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Fuzzy Ideals in BE-Algebras

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Abstract. The fuzzification of ideals in BE-algebras is considered, and several properties are investigated. Characterizations of a fuzzy ideal are provided.

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1. Introduction

In 1966, Imai and Iséki [2, 3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. As a generalization of a BCK-algebra, Kim and Kim [4] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. In this paper, we consider the fuzzification of ideals in BE-algebras. We introduce the notion of fuzzy ideals in BE-algebras, and investigate related properties. We give characterizations of a fuzzy ideal in BE-algebras.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BE-algebra* we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms hold (see [4]):

(2.1) $(\forall x \in X)\ (x * x = 1)$;

(2.2) $(\forall x \in X)\ (x * 1 = 1)$;

(2.3) $(\forall x \in X)\ (1 * x = x)$;

(2.4) $(\forall x, y, z \in X)\ (x * (y * z) = y * (x * z))$. (exchange)

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A relation "$\leq$" on a BE-algebra X is defined by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 1). \tag{2.5}$$

A BE-algebra $(X; *, 1)$ is said to be *transitive* (see [1]) if it satisfies:

$$(\forall x, y, z \in X)\,(y * z \leq (x * y) * (x * z)). \tag{2.6}$$

A BE-algebra $(X; *, 1)$ is said to be *self distributive* (see [4]) if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = (x * y) * (x * z)). \tag{2.7}$$

Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [1]).

A nonempty subset I of a BE-algebra X is called an *ideal* of X (see [1]) if it satisfies:

$$(\forall x \in X)\ (\forall a \in I)\ (x * a \in I); \tag{2.8}$$

$$(\forall x \in X)\ (\forall a, b \in I)\ ((a * (b * x)) * x \in I). \tag{2.9}$$

3. Fuzzy ideals

In what follows, let X denote a BE-algebra unless otherwise specified.

Definition 3.1. *A fuzzy set μ in X is called a fuzzy ideal of X if it satisfies:*

$$(\forall x, y \in X)\ (\mu(x * y) \geq \mu(y)); \tag{3.1}$$

$$(\forall x, y, z \in X)\ (\mu((x * (y * z)) * z) \geq \min\{\mu(x), \mu(y)\}). \tag{3.2}$$

Theorem 3.1. *Let μ be a fuzzy set in X. Then μ is a fuzzy ideal of X if and only if it satisfies:*

$$(\forall \alpha \in [0,1])(U(\mu; \alpha) \neq \emptyset \implies U(\mu; \alpha) \text{ is an ideal of } X) \tag{3.3}$$

where $U(\mu; \alpha) := \{x \in X \mid \mu(x) \geq \alpha\}$.

Proof. Assume that μ is a fuzzy ideal of X. Let $\alpha \in [0, 1]$ be such that $U(\mu; \alpha) \neq \emptyset$. Let $x, y \in X$ be such that $y \in U(\mu; \alpha)$. Then $\mu(y) \geq \alpha$, and so $\mu(x * y) \geq \mu(y) \geq \alpha$ by (3.1). Thus $x * y \in U(\mu; \alpha)$. Let $x \in X$ and $a, b \in U(\mu; \alpha)$. Then $\mu(a) \geq \alpha$ and $\mu(b) \geq \alpha$. It follows from (3.2) that

$$\mu((a * (b * x)) * x) \geq \min\{\mu(a), \mu(b)\} \geq \alpha$$

so that $(a * (b * x)) * x \in U(\mu; \alpha)$. Hence $U(\mu; \alpha)$ is an ideal of X.

Conversely, suppose that μ satisfies (3.3). If $\mu(a * b) < \mu(b)$ for some $a, b \in X$, then $\mu(a * b) < \alpha_0 < \mu(b)$ by taking $\alpha_0 := (\mu(a * b) + \mu(b))/2$. Hence $a * b \notin U(\mu; \alpha_0)$ and $b \in U(\mu; \alpha_0)$, which is a contradiction. Let $a, b, c \in X$ be such that

$$\mu((a * (b * c)) * c) < \min\{\mu(a), \mu(b)\}.$$

Taking $\beta_0 := (\mu((a * (b * c)) * c)/2 + \min\{\mu(a), \mu(b)\})$, we have $\beta_0 \in [0, 1]$ and

$$\mu((a * (b * c)) * c) < \beta_0 < \min\{\mu(a), \mu(b)\}.$$

It follows that $a, b \in U(\mu; \beta_0)$ and $(a * (b * c)) * c \notin U(\mu; \beta_0)$. This is a contradiction, and therefore μ is a fuzzy ideal of X. ∎

Example 3.1. Let $X = \{1, a, b, c, d, 0\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra (see [4]).

