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Analytical Approximation to Solutions of Singularly Perturbed Boundary Value Problems

[1]FAZHAN GENG AND [2]MINGGEN CUI
[1]Department of Mathematics, Changshu Institute of Technology,
Changshu, Jiangsu 215500, China
[2]Department of Mathematics, Harbin Institute of Technology,
Weihai, Shandong 264209, China
[1]gengfazhan@sina.com, [2]cmgyfs@263.net

Abstract. In this paper, a computational method is presented for solving a class of nonlinear singularly perturbed two-point boundary value problems with a boundary layer at the left of the underlying interval. First a zeroth order asymptotic expansion for the solution of the given singularly perturbed boundary value problem is constructed. Then the reduced terminal value problem is solved analytically using reproducing kernel Hilbert space method. This method is effective and easy to implement. Two numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other not only in the boundary layer, but also away from the layer.

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1. Introduction

Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, optimal control, etc. It is well known that the solution of singularly perturbed boundary value problem has a multiscale character; that is, there are thin transition layers where the solution varies rapidly, while away from the layers the solution behaves regularly and varies slowly.

Such problems have been investigated by many researchers. The existence and uniqueness of such problems are discussed in [13, 19]. The numerical treatment of singularly perturbed problems present some major computational difficulties, and in recent years a large number of special-purpose methods have been proposed to

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provide accurate numerical solutions. For the past two decades, many numerical method have appeared in the literature. Most notable among these are collocation methods [2, 15], finite-difference methods [14, 18], finite-element methods [20], boundary-value techniques [22], initial-value techniques [17, 21], Spline techniques [3, 4], and so on. However, there are few effective analytical methods to find the solutions of such problems. Mohamed El-Gamel and John R. Cannon gave the solution of a singularly perturbed boundary value problem via the Sinc-Galerkin method [10].

In this paper, we consider the following nonlinear singularly perturbed two-point boundary value problem in the reproducing kernel Hilbert space

$$\begin{cases} \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) + N(u) = f(x), \quad 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $f(x) \in W_2^1[0,1]$, $N(u)$ is a nonlinear function of u and $N(u)$ is continuous with respect to u, $a(x)$, $b(x)$ are continuous and $a(x) \geq \alpha > 0$, $b(x) \geq 0$, α is constant. For $0 < \varepsilon \ll 1$, the problem (1.1) is singularly perturbed and has a solution with a boundary layer at $x = 0$, where the size of the layer is of order $O(\varepsilon|Ln\varepsilon|)$ (see [6] for details).

Reproducing kernel Hilbert spaces (RKHS) are wonderful objects and can be used in a wide variety of curve fitting, function estimation and model description, differential equation, probability, statistics, and so on [1, 5]. Recently, using RKHS method, we discussed singular linear two-point boundary value problem, singular nonlinear two-point periodic boundary value problem, nonlinear system of boundary value problems and nonlinear Burgers equation [7, 8, 11, 12]. Nowadays, kernel method is one of the fastest growing and most exciting areas in machine learning.

2. The solution of the singularly perturbed problem (1.1)

In this section the asymptotic expansion approximation to the solution of singularly perturbed boundary value problem (SPBVP) (1.1) is constructed.

Let $u(x)$ and $u_0(x)$ be the solutions of SPBVP (1.1) and its reduced problem, respectively

$$\begin{cases} a(x)u_0'(x) - b(x)u_0(x) + N(u_0) = f(x), \ \ 0 < x < 1, \\ u_0(1) = 0. \end{cases} \tag{2.1}$$

Then, the zeroth order asymptotic expansion approximation

$$u_{as} = u_0(x) + v_0(x) \tag{2.2}$$

where

$$v_0(x) = [u(0) - u_0(0)]e^{-a(0)x/\varepsilon} = -u_0(0)e^{-a(0)x/\varepsilon}.$$

Theorem 2.1. *The zeroth order asymptotic expansion approximation u_{as} satisfies the inequality*

$$|u(x) - u_{as}(x)| \leq c\varepsilon, \quad \textit{for} \quad 0 \leq x \leq 1,$$

where $u(x)$ is the solution of SPBVP (1.1).

