rangle^r \exp \left\langle \ln \left( A^{-r} \right) x, x \right\rangle \leq \exp \left[ r \left( \langle Ax, x \rangle \cdot \left\langle A^{-1} x, x \right\rangle - 1 \right) \right]$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following result that provides both a refinement and a reverse of the multiplicative version of Jensen's inequality can be stated as well.

**Theorem 3.2.** *Let* $J$ *be an interval and* $g : J \to \mathbb{R}$ *be a log-convex differentiable function on* $\mathring{J}$ *whose derivative* $g'$ *is continuous on* $\mathring{J}$. *If* $A$ *is a selfadjoint operator on the Hilbert space* $H$ *with* $Sp(A) \subseteq [m, M] \subset \mathring{J}$, *then*

$$(3.4) \qquad 1 \leq \left\langle \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} (A - \langle Ax, x \rangle 1_H) \right] x, x \right\rangle \leq \frac{\langle g(A) x, x \rangle}{g(\langle Ax, x \rangle)} \leq \left\langle \exp \left[ g'(A) [g(A)]^{-1} (A - \langle Ax, x \rangle 1_H) \right] x, x \right\rangle$$

*for each* $x \in H$ *with* $\|x\| = 1$, *where* $1_H$ *denotes the identity operator on* $H$.

*Proof.* It is well known that if $h : J \to \mathbb{R}$ is a convex differentiable function on $\mathring{J}$, then the following *gradient inequality* holds

$$h(t) - h(s) \geq h'(s)(t - s)$$

for any $t, s \in \mathring{J}$.

Now, if we write this inequality for the convex function $h = \ln g$, then we get

$$(3.5) \qquad \ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)} (t - s)$$

which is equivalent with

$$g(t) \geq g(s)\exp\left[\frac{g'(s)}{g(s)}(t-s)\right] \tag{3.6}$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax, x\rangle \in [m, M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (3.6), then we get

$$g(t) \geq g(\langle Ax, x\rangle)\exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(t - \langle Ax, x\rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (1.1) for the operator $A$ and the Mond-Pečarić inequality for the exponential function, we can state the following inequality that is of interest in itself as well:

$$\begin{aligned} \langle g(A)y, y\rangle &\geq g(\langle Ax, x\rangle)\left\langle \exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(A - \langle Ax, x\rangle 1_H)\right]y, y\right\rangle \\ &\geq g(\langle Ax, x\rangle)\exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(\langle Ay, y\rangle - \langle Ax, x\rangle)\right] \end{aligned} \tag{3.7}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Further, if we put $y = x$ in (3.7), then we deduce the first and the second inequality in (3.4).

Now, if we replace $s$ with $t$ in (3.6) we can also write the inequality

$$g(t)\exp\left[\frac{g'(t)}{g(t)}(s-t)\right] \leq g(s)$$

which is equivalent with

$$g(t) \leq g(s)\exp\left[\frac{g'(t)}{g(t)}(t-s)\right] \tag{3.8}$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax, x\rangle \in [m, M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (3.8), then we get

$$g(t) \leq g(\langle Ax, x\rangle)\exp\left[\frac{g'(t)}{g(t)}(t - \langle Ax, x\rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (1.1) for the operator $A$, then we can state the following inequality that is of interest in itself as well:

$$\langle g(A)y, y\rangle \leq g(\langle Ax, x\rangle)\left\langle \exp\left[g'(A)[g(A)]^{-1}(A - \langle Ax, x\rangle 1_H)\right]y, y\right\rangle \tag{3.9}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we put $y = x$ in (3.9), then we deduce the last inequality in (3.4). ∎

**Remark 3.2.** Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible, then

$$1 \leq \left\langle \exp\left[r\left(1_H - \langle Ax, x\rangle^{-1} A\right)\right]x, x\right\rangle \tag{3.10}$$

$$\leq \left\langle A^{-r}x, x\right\rangle \langle Ax, x\rangle^{r} \leq \left\langle \exp\left[r\left(1_H - \langle Ax, x\rangle A^{-1}\right)\right] x, x\right\rangle$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse inequality may be proven as well.

