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An Introduction to the Theory of H_v-Semilattices

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Abstract. In this paper, we introduce the concept of H_v-semilattice and obtain some characterizations of it. We give the definitions of ideal and of hyperorder on an H_v-semilattice. We also study some of their related properties.

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1. Introduction

Hyperstructure theory was born in 1934 when Marty [32] defined hypergroups as a generalization of groups. A hypergroupoid is a non-empty set H together with a map $* : H \times H \to \mathcal{P}^*(H)$ which is called hyperoperation, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. The image of pair (x, y) is denoted by $x * y$. If $x \in H$ and A, B are non-empty subsets of H, then by $A * B$, $A * x$ and $x * B$ we mean

$$A * B = \bigcup_{a\in A, b\in B} a * b, \quad A * x = A * \{x\} \text{ and } x * B = \{x\} * B,$$

respectively. A hypergroupoid $(H, *)$ is called a hypergroup if for all $x, y, z \in H$ the following two conditions hold:

(i) $x * (y * z) = (x * y) * z$,
(ii) $x * H = H * x = H$.

Seventy years have elapsed since Marty's pioneer paper. During this period, numerous papers on algebraic hyperstructures have been published, the field has experimented an enormous growth. A recent book [6] contains a wealth of applications. There are applications to the following subjects: Geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilistic. H_v-structures were for the first time introduced by Vougiouklis in Fourth AHA congress (1990)[49]. The concept of H_v-structures constitute a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning

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the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. Since the quotients of the H_v-structures with respect to the fundamental equivalence relations ($\beta^*, \gamma^*, \epsilon^*$, etc.) are always ordinary structures, we can say that they are by virtue structures and this is why they are called H_v-structures. Many authors have published papers relating different "H_v-structures". In particular a variety of H_v-structures theory have been defined such as: Partial abelian H_v-monoids [12], H_v-semigroups [5, 38], H_v-groups [2, 5, 10, 13, 20, 33, 37, 47, 52], H_v-rings [7, 8, 16, 17, 20, 21, 36, 39–44, 49], H_v-modules [9, 11, 15, 19, 51], H_v-vector spaces [50], H_v-fields [53], and other papers on H_v-structures are [6, 14, 24, 31, 45, 46, 48, 54, 55]. The reader will find in [43] some basic definitions and theorems about the H_v-structures. In [14], Davvaz surveyed the theory of H_v-structures. Hyperlattices were for the first time introduced by Konstantinidou and Mittas [29]. The concept of hyperlattice is a generalization of the concept of lattice [3]. Other contributor to the development of hyperlattice theory were Konstantinidou [25–30], Ashrafi [1], Rahnamai-Barghi [34, 35], Xiao and Zhao [56].

2. About the definitions of semilattice

A semilattice is a mathematical concept with two definitions, one as a type of ordered set, the other as an algebraic structure. In mathematical order theory, a semilattice is a partially ordered set (poset) closed under one of two binary operations, either supremum (join) or infimum (meet). Hence we speak of either a join-semilattice or a meet-semilattice. If an ordered set is both a meet- and join-semilattice, it is also a lattice.

Semilattices as posets: Let S be a set partially ordered by the binary relation $\leq$. $(S, \leq)$ is a meet-semilattice if for all elements x and y of S, the greatest lower bound of the set $\{x, y\}$ exists. The greatest lower bound of the set $\{x, y\}$ is called the meet of x and y, denoted $x \wedge y$. Replacing "greatest lower bound" with "least upper bound" results in the dual concept of a join-semilattice. The least upper bound of $\{x, y\}$ is called the join of x and y, denoted $x \vee y$. Meet and join are binary operations on S. A simple induction argument shows that the existence of all possible pairwise suprema (infima), for each the definition, implies the existence of all non-empty finite suprema (infima). A join-semilattice is bounded if it has a least element, the join of the empty set. Dually, a meet-semilattice is bounded if it has a greatest element, the meet of the empty set. Other properties may be assumed; see the article on completeness in order theory for more discussion on this subject. That article also discusses how we may rephrase the above definition in terms of the existence of suitable Galois connections between related posets, an approach of special interest for category theoretic investigations of the concept.

Semilattices as algebraic structures: A "meet-semilattice" is an algebraic structure consisting of a set S with the binary operation $\wedge$, called meet, such that for all members x, y, and z of S, the following identities hold:

(i) $x \wedge x = x$ (idempotency),
(ii) $x \wedge y = y \wedge x$ (commutativity),
(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associativity).

If $\vee$, denoting join, replaces $\wedge$ in the definition just given, a join-semilattice results. Meet and join form a dual pair of binary operations, and meet-semilattice and join-semilattice are dual algebraic structures. A meet-semilattice is bounded if $(S, \wedge)$ includes the distinguished element 1 such that for all x in S, $x \wedge 1 = x$. 1 is the greatest element of S. Dually, $(S, \vee, 0)$ is a join-semilattice with least element 0 if $\vee$ and 0 replace $\wedge$ and 1, respectively, in the definition just given. A semilattice is an idempotent, commutative semigroup, and a bounded semilattice is an idempotent commutative monoid. Alternatively, a semilattice is a commutative band. Hence semilattices are magmas.

Connection between both definitions: An order theoretic meet-semilattice $(S, \leq)$ gives rise to a binary operation $\wedge$ such that $(S, \wedge)$ is an algebraic meet-semilattice. Conversely, the meet-semilattice $(S, \wedge)$ gives rise to a binary relation $\leq$ that partially orders S in the following way. For all elements x and y in S,

$$x \leq y \iff x = x \wedge y.$$

The relation $\leq$ introduced in this way defines a partial ordering from which the binary operation $\wedge$ may be recovered. Conversely, the order induced by the algebraically defined semilattice $(S, \wedge)$ coincides with that induced by $\leq$. Hence both definitions may be used interchangeably, depending on which one is more convenient for a particular purpose. A similar conclusion holds for join-semilattices and the dual ordering $\geq$.