For the proof see [9].

In order to obtain the zeroth order asymptotic expansion approximation u_{as}, it remains only to obtain the solution $u_0(x)$ of terminal value problem (TVP) (2.1).

In the following section, we will give the solution of TVP (2.1) analytically using RKHS method.

3. The solution of TVP (2.1)

In this section, the solution to the TVP (2.1) will be obtained in the RKHS $W_2^2[0,1]$. The space $W_2^2[0,1]$ will be defined in the following.

Put $Lu_0(x) \equiv a(x)u_0'(x) - b(x)u_0(x)$ and write $F(x,u_0(x)) = f(x) - N(u_0(x))$ simply. Then TVP (2.1) can be converted into the following form

$$\begin{cases} Lu_0(x) = F(x,u_0(x)), & 0 \le x \le 1, \\ u_0(1) = 0, \end{cases} \tag{3.1}$$

where $u_0(x) \in W_2^2[0,1]$, $F(x,u_0(x)) \in W_2^1[0,1]$.

3.1. The RKHS $W_2^2[0,1]$

The inner product space $W_2^2[0,1]$ is defined as $W_2^2[0,1] = \{u(x) \mid u, u'$ are absolutely continuous real valued functions, $u, u', u'' \in L^2[0,1], u(1) = 0\}$. The inner product in $W_2^2[0,1]$ is given by

$$(u(x), v(x))_{W_2^2} = u(0)v(0) + \int_0^1 u''v''dx, \tag{3.2}$$

and the norm $\| u \|_{W_2^2}$ is denoted by $\| u \|_{W_2^2} = \sqrt{(u,u)_{W_2^2}}$, where $u, v \in W_2^2[0,1]$.

Theorem 3.1. *The space $W_2^2[0,1]$ is a* RKHS. *That is, for any $u(y) \in W_2^2[0,1]$ and each fixed $x \in [0,1]$, there exists $R_x(y) \in W_2^2[0,1]$, $y \in [0,1]$, such that $(u(y), R_x(y))_{W_2^2} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} \frac{(-1+x)\left[-6+\left(6-2x+x^2\right)y+y^3\right]}{6}, & y \le x \\ \frac{(-1+y)\left[-6+x^3+x\left(6-2\,y+y^2\right)\right]}{6}, & y > x. \end{cases} \tag{3.3}$$

The proof of Theorem 3.1 is given in Appendix A.

3.2. The RKHS $W_2^1[0,1]$

The inner product space $W_2^1[0,1]$ is defined by $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real valued function, $u, u' \in L^2[0,1]\}$. The inner product and norm in $W_2^1[0,1]$ are given respectively by

$$(u(x), v(x))_{W_2^1} = \int_0^1 (uv + u'v')dx, \ \| u \|_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

where $u(x), v(x) \in W_2^1[0,1]$. In [16], the authors proved that $W_2^1[0,1]$ is a RKHS and its reproducing kernel is

$$\overline{R}_x(y) = \frac{1}{2\sinh(1)}\left[\cosh(x+y-1) + \cosh(|x-y|-1)\right].$$

3.3. The representation of the solution of TVP (3.1)

In this section, we will give the representation of analytical solution of TVP (3.1) and the implementation method in the RKHS $W_2^2[0,1]$.

In TVP (3.1), it is clear that $L : W_2^2[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{R}_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where L^* is the adjoint operator of L (L^* is defined as $< L^*u, v >=< u, Lv >$, $< \cdot, \cdot >$ denotes inner product). The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^2[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0,\ i = 1, 2, ...) \tag{3.4}$$

where $\{\beta_{ik}\}$ are coefficients of orthogonalization.

Theorem 3.2. *For* TVP (3.1)*, if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$*, then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete system of* $W_2^2[0,1]$ *and* $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.