(1) Let μ be a fuzzy set in X defined by

$$\mu(x) := \begin{cases} 0.7 & \text{if } x \in \{1, a, b\}, \\ 0.2 & \text{if } x \in \{c, d, 0\}. \end{cases}$$

Then

$$U(\mu; \alpha) = \begin{cases} \emptyset & \text{if } \alpha \in (0.7, 1], \\ \{1, a, b\} & \text{if } \alpha \in (0.2, 0.7], \\ X & \text{if } \alpha \in [0, 0.2]. \end{cases}$$

Note that $\{1, a, b\}$ and X are ideals of X, and so μ is a fuzzy ideal of X.

(2) Let ν be a fuzzy set in X defined by

$$\nu(x) := \begin{cases} 0.6 & \text{if } x \in \{1, a\}, \\ 0.4 & \text{if } x \in \{b, c, d, 0\}. \end{cases}$$

Then

$$U(\nu; \beta) = \begin{cases} \emptyset & \text{if } \beta \in (0.6, 1], \\ \{1, a\} & \text{if } \beta \in (0.4, 0.6], \\ X & \text{if } \beta \in [0, 0.4]. \end{cases}$$

Note that $\{1, a\}$ is not an ideal of X since

$$(a * (a * b)) * b = (a * a) * b = 1 * b = b \notin \{1, a\}.$$

Hence ν is not a fuzzy ideal of X.

Lemma 3.1. *Every fuzzy ideal μ of X satisfies the following inequality:*

$$(\forall x \in X)\ (\mu(1) \geq \mu(x)). \tag{3.4}$$

Proof. Using (2.1) and (3.1), we have

$$\mu(1) = \mu(x * x) \geq \mu(x)$$

for all $x \in X$. ∎

Proposition 3.1. *If μ is a fuzzy ideal of X, then*

$$(\forall x, y \in X)\ (\mu((x * y) * y) \geq \mu(x)). \tag{3.5}$$

Proof. Taking $y = 1$ and $z = y$ in (3.2) and using (2.3) and Lemma 3.1, we get

$$\mu((x * y) * y) = \mu((x * (1 * y)) * y) \geq \min\{\mu(x), \mu(1)\} = \mu(x)$$

for all $x, y \in X$. ∎

Corollary 3.1. *Every fuzzy ideal μ of X is order preserving, that is, μ satisfies:*

$$(\forall x, y \in X)\ (x \leq y \implies \mu(x) \leq \mu(y)). \tag{3.6}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$\mu(y) = \mu(1 * y) = \mu((x * y) * y) \geq \mu(x)$$

by (2.3) and (3.5). ∎

Proposition 3.2. *Let μ be a fuzzy set in X which satisfies* (3.4) *and*

$$(\forall x, y, z \in X)\ (\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}). \tag{3.7}$$

Then μ is order preserving.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$\mu(y) = \mu(1 * y) \geq \min\{\mu(1 * (x * y)), \mu(x)\} = \min\{\mu(1 * 1), \mu(x)\} = \mu(x)$$

by (2.1), (2.3), (3.7) and (3.4). ∎

We give a characterization of fuzzy ideals.

Theorem 3.2. *Let X be a transitive BE-algebra. A fuzzy set μ in X is a fuzzy ideal of X if and only if it satisfies conditions* (3.4) *and* (3.7).

Proof. Assume that μ is a fuzzy ideal of X. By Lemma 3.1, μ satisfies (3.4). Since X is transitive, we have

$$(y * z) * z \leq (x * (y * z)) * (x * z), \tag{3.8}$$

i.e., $((y * z) * z) * ((x * (y * z)) * (x * z)) = 1$ for all $x, y, z \in X$. It follows from (2.3), (3.2) and Proposition 3.1 that

$$\begin{aligned}
\mu(x * z) &= \mu(1 * (x * z)) \\
&= \mu((((y * z) * z) * ((x * (y * z)) * (x * z))) * (x * z)) \\
&\geq \min\{\mu((y * z) * z), \mu(x * (y * z))\} \\
&\geq \min\{\mu(x * (y * z)), \mu(y)\}.
\end{aligned}$$