3. H_v-semilattices

In this section, we introduce the notion of H_v-semilattice. The notion of H_v-semilattice is a generalization of the semilattice notion in classical theory as well as hypersemilattice.

Definition 3.1. *Let L be a nonempty set with a binary hyperoperation $*$ on L such that, for all $a, b, c \in L$, the following conditions hold:*

(i) $a \in a * a$ (*idempotent*)
(ii) $a * b = b * a$ (*commutative*)
(iii) $(a * b) * c \cap a * (b * c) \neq \phi$ (*weak associative*).

Then $(L, *)$ is called an H_v-semilattice. When in the condition (iii) *we have equality, then $(L, *)$ is called a hypersemilattice* [56].

Now, we present some examples of H_v-semilattices.

Example 3.1.

(i) Consider $L = \{a, b, c\}$ and define hyperoperation $*$ on L by the following table:

$*$	a	b	c
a	$\{a\}$	$\{c, a\}$	$\{b, a\}$
b	$\{a, c\}$	$\{b, a\}$	$\{a\}$
c	$\{b, a\}$	$\{a\}$	$\{c, a\}$

Then $(L, *)$ is an H_v-semilattice which is not a hypersemilattice. Indeed, we have $b*(c*a) = \{a, b, c\}$ and $(b*c)*a = \{a\}$. Therefore $*$ is not associative, but $*$ is weak associative for all $a, b, c \in L$.

(ii) We consider the classical differential ring of real functions $f \in C^{\infty}(J), J = (a,b) \subseteq \mathbb{R}$ (not excluding the case $J = \mathbb{R}$) with the usual differentiation. For any $f, g \in C^{\infty}(J)$ we define a hyperoperation $*$ on the ring $C^{\infty}(J)$ by, for all $x \in J, (f * g)(x) = \{f(x), g(x), f^{'}(x), g^{'}(x)\}$. For any $f, g, h \in C^{\infty}(J)$, we have

$$f \in f * f = \{f, f^{'}\} \quad \text{and} \quad f * g = \{f, g, f^{'}, g^{'}\} = g * f.$$

Also

$$\begin{aligned}(f * g) * h &= \{f, g, f^{'}, g^{'}\} * h \\ &= \{f, g, hf^{'}, g^{'}, h^{'}, f^{''}, g^{''}\} \\ &\neq f * (g * h) \\ &= f * \{g, h, g^{'}, h^{'}\} \\ &= \{f, g, h, f^{'}, g^{'}, h^{'}, g^{''}, h^{''}\}.\end{aligned}$$

But $f * (g * h) \cap (f * g) * h = \{f, g, h, f^{'}, g^{'}, h^{'}, g^{''}\} \neq \phi$. Therefore $(C^{\infty}(J), *)$ is an H_v-semilattice.

(iii) Let f be a function from L into $\mathcal{P}^*(\mathbb{R})$. We define the hyperoperation $*_f$ as follows:
$a *_f b = \{x \in L \mid f(x) \subseteq f(a) \cup f(b)\}$ for $a, b \in L$. It is clear that $(L, *_f)$ is an H_v-semilattice.

All properties of H_v-semilattices are also true for subsets. So we have:

Proposition 3.1. *Let L be a nonempty set and let $*$ be a binary hyperoperation on L. Then $(L, *)$ is an H_v-semilattice if and only if for all $A, B, C \in \mathcal{P}^*(L)$ the following conditions hold:*

(i) $A \subseteq A * A$,
(ii) $A * B = B * A$,
(iii) $(A * B) * C \cap A * (B * C) \neq \phi$.

Proof. The proof is straightforward. ∎

To each binary relation R on a set L, a partial hyperoperation $L_R = (L, \odot)$ is associated, as follows: for all $x, y \in L$, $x \odot x = \{y \in L \mid (x, y) \in R\}$, $x \odot z = x \odot x \cup z \odot z$.

Proposition 3.2. *If R is a reflexive relation on L, then L_R is an H_v-semilattice.*

Proof. For arbitrary $x, y, z \in L$, we have

(i) $(x, x) \in R$, so $x \in \{y \in L | (x, x) \in R\}, i.e, x \in x * x$,
(ii) $x \odot z = (x \odot x) \cup (z \odot z) = (z \odot z) \cup (x \odot x) = z \odot x$,

and we suppose that $Q = (x \odot y) \odot z, Q' = x \odot (y \odot z)$. By the definition of above hyperoperation $\odot$, we obtain

$$Q = [(x \odot x) \odot (x \odot x)] \cup (z \odot z) \cup [(y \odot y) \odot (y \odot y)],$$

and

$$Q' = (x \odot x) \cup [y \odot y) \odot (y \odot y)] \cup [(z \odot z) \odot (z \odot z)].$$

By (i), it is easy to see that $x, y, z \in Q \cap Q'$. Therefore, this completes the proof. ∎

Proposition 3.3. *Let $(\mathbb{Z}, +, \leqslant)$ be the additive group of all integers with a usual ordering "$\leq$". Then, the hyperstructure $(\mathbb{N}, *)$, where $* : \mathbb{N} \times \mathbb{N} \longrightarrow \mathcal{P}^*(\mathbb{N}), k * l = \{u \in \mathbb{N} \mid k + l \leq 2u\}$ is an H_v-semilattice.*

Proof. Here we only prove the week associativity and the other conditions (idempotency, commutativity) are trivial. For any $k, l, s \in \mathbb{N}$, we have

(i) $k * (l * s) = [(2k + l + s + i)/4, +\infty) \cap \mathbb{N}$, where $i = 0, 1, 2, 3$ is chosen that $(2k + l + s + i)/4 \in \mathbb{N}$, and

(ii) $(k * l) * s = [(k + l + 2s + i)/4, +\infty) \cap \mathbb{N}$, where $i = 0, 1, 2, 3$ is chosen that $(k + l + 2s + i)/4 \in \mathbb{N}$.