Proof. Notice that

$$\begin{aligned}\psi_i(x) &= (L^*\varphi_i)(x) = ((L^*\varphi_i)(y), R_x(y))_{W_2^2} \\ &= (\varphi_i(y), L_y R_x(y))_{W_2^1} = L_y R_x(y)|_{y=x_i}.\end{aligned}$$

The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly, $\psi_i(x) \in W_2^2[0,1]$. For each fixed $u(x) \in W_2^2[0,1]$, let $(u(x), \psi_i(x))_{W_2^2} = 0$, $(i = 1, 2, ...)$, which means that,

$$(u(x), (L^*\varphi_i)(x))_{W_2^2} = (Lu(\cdot), \varphi_i(\cdot))_{W_2^1} = (Lu)(x_i) = 0. \tag{3.5}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1} (L^{-1} exists since the problem we discuss has a unique solution [1, 2]). So the proof of the Theorem 3.2 is complete. ∎

Theorem 3.3. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *and the solution of* TVP (3.1) *is unique, then the solution of* TVP (3.1) *satisfies the form*

$$u_0(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u_0(x_k))\overline{\psi}_i(x), \tag{3.6}$$

where $F(x,u) = f(x) - N(u)$.

Proof. Applying Theorem 3.2, it is easy to see that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^2[0,1]$. Note that $(v(x), \varphi_i(x)) = v(x_i)$ for each $v(x) \in W_2^1[0,1]$, hence we have

$$\begin{aligned}u_0(x) &= \sum_{i=1}^{\infty} (u_0(x), \overline{\psi}_i(x))_{W_2^2}\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(u_0(x), L^*\varphi_k(x))_{W_2^2}\overline{\psi}_i(x)\end{aligned}$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (Lu_0(x), \varphi_k(x))_{W_2^1} \overline{\psi}_i(x) \tag{3.7}$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (F(x, u_0(x)), \varphi_k(x)) \overline{\psi}_i(x)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u_0(x_k)) \overline{\psi}_i(x)$$

and the proof of the theorem is complete. ∎

Remark 3.1.
Case I: TVP (3.1) is linear, that is, $N(u_0(x)) = 0$. Then the analytical solution to TVP (3.1) can be obtained directly from (3.7).

Case II: TVP (3.1) is nonlinear. In this case, the analytical solution to TVP(3.1) can be obtained using the following method.

3.4. The implementation method

(3.6) can be denoted by

$$u_0(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x), \tag{3.8}$$

where $A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u_0(x_k))$. Let $x_1 = 1$, it follows that $F(x_1, u_0(x_1))$ is known. Considering the numerical computation, we put $u_{00}(x_1) = u_0(x_1)$ and define the n-term approximation to $u_0(x)$ by

$$u_{0n}(x) = \sum_{i=1}^{n} B_i \overline{\psi}_i(x), \tag{3.9}$$

where

$$\begin{aligned}
&B_1 = \beta_{11} F(x_1, u_{00}(x_1)),\\
&u_{01}(x) = B_1 \overline{\psi}_1(x),\\
&B_2 = \sum_{k=1}^{2} \beta_{2k} F(x_k, u_{0(k-1)}(x_k)),\\
&u_{02}(x) = \sum_{i=1}^{2} B_i \overline{\psi}_i(x),\\
&\ldots\ldots\\
&u_{0(n-1)}(x) = \sum_{i=1}^{n-1} B_i \overline{\psi}_i(x),\\
&B_n = \sum_{k=1}^{n} \beta_{nk} F(x_k, u_{0(k-1)}(x_k)).
\end{aligned} \tag{3.10}$$

Next, the convergence of $u_{0n}(x)$ will be proved.
Now, two lemmas are given first.