**Theorem 3.3.** *Let $J$ be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative $g'$ is continuous on $\mathring{J}$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $Sp\,(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$(1 \leq)\frac{\left\langle [g\,(M)]^{\frac{A-m1_H}{M-m}}\,[g\,(m)]^{\frac{M1_H-A}{M-m}}\,x, x\right\rangle}{\langle g\,(A)\,x, x\rangle} \tag{3.11}$$
$$\leq \frac{\left\langle g\,(A)\exp\left[\frac{(M1_H-A)(A-m1_H)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]x, x\right\rangle}{\langle g\,(A)\,x, x\rangle}$$
$$\leq \exp\left[\frac{1}{4}\,(M-m)\left(\frac{g'\,(M)}{g\,(M)} - \frac{g'\,(m)}{g\,(m)}\right)\right]$$

*for each $x \in H$ with $\|x\| = 1$.*

*Proof.* Utilising the inequality (3.5) we have successively

$$\frac{g\,((1-\lambda)\,t + \lambda s)}{g\,(s)} \geq \exp\left[(1-\lambda)\,\frac{g'\,(s)}{g\,(s)}\,(t-s)\right] \tag{3.12}$$

and

$$\frac{g\,((1-\lambda)\,t + \lambda s)}{g\,(t)} \geq \exp\left[-\lambda\frac{g'\,(t)}{g\,(t)}\,(t-s)\right] \tag{3.13}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0, 1]$.

Now, if we take the power $\lambda$ in the inequality (3.12) and the power $1-\lambda$ in (3.13) and multiply the obtained inequalities, we deduce

$$\frac{[g\,(t)]^{1-\lambda}\,[g\,(s)]^{\lambda}}{g\,((1-\lambda)\,t + \lambda s)} \leq \exp\left[(1-\lambda)\,\lambda\left(\frac{g'\,(t)}{g\,(t)} - \frac{g'\,(s)}{g\,(s)}\right)(t-s)\right] \tag{3.14}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0, 1]$.

Further on, if we choose in (3.14) $t = M, s = m$ and $\lambda = (M-u)/(M-m)$, then, from (3.14) we get the inequality

$$\frac{[g\,(M)]^{\frac{u-m}{M-m}}\,[g\,(m)]^{\frac{M-u}{M-m}}}{g\,(u)} \leq \exp\left[\frac{(M-u)\,(u-m)}{M-m}\left(\frac{g'\,(M)}{g\,(M)} - \frac{g'\,(m)}{g\,(m)}\right)\right] \tag{3.15}$$

which, together with the inequality

$$\frac{(M-u)\,(u-m)}{M-m} \leq \frac{1}{4}\,(M-m)$$

produce

$$[g\,(M)]^{\frac{u-m}{M-m}}\,[g\,(m)]^{\frac{M-u}{M-m}} \tag{3.16}$$
$$\leq g\,(u)\exp\left[\frac{(M-u)\,(u-m)}{M-m}\left(\frac{g'\,(M)}{g\,(M)} - \frac{g'\,(m)}{g\,(m)}\right)\right]$$

$$\leq g(u)\exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]$$

for any $u\in[m,M]$.

If we apply the property (1.1) to the inequality (3.16) and for the operator $A$ we deduce the desired result. ∎

**Remark 3.3.** Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $Sp(A)\subseteq[m,M]\,(0<m<M)$, then

$$(1\leq)\frac{\left\langle [g(M)]^{\frac{r(m1_H-A)}{M-m}}[g(m)]^{\frac{r(A-M1_H)}{M-m}}x,x\right\rangle}{\langle A^{-r}x,x\rangle} \tag{3.17}$$
$$\leq\frac{\left\langle A^{-r}\exp\left[\frac{r(M1_H-A)(A-m1_H)}{Mm}\right]x,x\right\rangle}{\langle A^{-r}x,x\rangle}\leq\exp\left[\frac{1}{4}r\frac{(M-m)^2}{mM}\right]$$