It is easy to see that $(k * l) * s \cap k * (l * s) \neq \emptyset$. But, it is not associative. Because, for example, we have $(1 * 2) * 5 = [4, +\infty) \cap \mathbb{N} \neq [3, +\infty) \cap \mathbb{N} = 1 * (2 * 5)$. ∎

Definition 3.2. *Let $(\bullet)$ and $(*)$ be two hyperoperations on H. We call $(*)$ the dual of $(\bullet)$ if and only if for all $x, y \in H, x \bullet y = y * x$.*

Proposition 3.4. *Let $(*)$ be the dual of $(\bullet)$. Therefore $(H, \bullet)$ is an H_v-semilattice if and only if $(H, *)$ is an H_v-semilattice.*

Proof. The proof is straightforward. ∎

Let $(L, \oplus), (S, \otimes)$ be two H_v-semilattices. A map $f : L \longrightarrow S$ is called a weak homomorphism if $f(x \oplus y) \cap (f(x) \otimes f(y)) \neq \phi$ for all $x, y \in L$. f is called an inclusion homomorphism if $f(x \oplus y) \subseteq (f(x) \otimes f(y))$ for all $x, y \in L$. Finally, f is called a strong homomorphism (preserving binary hyperoperation) if $f(x \oplus y) = f(x) \otimes f(y)$ for all $x, y \in L$. Let α be a strong homomorphism from an H_v-semilattice L onto H_v-semilattice S. The relation $\alpha^{-1} o \alpha$ is an equivalence ρ on $L(a\rho b$ if and only if $\alpha(a) = \alpha(b)$) known as the kernel of α. The natural mapping associated with ρ is $\phi : L \longrightarrow L/ker\alpha$ where $\phi(a) = \rho(a)$. The mapping $\psi : L/\rho \longrightarrow S$, where $\psi(\rho(a)) = \alpha(a)$ is then the unique bijection that makes the following diagram commute: $\psi : L/\rho \longrightarrow S$, $\phi : L \longrightarrow L/ker\alpha$, $\alpha : L \longrightarrow S$. If f is an onto, one to one and strong homomorphism, then it is called isomorphism, if moreover f defined on the same H_v-semilattice then it is called automorphism, if moreover f defined on the same H_v-semilattice then it is called automorphism. It is easy verification that the set of all automorphism in L, written $AutL$, is a group. If f is injective as a map of sets, then f is said to be a monomorphism. If f is surjective, then f is called an epimorphism.

If $f : L \longrightarrow S$ and $g : S \longrightarrow T$ are homomorphisms of H_v-semilattices, it is easy to see that $gof : L \longrightarrow T$, is also a homomorphism. Like wise the composition of monomorphisms is also a monomorphism, similarly to epimorphisms and isomorphisms.

Proposition 3.5. *Let $(L, \oplus)$ be an H_v-semilattice and S be a nonempty set with a binary hyperoperation $\otimes$. If a function $f : L \longrightarrow S$ is surjective and preserving binary weak hperoperation, then $(S, \otimes)$ is an H_v-semilattice.*

Proof. For all $a_1, b_1, c_1 \in S$ we have

(i) $a_1 = f(a) \in f(a \oplus a) = f(a) \otimes f(a) = a_1 \otimes a_1$, i.e. $a_1 \in a_1 \otimes a_1$

(ii) $a_1 \otimes b_1 = f(a) \otimes f(b) = f(a \oplus b) = f(b \oplus a) = f(b) \otimes f(a) = b_1 \otimes a_1$.

(iii) $[(a_1 \otimes b_1) \otimes c_1] \cap [a_1 \otimes (b_1 \otimes c_1)] = [(f(a) \otimes f(b)) \otimes f(c)] \cap [f(a) \otimes (f(b) \otimes f(c))] = [f((a \oplus b) \oplus c)] \cap [f(a \oplus (b \oplus c))] \neq \phi$, since we have $(a * b) * c \cap a * (b * c) \neq \phi$ for all $a, b, c, \in L$.

Therefore $(S, \otimes)$ is an H_v-semilattice. ∎

On a set L several H_v-semilattices can be defined. A partial order on those H_v-semilattices is introduced as follows.

Definition 3.3. *Let $(L, \oplus), (L, \otimes)$ be two H_v-semilattices defined on the same set L. We call $\oplus$ less than or equal to $\otimes$, and write $\oplus \leq \otimes$, if there is $f \in Aut(L, \otimes)$ such that $x \oplus y \subseteq f(x \otimes y)$ for all $x, y \in L$.*

Definition 3.4. *Let $(L, *)$ be an H_v-semilattice. An element $a \in L$ is called an absorbent element of L if it satisfies $c \in a * c$ for all $c \in L$. An element $b \in L$ is called a fixed element of L if it satisfies $b * c = \{b\}$ for all $c \in L$.*

Example 3.2. Let L be a non-empty set and define a binary hyperoperation on L by $a * b = L$ for all $a, b \in L$. It is easy to verify that $(L, *)$ is an H_v-semilattice, and we call it trivial H_v-semilattice.