Lemma 3.1. *If $u(x) \in W_2^2[0,1]$, then there exists a positive constant c such that $|u(x)| \le c \parallel u(x) \parallel_{W_2^2}$ and $|u'(x)| \le c \parallel u(x) \parallel_{W_2^2}$.*

By Lemma 3.1, it is easy to obtain the following Lemma 3.2.

Lemma 3.2. *If $u_n \to \overline{u}(n \to \infty)$ in the sense of $\| \cdot \|_{W_2^2}$, $\| u_n \|_{W_2^2}$ is bounded, $x_n \to y(n \to \infty)$ and $F(x, u(x))$ is continuous, then $F(x_n, u_{n-1}(x_n)) \to F(y, \overline{u}(y))(n \to \infty)$.*

Theorem 3.4. *Suppose that $\| u_{0n} \|_{W_2^2}$ is bounded in (3.9) and TVP (3.1) has a unique solution. If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, then the n-term approximate solution $u_{0n}(x)$ derived from the above method converges to the analytical solution $u_0(x)$ of TVP (3.1) and*

$$u_0(x) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i(x), \tag{3.11}$$

where B_i is given by (3.10).

Proof. First of all, we will prove the convergence of $u_{0n}(x)$. From (3.9), we infer that

$$u_{0(n+1)}(x) = u_{0n}(x) + B_{n+1}\overline{\psi}_{n+1}(x). \tag{3.12}$$

The orthonormality of $\{\overline{\psi}_i\}_{i=1}^{\infty}$ yields that

$$\| u_{0(n+1)} \|_{W_2^2}^2 = \| u_{0n} \|_{W_2^2}^2 + (B_{n+1})^2 = \cdots = \sum_{i=1}^{n+1} (B_i)^2. \tag{3.13}$$

In terms of (3.13), it holds that $\| u_{0(n+1)} \| \geq \| u_{0n} \|_{W_2^2}$. Due to the condition that $\| u_{0n} \|_{W_2^2}$ is bounded, $\| u_{0n} \|$ is convergent and there exists a constant c such that

$$\sum_{i=1}^{\infty} (B_i)^2 = c.$$

This implies that

$$\{B_i\}_{i=1}^{\infty} \in l^2.$$

If $m > n$, then

$$\| u_{0m} - u_{0n} \|_{W_2^2}^2 = \| u_{0m} - u_{0(m-1)} + u_{0(m-1)} - u_{0(m-2)} + \cdots + u_{0(n+1)} - u_{0n} \|_{W_2^2}^2 .$$

In view of $(u_{0m} - u_{0(m-1)}) \perp (u_{0(m-1)} - u_{0(m-2)}) \perp \cdots \perp (u_{0(n+1)} - u_{0n})$, it follows that

$$\| u_{0m} - u_{0n} \|_{W_2^2}^2 = \| u_{0m} - u_{0(m-1)} \|_{W_2^2}^2 + \cdots + \| u_{0(n+1)} - u_{0n} \|_{W_2^2}^2 .$$

Furthermore

$$\| u_{0m} - u_{0(m-1)} \|_{W_2^2}^2 = (B_m)^2.$$

Consequently,

$$\| u_{0m} - u_{0n} \|_{W_2^2}^2 = \sum_{l=n+1}^{m} (B_l)^2 \to 0 \quad \text{as } n \to \infty.$$

The completeness of $W_2^2[0, 1]$ shows that $u_{0n} \to \overline{u}_0$ as $n \to \infty$ in the sense of $\| \cdot \|_{W_2^2}$.