Hence μ satisfies (3.7). Conversely suppose that μ satisfies two conditions (3.4) and (3.7). Using (3.7), (2.1), (2.2) and (3.4), we have

$$\begin{aligned}
\mu(x * y) &\geq \min\{\mu(x * (y * y)), \mu(y)\} \\
&= \min\{\mu(x * 1), \mu(y)\} \\
&= \min\{\mu(1), \mu(y)\} = \mu(y)
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\mu((x * y) * y) &\geq \min\{\mu((x * y) * (x * y)), \mu(x)\} \\
&= \min\{\mu(1), \mu(x)\} = \mu(x)
\end{aligned} \tag{3.10}$$

for all $x, y \in X$. Since μ is order preserving by Proposition 3.2, it follows from (3.8) that

$$\mu((y * z) * z) \leq \mu((x * (y * z)) * (x * z))$$

so from (3.7) and (3.10) that

$$\mu((x * (y * z)) * z) \geq \min\{\mu(((x * (y * z)) * (x * z)), \mu(x)\}$$

$$\geq \min\{\mu((y * z) * z), \mu(x)\}$$
$$\geq \min\{\mu(x), \mu(y)\}$$

for all $x, y, z \in X$. Hence μ is a fuzzy ideal of X. ∎

Corollary 3.2. *Let X be a self distributive BE-algebra. A fuzzy set μ in X is a fuzzy ideal of X if and only if it satisfies conditions* (3.4) *and* (3.7)*.*

Proof. Straightforward. ∎

For every $a, b \in X$, let μ_a^b be a fuzzy set in X defined by

$$\mu_a^b(x) := \begin{cases} \alpha & \text{if } a * (b * x) = 1, \\ \beta & \text{otherwise} \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$.

The following example shows that there exist $a, b \in X$ such that μ_a^b is not a fuzzy ideal of X.

Example 3.2. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.1. Then μ_1^a is not a fuzzy ideal of X since

$$\begin{aligned} \mu_1^a((a * (a * b)) * b) &= \mu_1^a((a * a) * b) = \mu_1^a(1 * b) \\ &= \mu_1^a(b) = \beta < \alpha = \mu_1^a(a) \\ &= \min\{\mu_1^a(a), \mu_1^a(a)\}. \end{aligned}$$

Theorem 3.3. *If X is self distributive, then the fuzzy set μ_a^b in X is a fuzzy ideal of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. For every $x, y \in X$, if $a * (b * y) \neq 1$, then $\mu_a^b(y) = \beta \leq \mu_a^b(x * y)$. Assume that $a * (b * y) = 1$. Then

$$\begin{aligned} a * (b * (x * y)) &= a * ((b * x) * (b * y)) \\ &= (a * (b * x)) * (a * (b * y)) \\ &= (a * (b * x)) * 1 = 1, \end{aligned}$$

and so $\mu_a^b(x * y) = \alpha = \mu_a^b(y)$. Hence $\mu_a^b(x * y) \geq \mu_a^b(y)$ for all $x, y \in X$. Now, for every $x, y, z \in X$, if $a * (b * x) \neq 1$ or $a * (b * y) \neq 1$, then $\mu_a^b(x) = \beta$ or $\mu_a^b(y) = \beta$. Thus

$$\mu_a^b((x * (y * z)) * z) \geq \beta = \min\{\mu_a^b(x), \mu_a^b(y)\}.$$

Suppose that $a * (b * x) = 1$ and $a * (b * y) = 1$. Then

$$\begin{aligned} a * (b * ((x * (y * z)) * z)) &= a * ((b * (x * (y * z))) * (b * z)) \\ &= (a * (b * (x * (y * z)))) * (a * (b * z)) \\ &= ((a * (b * x)) * (a * (b * (y * z)))) * (a * (b * z)) \\ &= (1 * (a * (b * (y * z)))) * (a * (b * z)) \\ &= (a * (b * (y * z))) * (a * (b * z)) \\ &= ((a * (b * y)) * (a * (b * z))) * (a * (b * z)) \\ &= (1 * (a * (b * z))) * (a * (b * z)) \\ &= (a * (b * z)) * (a * (b * z)) = 1, \end{aligned}$$

which implies that

$$\mu_a^b((x*(y*z))*z) = \alpha = \min\{\mu_a^b(x), \mu_a^b(y)\}.$$

Therefore $\mu_a^b((x*(y*z))*z) \geq \min\{\mu_a^b(x), \mu_a^b(y)\}$ for all $x, y, z \in X$. Consequently, μ_a^b is a fuzzy ideal of X for all $a, b \in X$. ∎

For any $a, b \in X$, the set

$$A(a, b) := \{x \in X \mid a*(b*x) = 1\}$$

is called the *upper set* of a and b (see [4]). Clearly, $1, a, b \in A(a, b)$ for all $a, b \in X$ (see [4]). Note that $A(a, b)$ is not an ideal of X in general (see [1]).