Let $(L, +)$ be an H_v-semilattice, θ an equivalence relation on L and $\theta(x)$ the θ-equivalence class of the element $x \in L$. In L/θ consider the hyperoperation $\oplus$ defined on the usual manner: $\theta(x) \oplus \theta(y) = \{\theta(z) \mid z \in \theta(x) + \theta(y)\}$ for all $x, y \in L$.

Proposition 3.6. *$(L/\theta, \oplus)$ is an H_v-semilattice.*

Proof. For all $x \in L$ we have $\theta(x) = \{y \in L \mid x\theta y\}$. It is easy to verify that the hyperoperation $\oplus$ is idempotent, commutative and weak associative. For example, we show that this is weak associative. For all $x, y, z \in L$, we have $(x + y) + z \subseteq (\theta(x) \oplus \theta(y)) \oplus \theta(z), x + (y + z) \subseteq \theta(x) \oplus (\theta(y) \oplus \theta(z))$. Since $x + (y + z) \cap (x + y) + z \neq \phi$, so $(\theta(x) \oplus \theta(y)) \oplus \theta(z) \cap \theta(x) \oplus (\theta(y) \oplus \theta(z)) \neq \phi$. Thus $\oplus$ is weak associative. ∎

Proposition 3.7. *Let ϕ_1 and ϕ_2 be two strong homomorphisms of H_v-semilattice L upon H_v-semilattices L_1 and L_2 respectively, such that ${\phi_1}^{-1} \circ \phi_1 \subseteq {\phi_2}^{-1} \circ \phi_2$. Then, a unique strong homomorphism θ of L_1 upon L_2 such that $\theta \circ \phi_1 = \phi_2$, exists.*

Proof. We show that θ is a strong homomorphism of L_1 upon L_2. For all $a_1, b_1 \in L_1$ we have $\theta(a_1 \oplus b_1) = \theta(\phi_1(a) \oplus \phi_1(b)) = \theta(\phi_1(a + b)) = \phi_2(a + b) = \phi_2(a) \otimes \phi_2(b) = \theta(\phi_1(a)) \otimes \theta(\phi_1(b)) = \theta(a_1) \otimes \theta(b_1)$. ∎

Let $(L, \oplus)$ be an H_v-semilattice. By a congruence on L we mean an equivalence relation ρ such that $x\rho y$ if and only if for every $a \in L$ and for every $u \in x \oplus a$, there exists $v \in y \oplus a$ such that $u\rho v$.

Proposition 3.8. *Let $\alpha : L \longrightarrow S$ be a strong homomorphism of H_v-semilattices. Then $\rho = ker\alpha$ is a congruence and a strong homomorphism $f : L/\rho \longrightarrow S$ exists such that $f \circ \phi = \alpha$ (note that L/ρ is an H_v-semilattice).*

Proof. The proof is straightforward. ∎

Proposition 3.9. *If ρ_1 and ρ_2 are congruences on an H_v-semilattice L such that $\rho_1 \subseteq \rho_2$, then a strong homomorphism from L/ρ_1 upon L/ρ_2, exists .*

Proof. The proof is straightforward. ∎

Proposition 3.10. *Let $f : L \longrightarrow S$ be an epimorphism of H_v-semilattices. If a is an absorbent element of L, then $f(a)$ is also an absorbent element of S. An analogous result holds for a fixed element.*

Proof. There exists $c \in L$ such that $f(c) = s$ for any $s \in S$ because f is surjective, a is an absorbent element provided $c \in a \oplus c$ for any $c \in L$. Then we have since $s = f(c) \in f(a \oplus c) = f(a) \otimes f(c) = f(a) \oplus s$ for all $s \in S$. Therefore $f(a)$ is an absorbent element of S. The proof for a fixed element is analogous. ∎

The product operation plays an important role as a means of constructing new construction from the old. Now, we consider the product of H_v-semilattices $(L, \oplus)$ and $(S, \otimes)$. Let $(L, \oplus)$, and $(S, \otimes)$ be two H_v-semilattices. Define a binary hyperoperation on the cartesian product $L \times S$ as follows: $(a_1, b_1) \times (a_2, b_2) = \{(c, d)|c \in a_1 \oplus a_2, d \in b_1 \otimes b_2\}$ for all $(a_1, b_1), (a_2, b_2) \in L \times S$, then $(L \times S, \times)$ is called the direct product of H_v-semilattices $(L, \oplus)$, and $(S, \otimes)$.

Proposition 3.11. *The direct product of two H_v-semilattices is also an H_v-semilattice.*

Proof. Let $(L, \oplus)$, and $(S, \otimes)$ be two H_v-semilattices. Then

(i) For each $(a, b) \in L \times S$, we have that $a \in a \oplus a$ and $b \in b \oplus b$, then $(a, b) \in \{(c, d)|c \in a \oplus a, d \in b \otimes b\} = (a, b) \times (a, b)$

(ii) For all $(a_1, b_1), (a_2, b_2) \in L \times S$, we have $a_1 \oplus a_2 = a_2 \oplus a_1$ and $b_1 \oplus b_2 = b_2 \oplus b_1$ by the definition of H_v-semilattice. Then it is clear that

$$\begin{aligned}(a_1, b_1) \times (a_2, b_2) &= \{(c, d)|c \in a_1 \oplus a_2, d \in b_1 \otimes b_2\} \\ &= \{(c, d)|c \in a_2 \oplus a_1, d \in b_2 \otimes b_1\} \\ &= (a_2, b_2) \times (a_1, b_1).\end{aligned}$$