Secondly, we will prove that $\overline{u}_0$ is the solution of TVP (3.1). Taking limits in (3.9), we get

$$\overline{u}_0(x) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i(x). \tag{3.14}$$

Note here that

$$L\overline{u}_0(x) = \sum_{i=1}^{\infty} B_i L\overline{\psi}_i(x)$$

and

$$\begin{aligned}(L\overline{u}_0)(x_n) &= \sum_{i=1}^{\infty} B_i (L\overline{\psi}_i, \varphi_n)_{W_2^1} \\ &= \sum_{i=1}^{\infty} B_i (\overline{\psi}_i, L^*\varphi_n)_{W_2^2} \\ &= \sum_{i=1}^{\infty} B_i (\overline{\psi}_i, \psi_n)_{W_2^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{n} \beta_{nj} (L\overline{u}_0)(x_j) &= \sum_{i=1}^{\infty} B_i (\overline{\psi}_i, \sum_{j=1}^{n} \beta_{nj}\psi_j)_{W_2^2} \\ &= \sum_{i=1}^{\infty} B_i (\overline{\psi}_i, \overline{\psi}_n)_{W_2^2} \\ &= B_n.\end{aligned} \tag{3.15}$$

If $n = 1$, then

$$(L\overline{u}_0)(x_1) = F(x_1, u_{00}(x_1)).$$

If $n = 2$, then

$$\beta_{21}(L\overline{u}_0)(x_1) + \beta_{22}(L\overline{u}_0)(x_2) = \beta_{21}F(x_1, u_{00}(x_1)) + \beta_{22}F(x_2, u_{01}(x_2)).$$

It is clear that

$$(L\overline{u}_0)(x_2) = F(x_2, u_{01}(x_2)).$$

Moreover, it is easy to see by induction that

$$(L\overline{u}_0)(x_j) = F(x_j, u_{0(j-1)}(x_j)), j = 1, 2, \cdots. \tag{3.16}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, for $\forall\ Y \in [0, 1]$, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that

$$x_{n_j} \to Y \quad \text{as } j \to \infty.$$

From (3.16), it is easy to see that $(L\overline{u}_0)(x_{n_j}) = F(x_{n_j}, u_{0(n_j-1)}(x_{n_j}))$. Let $j \to \infty$, by Lemma 3.2 and the continuity of $F(x, u_0(x))$, we have

$$(L\overline{u}_0)(Y) = F(Y, \overline{u}_0(Y)). \tag{3.17}$$

From (3.17), it follows that $\overline{u}_0(x)$ satisfies TVP (3.1). Since $\overline{\psi}_i(x) \in W_2^2[0, 1]$, clearly, $\overline{u}(Y)$ satisfies the boundary condition of TVP (3.1). That is, $\overline{u}_0(x)$ is the solution of

TVP (3.1). The application of the uniqueness of solution to TVP (3.1) then yields that

$$u_0(x) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i(x). \tag{3.18}$$

The proof is complete. ∎

Now the zeroth order asymptotic expansion approximation u_{as} can be obtained

$$u_{as} = u_0(x) + v_0(x) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i(x) - u_0(0) e^{\frac{-a(0)x}{/}\varepsilon} \tag{3.19}$$

and the approximate solution of SPBVP (1.1) can be obtained by the n-term intercept of u_{as} and

$$u_n = \sum_{i=1}^{n} B_i \overline{\psi}_i(x) - u_0(0) e^{-a(0)x/\varepsilon}.$$

4. Numerical examples

In this section, two numerical examples are studied to demonstrate the accuracy of the present method. These examples are computed using Mathematica 5.0. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 4.1. Consider the nonlinear singularly perturbed two-point boundary value problem

$$\begin{cases} \varepsilon u''(x) + \cos(x) u'(x) - x u(x) + x u^2 = f(x), & 0 < x < 1, \\ u(0) = 0, u(1) = 0, \end{cases}$$

where

$$f(x) = x \left(\frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} - x \right)^2 - x \left(\frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} - x \right) + \left(\frac{e^{-x/\epsilon}}{\left(1 - e^{-1/\epsilon}\right)\epsilon} - 1 \right) \cos(x) - \frac{e^{-x/\epsilon}}{\left(1 - e^{-1/\epsilon}\right)\epsilon}.$$

The true solution is

$$\frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}} - x.$$

Using our method, we choose 51 points on $[0, 1]$ and take $\varepsilon = 10^{-4}$, $\varepsilon = 10^{-6}$, $\varepsilon = 10^{-8}$ respectively. The numerical results are given in Tables 1, 2 and 3.