## 4. Applications for Ky Fan's inequality

Consider the function $g:(0,1)\to\mathbb{R}$, $g(t)=((1-t)/(t))^r$, $r>0$. Observe that for the new function $f:(0,1)\to\mathbb{R}$, $f(t)=\ln g(t)$ we have

$$f'(t)=\frac{-r}{t(1-t)}\quad\text{and}\quad f''(t)=\frac{2r\left(\frac{1}{2}-t\right)}{t^2(1-t)^2}\quad\text{for}\quad t\in(0,1)$$

showing that the function $g$ is log-convex on the interval $(0,1/2)$.

If $p_i>0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and $t_i\in(0,1/2)$ for $i\in\{1,...,n\}$, then by applying the Jensen inequality for the convex function $f$ (with $r=1$) on the interval $(0,1/2)$ we get

$$\frac{\sum_{i=1}^n p_it_i}{1-\sum_{i=1}^n p_it_i}\geq\prod_{i=1}^n\left(\frac{t_i}{1-t_i}\right)^{p_i}, \tag{4.1}$$

which is the weighted version of the celebrated *Ky Fan's inequality*, see [2, p. 3].

This inequality is equivalent with

$$\prod_{i=1}^n\left(\frac{1-t_i}{t_i}\right)^{p_i}\geq\frac{1-\sum_{i=1}^n p_it_i}{\sum_{i=1}^n p_it_i},$$

where $p_i>0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and $t_i\in(0,1/2)$ for $i\in\{1,...,n\}$.

By the weighted arithmetic mean-geometric mean inequality we also have that

$$\sum_{i=1}^n p_i(1-t_i)t_i^{-1}\geq\prod_{i=1}^n\left(\frac{1-t_i}{t_i}\right)^{p_i}$$

giving the double inequality

$$\sum_{i=1}^n p_i(1-t_i)t_i^{-1}\geq\prod_{i=1}^n\left((1-t_i)t_i^{-1}\right)^{p_i}\geq\sum_{i=1}^n p_i(1-t_i)\left(\sum_{i=1}^n p_it_i\right)^{-1}. \tag{4.2}$$

The following operator inequalities generalizing (4.2) may be stated.

**Proposition 4.1.** *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $Sp(A) \subset (0, 1/2)$, then*

$$\left\langle \left(A^{-1}(1_H - A)\right)^r x, x \right\rangle \geq \exp\left\langle \ln\left(A^{-1}(1_H - A)\right)^r x, x \right\rangle \geq \left(\langle (1_H - A)x, x\rangle \langle Ax, x\rangle^{-1}\right)^r \tag{4.3}$$

*for each $x \in H$ with $\|x\| = 1$ and $r > 0$.*

*In particular,*

$$\left\langle A^{-1}(1_H - A)x, x \right\rangle \geq \exp\left\langle \ln\left(A^{-1}(1_H - A)\right)x, x \right\rangle \geq \langle (1_H - A)x, x\rangle \langle Ax, x\rangle^{-1} \tag{4.4}$$

*for each $x \in H$ with $\|x\| = 1$.*

The proof follows by Theorem 2.2 applied for the log-convex function $g(t) = ((1-t)(t))^r, r > 0, t \in (0, 1/2)$.