(iii) For all $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in L \times S$ $(a_1 \oplus a_2) \oplus a_3 \cap a_1 \oplus (a_2 \oplus a_3) \neq \phi$ and $(b_1 \oplus b_2) \oplus b_3 \cap b_1 \oplus (b_2 \oplus b_3) \neq \phi$ hold. Then, we may obtain that

$$\begin{aligned}M &= ((a_1, b_1) \times (a_2, b_2)) \times (a_3, b_3) \\ &= \{(c, d)|c \in a_1 \oplus a_2, d \in b_1 \otimes b_2\} \times (a_3, b_3) \\ &= \{(e, f)|e \in c \oplus a_3, f \in d \otimes b_3, c \in a_1 \oplus a_2, d \in b_1 \otimes b_2\} \\ &= \{(e, f)|e \in (a_1 \oplus a_2) \oplus a_3, f \in (b_1 \otimes b_2) \otimes b_3\}\end{aligned}$$

and

$$\begin{aligned}N &= (a_1, b_1) \times ((a_2, b_2) \times (a_3, b_3)) \\ &= (a_1, b_1) \times \{(c, d)|c \in a_2 \oplus a_3, d \in b_2 \otimes b_3\} \\ &= \{(e, f)|e \in a_1 \oplus (a_2, a_3), f \in b_1 \otimes (b_2, b_3)\}.\end{aligned}$$

Since L, S are H_v-semilattices, so we have $(a_1 \oplus a_2) \oplus a_3 \cap a_1 \oplus (a_2 \oplus a_3) \neq \phi$ and $(b_1 \oplus b_2) \oplus b_3 \cap b_1 \oplus (b_2 \oplus b_3) \neq \phi$.

Therefore $(L \times S, \times)$ is an H_v-semilattice. ∎

Proposition 3.12. *Let $(L, \oplus)$ and $(S, \otimes)$ be two H_v-semilattices. If a and b are absorbent elements of L and S respectively, then (a, b) is an absorbent element of $(L \times S, \times)$.*

Proof. Since a and b are absorbent elements, we have that $c \in a \oplus c$, for all $c \in L$ and $d \in b \otimes d$ for all $d \in S$. Then for all $(c,d) \in L \times S, (c,d) \in \{(e,f) \mid e \in a \oplus c, f \in b \otimes d\} = (a,b) \times (c,d)$. So (a,b) is an absorbent element of $(L \times S, \times)$. ∎

Proposition 3.13. *Let $(L, \oplus)$ be an H_v-semilattice, and N be a non-empty subset of L. Then there exists a well-defined binary operation $\boxplus$ on $L/N = \{N \oplus a \mid a \in L\}$ given by $(N \oplus a) \boxplus (N \oplus b) = \{N \oplus n \mid n \in a \oplus b\}$ for all $a, b \in L$, such that $(L/N, \boxplus)$ is an H_v-semilattice.*

Proof. To begin, let us show $\boxplus$ is a binary hyperoperation on L/N. Obviously, $\boxplus$ is a function $L/N \times L/N \longrightarrow \mathcal{P}^*(L/N)$. Then, for all $N \oplus a, N \oplus b, N \oplus c \in L/N$ and all $A, B \in P^*(L/N)$, we have

$$\begin{aligned}
(N \oplus a) \boxplus (N \oplus a) &= \{(N \oplus n) \mid n \in (a \oplus a)\} \in P^*(L/N),\\
(N \oplus c) \boxplus A &= \cup_{N \oplus a \in A}((N \oplus c) \boxplus (N \oplus a)) \in P^*(L/N),\\
A \boxplus (N \oplus c) &= \cup_{N \oplus a \in A}((N \oplus a) \boxplus (N \oplus c)) \in P^*(L/N),\\
A \boxplus B &= \cup_{N \oplus a \in A, N \oplus b \in B}((N \oplus a) \boxplus (N \oplus b)) \in P^*(L/N).
\end{aligned}$$

That is, $\boxplus$ can be seen as a binary hyperoperation on L/N. Now, we prove that $(L/N, \boxplus)$ is an H_v-semilattice. We have:

(i) Since $a \in a \oplus a$, then $N \oplus a \in \{N \oplus n \mid n \in a \oplus a\} = (N \oplus a) \boxplus (N \oplus a)$.

(ii) Since $a \oplus b = b \oplus a$, then

$$\begin{aligned}
(N \oplus a) \boxplus (N \oplus b) &= \{N \oplus n \mid n \in a \oplus b\}\\
&= \{N \oplus n \mid n \in b \oplus a\}\\
&= (N \oplus b) \boxplus (N \oplus a).
\end{aligned}$$

(iii) Since $(a \oplus b) \oplus c \cap a \oplus (b \oplus c) \neq \phi$, then

$$\begin{aligned}
Q &= ((N \oplus a) \boxplus (N \oplus b)) \boxplus (N \oplus c)\\
&= \{N \oplus n \mid n \in a \oplus b\} \boxplus (N \oplus c)\\
&= \cup_{n \in a \oplus b}((N \oplus n) \boxplus (N \oplus c))\\
&= \cup_{n \in (a \oplus b)}\{N \oplus m \mid m \in n \oplus c\}\\
&= \{N \oplus m \mid m \in (a \oplus b) \oplus c\}, \text{ and}\\
Q' &= (N \oplus a) \boxplus ((N \oplus b) \boxplus (N \oplus c))\\
&= (N \oplus c) \boxplus \{N \oplus l \mid l \in b \oplus c\}\\
&= \cup_{l \in (b \oplus c)}((N \oplus a) \boxplus (N \oplus l))\\
&= \cup_{n \in (b \oplus c}\{N \oplus s \mid s \in a \oplus l\}\\
&= \{N \oplus s \mid s \in a \oplus (b \oplus c)\}.
\end{aligned}$$

Since for all $a, b, c \in L$ we have $(a \oplus b) \oplus c \cap a \oplus (b \oplus c) \neq \phi$, then $Q \cap Q' \neq \phi$. Therefore hyperoperation $\oplus$ is weak associative. ∎

4. Generalized action and H_v-semilattices

In this section, we generalize in a certain sense the classical concept of action of a hypergroup on a given phase space. This subject plays a very important role in the current progress of concrete mathematics (especially geometry). Now, let us recall the definition of a generalized action of G (as a hypergroup) on set X.