Example 4.2. Consider the linear singularly perturbed two-point boundary value problem

$$\begin{cases} \varepsilon u''(x) + e^x u'(x) + x^2 u(x) = f(x), & 0 < x < 1, \\ u(0) = 0, u(1) = 0 \end{cases}$$

where

$$f(x) = x^2 \left(\frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} - x^2 \right) + e^x \left(-2x + \frac{e^{(1-x)/\epsilon}}{(-1 + e^{1/\epsilon})\epsilon} \right) + \left(-2 - \frac{e^{(1-x)/\epsilon}}{(-1 + e^{1/\epsilon})\epsilon^2} \right) \epsilon.$$

The true solution is

$$\frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}} - x^2.$$

Using our method, we choose 51 points ($x_j = 1 - 0.02(j-1)$, $j = 1, 2, \cdots, n$, $n = 11, 51$) on $[0, 1]$ and take $\varepsilon = 10^{-4}$, $\varepsilon = 10^{-6}$ respectively. The numerical results are given in Figures 1 and 2. From Figures 1 and 2, it is shown that the larger the number of nodes we choose on $[0, 1]$, the higher the accuracy of solution.

Table 1. Numerical results for example 1($\varepsilon = 10^{-4}$).

x	True solution $u(x)$	Approximate solution u_{51}	Relative error
1.0E-08	9.998E-05	9.998E-05	1.7E-05
1.0E-05	0.0951526	0.0951499	2.7E-05
9.0E-05	0.5933400	0.5933240	2.7E-05
3.6E-04	0.9723160	0.9722890	2.7E-05
0.001	0.9989550	0.9989270	2.7E-05
0.08	0.9200000	0.9199750	2.7E-05
0.32	0.6800000	0.6799830	2.4E-05
0.48	0.5200000	0.5199880	2.2E-05
0.64	0.3600000	0.3599930	2.0E-05
0.80	0.2000000	0.1999960	1.8E-05
0.96	0.0400000	0.0399993	1.7E-05
0.99	0.0100000	0.0099986	1.3E-05

Table 2. Numerical results for example 1($\varepsilon = 10^{-6}$).

x	True solution $u(x)$	Approximate solution u_{51}	Relative error
1.0E-08	0.0099501	0.0099498	2.7E-05
1.0E-07	0.0951625	0.0951598	2.7E-05
9.0E-07	0.5934290	0.5934130	2.7E-05
3.6E-06	0.9726730	0.9726460	2.7E-05
0.00001	0.9999450	0.9999170	2.7E-05
0.08	0.9200000	0.9199750	2.7E-05
0.32	0.6800000	0.6799830	2.4E-05
0.48	0.5200000	0.5199880	2.2E-05
0.64	0.3600000	0.3599930	2.0E-05
0.80	0.2000000	0.1999960	1.8E-05
0.96	0.0400000	0.0399993	1.7E-05
0.99	0.0100000	0.0099998	1.7E-05

Table 3. Numerical results for example 1($\varepsilon = 10^{-8}$).

x	True solution $u(x)$	Approximate solution u_{51}	Relative error
1.0E-10	0.0099501	0.0099498	2.7E-05
1.0E-09	0.0951626	0.0951599	2.7E-05
9.0E-09	0.5934300	0.5934140	2.7E-05
3.6E-08	0.9726760	0.9726490	2.7E-05
1.0E-07	0.9999550	0.9999270	2.7E-05
0.08	0.9200000	0.9199750	2.7E-05
0.32	0.6800000	0.6799830	2.4E-05
0.48	0.5200000	0.5199880	2.2E-05
0.64	0.3600000	0.3599930	2.0E-05
0.80	0.2000000	0.1999960	1.8E-05
0.96	0.0400000	0.0399993	1.7E-05
0.99	0.0100000	0.0099983	1.7E-05