Lemma 3.2. *A nonempty subset I of X is an ideal of X if and only if it satisfies*

$$1 \in I, \tag{3.11}$$

$$(\forall x, z \in X)\ (\forall y \in I)\ (x*(y*z) \in I \implies x*z \in I). \tag{3.12}$$

Proof. Let I be an ideal of X. Using (2.1) and (2.8), we have $1 = a*a \in I$ for all $a \in I$. We prove the following assertion:

$$(\forall x \in I)\ (\forall y \in X)\ (x*y \in I \implies y \in I). \tag{3.13}$$

Let $x \in I$ and $y \in X$ be such that $x*y \in I$. Then

$$y = 1*y = ((x*y)*(x*y))*y \in I$$

by (2.9). Now, let $x, z \in X$ and $y \in I$ be such that $x*(y*z) \in I$. Then $y*(x*z) \in I$ by (2.4). Since $y \in I$, it follows from (3.13) that $x*z \in I$. Hence (3.12) is valid.

Conversely, assume that (3.11) and (3.12) are valid. Let $x \in X$ and $a \in I$. Then $x*(a*a) = x*1 = 1 \in I$, and so $x*a \in I$ by (3.12). Since $(a*x)*(a*x) = 1 \in I$, we have $(a*x)*x \in I$ by (3.12). It follows that $(a*(b*x))*(b*x) \in I$ for all $a, b \in I$ and $x \in X$. Using (3.12), we get $(a*(b*x))*x \in I$. Therefore I is an ideal of X. ∎

Theorem 3.4. *Let μ be a fuzzy set in X. Then μ is a fuzzy ideal of X if and only if μ satisfies the following assertion:*

$$(\forall a, b \in X)\,(\forall \alpha \in [0,1])(a, b \in U(\mu;\alpha) \implies A(a, b) \subseteq U(\mu;\alpha)). \tag{3.14}$$

Proof. Assume that μ is a fuzzy ideal of X and let $a, b \in U(\mu;\alpha)$. Then $\mu(a) \geq \alpha$ and $\mu(b) \geq \alpha$. Let $x \in A(a, b)$. Then $a*(b*x) = 1$. Hence

$$\mu(x) = \mu(1*x) = \mu((a*(b*x))*x) \geq \min\{\mu(a), \mu(b)\} \geq \alpha,$$

and so $x \in U(\mu;\alpha)$. Thus $A(a, b) \subseteq U(\mu;\alpha)$.

Conversely, suppose that μ satisfies (3.14). Note that $1 \in A(a, b) \subseteq U(\mu;\alpha)$ for all $a, b \in X$. Let $x, y, z \in X$ be such that $x*(y*z) \in U(\mu;\alpha)$ and $y \in U(\mu;\alpha)$. Since

$$(x*(y*z))*(y*(x*z)) = (x*(y*z))*(x*(y*z)) = 1$$

by (2.4) and (2.1), we have $x*z \in A(x*(y*z), y) \subseteq U(\mu;\alpha)$. It follows from Lemma 3.2 that $U(\mu;\alpha)$ is an ideal of X. Hence μ is a fuzzy ideal of X by Theorem 3.1. ∎

Corollary 3.3. *If μ is a fuzzy ideal of X, then*

$$(\forall\alpha \in [0,1])\Big(U(\mu;\alpha) \neq \emptyset \implies U(\mu;\alpha) = \bigcup_{a,b\in U(\mu;\alpha)} A(a,b)\Big). \tag{3.15}$$

Proof. Let $\alpha \in [0,1]$ be such that $U(\mu;\alpha) \neq \emptyset$. Since $1 \in U(\mu;\alpha)$, we have

$$U(\mu;\alpha) \subseteq \bigcup_{a\in U(\mu;\alpha)} A(a,1) \subseteq \bigcup_{a,b\in U(\mu;\alpha)} A(a,b).$$

Now, let $x \in \bigcup_{a,b\in U(\mu;\alpha)} A(a,b)$. Then there exist $u, v \in U(\mu;\alpha)$ such that $x \in A(u,v) \subseteq U(\mu;\alpha)$ by Theorem 3.4. Thus $\bigcup_{a,b\in U(\mu;\alpha)} A(a,b) \subseteq U(\mu;\alpha)$. This completes the proof. ∎

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References

[1] S. S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, *Sci. Math. Jpn.* **68** (2008), no. 2, 279–285.

[2] Y. Imai and K. Iséki, On axiom systems of propositional calculi. XIV, *Proc. Japan Acad.* **42** (1966), 19–22.

[3] K. Iséki, An algebra related with a propositional calculus, *Proc. Japan Acad.* **42** (1966), 26–29.

[4] H. S. Kim and Y. H. Kim, On BE-algebras, *Sci. Math. Jpn.* **66** (2007), no. 1, 113–116.