Definition 4.1. [23] *Let $(G, \odot)$ be a hypergroup and X be a set. The map $\phi : G \times X \longrightarrow X$ is called a generalized action of G on X, if the following axiom hold: for all $g, h \in G$ and $x \in X$, $\phi(g, \phi(h, x)) \in \phi(g \odot h, x)$, where $\phi(g \odot h, x) = \{\phi(r, x) \mid r \in g \odot h\}$ for all $x \in X, g, h \in G$. Then the triple $\tau = (X, G, \phi)$ is called an action of the hypergroup G on the phase set X. Let θ be a tolerance relation(i.e., reflexive and symmetric binary relation). Then the pair (X, θ) is a tolerance space.*

Definition 4.2. [23] *Let (X, θ) be a tolerance space (so called phase tolerance space), $(G, \odot)$ be a semihypergroup and $\phi : G \times X \longrightarrow X$ a mapping such that*

(i) *$\phi(g, \phi(h, x)) \in \phi(g \odot h, x)$, where $\phi(g \odot h, x) = \{\phi(r, x); r \in g \odot h\}$ for each $x \in X, g, h \in G$;*
(ii) *if $x, y \in X$ are such that $x\theta y$, then $\phi(g, x)\ \theta\ \phi(g, y)$ holds for any $g \in G$.*

Then the triple $\tau = (X, G, \phi)$ is called an action of the semihypergroup G with phase tolerance space. Moreover if, the triple $\tau = (X, G, \phi)$ is an action of the hypergroup with phase tolerance space, in case the tolerance θ is trivial, in fact the preceding definition coincides with the above definition.

Let us define for arbitrary pair of elements $x, y \in X$ a binary hyperoperation $\otimes : X \times X \longrightarrow P^*(X)$, in this way: $x \otimes y = \phi(G, x) \cup \phi(G, y) \cup \{x, y\}$, where $\phi(G, x) = \{\phi(g, x) \mid g \in G\}$ and similarly for $\phi(G, y)$. For all $A \subseteq X, \phi(g, A) = \{\phi(g, a) \mid a \in A\}$.

Proposition 4.1. *The pair $(X, \otimes)$ is an H_v-subsemilattice.*

Proof. For arbitrary $x, y, z \in X$, we have

(i) $x \in x \otimes x = \phi(G, x) \cup \{x\}$,
(ii) $x \otimes y = \phi(G, x) \cup \phi(G, y) \cup \{x, y\} = \phi(G, y) \cup \phi(G, x) \cup \{y, x\} = y \otimes x$,
(iii) For the weak associativity axiom, we have

$$\begin{aligned}(x \otimes y) \otimes z &= \phi(G, x \otimes y) \cup \phi(G, z) \cup \{x \otimes y, z\} \\ &= \phi(G, \phi(G, x)) \cup \phi(G, \phi(G, y)) \cup \phi(G, \{x, y\})\phi(G, z) \\ &\quad \cup \{\phi(G, x) \cup \phi(G, y) \cup \{x, y\}, z\} \\ &= \phi(G, \phi(G, x)) \cup \phi(G, \phi(G, y)) \cup \phi(G, x) \\ &\quad \cup \phi(G, y) \cup \phi(G, z) \cup \{\phi(G, x) \cup \phi(G, y) \cup \{x, y\}, z\}\end{aligned}$$

and

$$\begin{aligned}x \otimes (y \otimes z) &= \phi(G, x) \cup \phi(G, y \otimes z) \cup \{x, y \otimes z\} \\ &= \phi(G, x) \cup \phi(G, \phi(G, y)) \cup \phi(G, \phi(G, z)) \cup \phi(G, \{y, z\}) \\ &\quad \cup \{x, \phi(G, y) \cup \phi(G, z) \cup \{x, y\}\} \\ &= \phi(G, x) \cup \phi(G, \phi(G, y)) \cup \phi(G, \phi(G, z)) \cup \phi(G, y) \\ &\quad \cup \phi(G, z) \cup \{x, \phi(G, y) \cup \phi(G, z) \cup \{x, y\}\}.\end{aligned}$$

Thus the weak associative condition is satisfied, since $\varnothing \neq \phi(G, x) \cup \phi(G, y) \cup \phi(G, z) \in (x \otimes y) \otimes z \cap x \otimes (y \otimes z)$. ∎

5. H_v-subsemilattices

Let $(L, \oplus)$ be a H_v-semilattice, and M be a non-empty subset of L. Then M is called H_v-subsemilattice of $(L, \oplus)$ if $a \oplus b \in \mathcal{P}^*(M)$ for all $a, b \in M$. That is to say, M is an H_v-subsemilattice of $(L, \oplus)$ if and only if M is closed under the binary hyperoperation on L. An H_v-subsemilattice M is a single point H_v-subsemilattice if $|M| = 1$, and H_v-subsemilattice M such that $M \neq L$ is called proper H_v-subsemilattice. We may easily get the conclusion as follows: M is an H_v-subsemilattice of $(L, \oplus)$ if and only if $M \oplus M = M$.