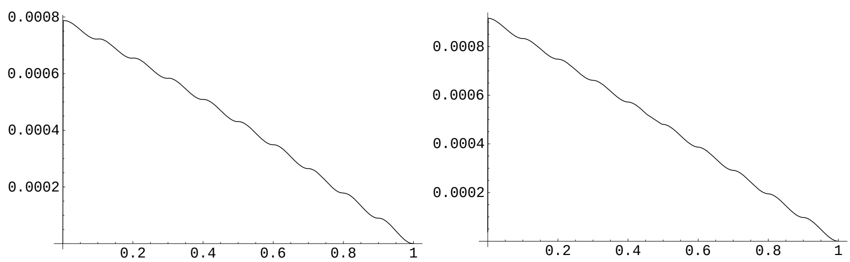

Figure 1. The absolute error between the true solution and approximate solution of Example 2 on $[0, 1]$ when $n = 11$, $\varepsilon = 10^{-4}$, 10^{-6} respectively.

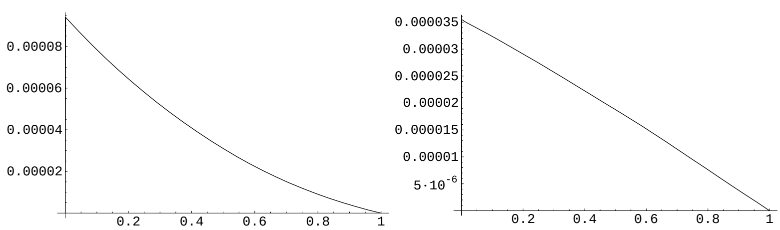

Figure 2. The absolute error between the true solution and approximate solution of Example 2 on $[0, 1]$ when $n = 51$, $\varepsilon = 10^{-4}$, 10^{-6} respectively.

Appendix A. The proof of Theorem 3.1

Through several integrations by parts for (3.2), then

$$(u(y), R_x(y))_{W_2^2} = u(0)R_x(0) + u(1)R_x(1) + \int_0^1 u(y)R_x^{(4)}(y)dy - u(y)R_x^{(3)}(y)|_0^1 + u'(y)R_x^{(2)}(y)|_0^1. \tag{A.1}$$

Since $R_x(y) \in W_2^2[0,1]$, it follows that

$$R_x(1) = 0. \tag{A.2}$$

Since $u \in W_2^2[0,1]$, $u(1) = 0$. If

$$R'_x(0) + R_x^{(3)}(0) = 0, R_x^{(2)}(0) = 0, \ \textit{and} \ R_x^{(2)}(1) = 0, \tag{A.3}$$

then (A.1) implies that

$$(u(y), R_x(y))_{W_2^2} = \int_0^1 u(y) R_x^{(4)}(y) dy.$$

For $\forall x \in [0,1]$, if $R_x(y)$ also satisfies

$$R_x^{(4)}(y) = \delta(y - x), \tag{A.4}$$

then

$$(u(y), R_x(y))_{W_2^2} = u(x).$$

Characteristic equation of (A.4) is given by

$$\lambda^4 = 0,$$

then we can obtain characteristic values $\lambda = 0$. So, let

$$R_x(y) = \begin{cases} c_1 + c_2 y + c_3 y^2 + c_4 y^3, & y \le x, \\ d_1 + d_2 y + d_3 y^2 + d_4 y^3, & y > x. \end{cases}$$

On the other hand, for (A.4), let $R_x(y)$ satisfy

$$R_x^{(k)}(x+0) = R_x^{(k)}(x-0), k = 0, 1, 2. \tag{A.5}$$

Integrating (A.4) from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $R_x^{(3)}(y)$ at $y = x$,

$$R_x^{(3)}(x+0) - R_x^{(3)}(x-0) = 1. \tag{A.6}$$

From (A.2), (A.3), (A.5), (A.6), the unknown coefficients of (3.3) can be obtained.

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