**Proposition 4.2.** *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $Sp(A) \subseteq [m, M] \subset \left(0, \frac{1}{2}\right)$, then*

$$\begin{aligned} &\left\langle \left((1_H - A)A^{-1}\right)^r x, x \right\rangle \\ &\leq \left\langle \left[ \left(\frac{1-m}{m}\right)^{\frac{r(M1_H - A)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(A - m1_H)}{M-m}} \right] x, x \right\rangle \\ &\leq \frac{M - \langle Ax, x\rangle}{M-m} \cdot \left(\frac{1-m}{m}\right)^r + \frac{\langle Ax, x\rangle - m}{M-m} \cdot \left(\frac{1-M}{M}\right)^r \end{aligned} \tag{4.5}$$

*and*

$$\begin{aligned} \left(\frac{1 - \langle Ax, x\rangle}{\langle Ax, x\rangle}\right)^r &\leq \left(\frac{1-m}{m}\right)^{\frac{r(M - \langle Ax, x\rangle)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(\langle Ax, x\rangle - m)}{M-m}} \\ &\leq \left\langle \left[ \left(\frac{1-m}{m}\right)^{\frac{r(M1_H - A)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(A - m1_H)}{M-m}} \right] x, x \right\rangle \end{aligned} \tag{4.6}$$

*for each $x \in H$ with $\|x\| = 1$ and $r > 0$.*

The proof follows by Theorem 2.5 applied for the log-convex function $g(t) = ((1-t)(t))^r, r > 0, t \in (0, 1/2)$.

Finally we have the following.

**Proposition 4.3.** *Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $Sp(A) \subset (0, 1/2)$, then*

$$\begin{aligned} (1 \leq) &\frac{\exp\left\langle \ln\left((1_H - A)A^{-1}\right)^r x, x \right\rangle}{\left((1 - \langle Ax, x\rangle)\langle Ax, x\rangle^{-1}\right)^r} \\ &\leq \exp\left[ r\left( \langle Ax, x\rangle \cdot \left\langle A^{-1}(1_H - A)^{-1} x, x \right\rangle - \left\langle (1_H - A)^{-1} x, x \right\rangle \right) \right] \end{aligned} \tag{4.7}$$

*and*

$$1 \le \left\langle \exp\left[ r\left(1 - \langle Ax, x\rangle\right)^{-1} \left(1_H - \langle Ax, x\rangle^{-1} A\right)\right] x, x \right\rangle \le \frac{\left\langle \left( (1_H - A) A^{-1} \right)^r x, x \right\rangle}{\left( (1 - \langle Ax, x\rangle) \langle Ax, x\rangle^{-1} \right)^r} \le \left\langle \exp\left[ r (1_H - A)^{-1} \left( \langle Ax, x\rangle A^{-1} - 1_H \right)\right] x, x \right\rangle \tag{4.8}$$

*for each* $x \in H$ *with* $\|x\| = 1$ *and* $r > 0$.

## References


[1] S. Abramovich, S. Ivelić and J. E. Pečarić, Improvement of Jensen-Steffensen's inequality for superquadratic functions, *Banach J. Math. Anal.* **4** (2010), no. 1, 159–169.

[2] E. F. Beckenbach and R. Bellman, *Inequalities*, Second revised printing. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Band 30 Springer, New York, Inc., 1965.

[3] S. S. Dragomir, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint, *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [Online: http://www.staff.vu.edu.au/RGMIA/v11(E).asp].

[4] S. S. Dragomir, Some new Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 12. [Online: http://www.staff.vu.edu.au/RGMIA/v11(E).asp].

[5] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 15. [Online: http://www.staff.vu.edu.au/RGMIA/v11(E).asp].

[6] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[7] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications, *Linear Algebra Appl.* **418** (2006), no. 2–3, 551–564.

[8] C. A. McCarthy, $c_p$, *Israel J. Math.* **5** (1967), 249–271.

[9] J. Mićić, Y. Seo, S. Takahasi and M. Tominaga, Inequalities of Furuta and Mond-Pečarić, *Math. Inequal. Appl.* **2** (1999), no. 1, 83–111.

[10] B. Mond and J. E. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.* **19** (1993), no. 3, 405–420.

[11] B. Mond and J. E. Pečarić, On some operator inequalities, *Indian J. Math.* **35** (1993), no. 3, 221–232.

[12] B. Mond and J. E. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.* **46** (1994), 155–166.