Example 5.1. Let L be a non-empty set, and define a binary hyperoperation on L as follows: $a \oplus b = \{a, b\}$ for all $a, b \in L$. Then $(L, \oplus)$ is an H_v-semilattice. Each nonempty subset of L is an H_v-subsemilattice of $(L, \oplus)$.

Proposition 5.1. *Let $L \longrightarrow S$ be a homomorphism of H_v-semilattices. Then the following conditions hold:*

(i) *If M is an H_v-subsemilattice of $(L, \oplus)$, then $f(M)$ is an H_v-subsemilattice of $(S, \otimes)$.*

(ii) *If f is surjective and N is an H_v-subsemilattice of $(S, \otimes)$, then $f^{-1}(N)$, which is defined by $f^{-1}(N) = \{a \in L \mid f(a) \in N\}$, is also an H_v-subsemilattice of $(L, \oplus)$.*

Proof. (i) Since f is a homomorphism of H_v-semilattice, there exist $a, b \in M$ such that $f(a) = a_1$, $f(b) = b_1$ for all $a_1, b_1 \in f(M)$. By the definition of H_v-subsemilattice $a \oplus b \subseteq M$ holds. So we have $a_1 \otimes b_1 = f(a) \otimes f(b) = f(a \oplus b) \subseteq f(M)$. Then $f(M)$ is an H_v-subsemilattice of $(S, \otimes)$.

(ii) Since f is a surjective function, $f^{-1}(N)$ always exists. For all $a, b \in f^{-1}(N)$, $f(a) \in N$, $f(b) \in N$, we have $f(a \oplus b) = f(a) \oplus f(b) \subseteq N$. So $f^{-1}(N)$ is an H_v-semilattice of $(L, \oplus)$. ∎

Proposition 5.2. *Let $(L, \oplus)$ be an H_v-semilattice and let M and N be H_v-subsemilattices of $(L, \oplus)$. Then $M \cap N$ is also an H_v-subsemilattice of $(L, \oplus)$ if $M \cap N$ is non-empty.*

Proof. The proof is straightforward. ∎

Note that, we can easily verify that the union of an H_v-subsemilattice of H_v-semilattice $(L, \oplus)$, may not be the H_v-subsemilattice of $(L, \oplus)$, because it is not closed under the binary hyperoperation on L.

Example 5.2. Let $L = \{a, b, c, d\}$ and define a binary hyperoperation $\lozenge$ on L with help of the following table:

$\lozenge$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, b\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, c\}$	$\{a, b, c\}$
d	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, d\}$

It is easy to prove that $(L, \lozenge)$ is an H_v-semilattice. If $M_1 = \{a, c\}, M_2 = \{a, d\}$, then they are H_v-subsemilattices of $(L, \lozenge)$. But we can easily verify that $M_1 \cup M_2 = \{a, c, d\}$ is not an H_v-subsemilattice of L because it isn't closed under the binary hyperoperation on L.

By the definitions of an H_v-semilattice and the product of H_v-semilattices, we have the following. Let $(L, \oplus)$ and $(S, \otimes)$ be H_v-semilattices and let M and N be H_v-subsemilattices of $(L, \oplus)$ and $(S, \otimes)$, respectively. Then $M \times N$ is also an H_v-subsemilattice of $(L \times S, \times)$.

6. Definition of an ideal of an H_v-semilattice

Ideal play an important role in the study of algebraic structures. In this section, we introduce the definition of an ideal of an H_v-semilattice and discuss some basic properties of it.

Definition 6.1. *Let $(L, \oplus)$ be an H_v-semilattice, and N be a nonempty subset of L. We say N is an ideal of $(L, \oplus)$ if $a \oplus N \subseteq N$ for all $a \in L$. If $N \neq L$, then N is called a proper ideal of $(L, \oplus)$.*

Obviously, any H_v-semilattice is an H_v-subsemilattice and ideal of itself. If N is an ideal of $(L, \oplus)$, then N is an H_v-subsemilattice of $(L, \oplus)$. But the converse is not true. We would like to give an example to illustrate it.

Example 6.1. Let us recall from Example 3.2, the set $N_1 = \{a, b\}$ is an H_v-subsemilattice, but not an ideal of $(L, \oplus)$, and $N_2 = \{a, d, c\}$ is an ideal of $(L, \oplus)$, but N_2 is not an H_v-subsemilattice of $(L, \oplus)$.

Next, we want to give some equivalent statements about ideal of H_v-semilattice and introduce some special ideals.

Proposition 6.1. *Let I be an ideal and M an H_v-subsemilattice of an H_v-semilattice L. Then $I \cap M$ is an ideal of $M, I \cup M$ is an H_v-subsemilattice of L, and there is an inclusion homomorphism from $M/(I \cap M)$ upon $(I \cup M)/I$.*

Proof. The proof is straightforward. ∎

Proposition 6.2. *Let I be an ideal of an H_v-semilattice L. We consider the Rees relation on L as follows: $x \rho y \Longleftrightarrow x = y$ or $(x \in I$ and $y \in I)$. Then ρ is congruence on L.*

Proof. The proof is straightforward. ∎

Proposition 6.3. *Let $(L, \oplus)$, be an H_v-semilattice and let N be a non-empty subset of L. Then the following conditions are equivalent:*

(i) *N is an ideal of $(L, \oplus)$.*
(ii) *$a \oplus n \in P^*(N)$ for all $a \in L$ and all $n \in N$.*
(iii) *$L \oplus N \subseteq N$.*

Proof. The proof is straightforward. ∎

Proposition 6.4. *Let N be an ideal of an H_v-semilattice $(L, \oplus)$. If a is an absorbent element of L, then the following condition hold:*

(i) *$N = L$ if and only if $a \in N$.*
(ii) *N is a proper ideal of $(L, \oplus)$ if and only if a isn't belong to N.*

Proof. The proof is straightforward. ∎

Proposition 6.5. *Let a be an element of an H_v-semilattice $(L, \oplus)$. Then $\{a\}$ is an ideal of $(L, \oplus)$.*

Proof. The proof is straightforward. ∎

Proposition 6.6. *Let M and N be ideals of an H_v-semilattice $(L, \oplus)$. Then, we have the following conclusions:*

(i) *$M \cap N$ is an ideal of $(L, \oplus)$ and $M \cap N = M \oplus N$.*
(ii) *$M \cup N$ is also an ideal of $(L, \oplus)$.*

Proof. (i) Let us prove that $M \cap N \neq \phi$. Suppose that $m \in M, n \in N$. Then $m \oplus n \subseteq M, m \oplus n \subseteq N$ by Proposition 6.3 (ii), that is $m \oplus n \subseteq M \cap N$. So, we have $M \cap N \neq \phi$. For all $n \in M \cap N$, i.e., $n \in M$ and $n \in N$, and for all $a \in L$, we have $a \oplus n \subseteq M$ and $a \oplus n \subseteq N$, i.e. $a \oplus n \in p^*(M \cup N)$. Therefore $M \cap N$ is an ideal of $(L, \oplus)$. By Proposition 6.3 (iii), we can easily get that $M \oplus N \subseteq M \cap N$. For all $a \in M \cap N$, $a \in a \oplus a \subseteq M \cap N$, i.e., $M \cap N \subseteq M \oplus N$. So $M \cap N = M \oplus N$.

(ii) For all $n \in M \cap N$ and for all $a \in L$, we have that $a \oplus n \subseteq M$ or $a \oplus n \subseteq N$, then $a \oplus n \subseteq M \cup N$, i.e., $a \oplus n \in P^*(M \cup N)$. So $M \cup N$ is an ideal of $(L, \oplus)$. ∎

Proposition 6.7. *Let M be an H_v-subsemilattice of an H_v-semilattice $(L, \oplus)$ and let I be an ideal of $(L, \oplus)$. If $M \cap I$ is non-empty, then $M \cap I$ is an ideal of $(N, \oplus)$.*

Proof. The proof is straightforward. ∎

Proposition 6.8. *Let I and J be ideals of an H_v-semilattices $(L, \oplus)$ and $(S, \otimes)$, respectively. Then $I \times J$ is also an ideal of $(L \times S, *)$.*

Proof. The proof is straightforward. ∎

Proposition 6.9. *Let $L \longrightarrow S$ be a homomorphism of H_v-semilattices. If a is a fixed element of S, then $f^{-1}(a) = \{n \in L \mid f(n) = a\}$ is an ideal of $(L, \oplus)$.*

Proof. For all $m \in L$ and all $n \in f^{-1}(a)$, $f(m \oplus n) = f(m) \otimes f(n) = f(m) \otimes a = \{a\}$, i.e., $m \oplus n \subseteq f^{-1}(a)$. Therefore $f^{-1}(a)$ is an ideal of $(L, \otimes)$. ∎

Proposition 6.10. *Let $L \longrightarrow S$ be an epimorphism of H_v-semilattices. Then, we can get the following results:*

(i) *If I is an ideal of $(L, \oplus)$, then $f(I)$ is also an ideal of $(S, \otimes)$.*
(ii) *If J is an ideal of $(S, \otimes)$, then $f^{-1}(J)$, which is denoted by $f^{-1}(J) = \{a \in L \mid f(a) \in J\}$ is also an ideal of $(L, \oplus)$.*

Proof. The proof is straightforward. ∎

7. Hyperorder on an H_v-semilattice

An ordered semilattice is a triple $(S, \cdot, \leq)$, where $(S, \cdot)$ is a semilattice and "$\leq$" is an ordering on the set S with subsituation property on $(S, \cdot)$ (i.e., for an arbitrary quadruple of elements $a, b, c, d \in S$ for which $a \leq b, c \leq d$ the relation $a \cdot c \leq b \cdot d$ holds). It should be noticed that the subsituation property is equivalent to a simpler condition for an arbitrary triple of elements $a, b, c \in S$ such that $a \leq b$, the relations $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ hold.

Below we will need the following result in which we denote for t from an ordered set H; $[t)_{\leq} = \{x \in H \mid t \leq x\}$ (principal upper and determined by t).

Proposition 7.1. *Let* $(H, \cdot, \leq)$ *be an ordered commutative semilattice and define a binary hyperoperation* $*$ *on* H *in this way;* $a * b = [a \cdot b)_{\leq}$ *for any* $a, b \in H$. *Then* $(H, *)$ *is an* H_v*-semilattice.*

Proof. See [22]. ∎

Now, we define a hyperorder $\leq_L$ on an H_v-semilattice $(L, \oplus)$ as follows:

Proposition 7.2. *Let* $(L, \oplus)$, *be an* H_v*-semilattice and* $a, b \in L$. *We say that* $a \leq_L b$ *if* $a \oplus c \subseteq b \oplus c$ *for all* $c \in L$, *and* $\leq_L$ *is called the hyperorder on* H_v*-semilattice* L.

Proposition 7.3. *Let* $(L, \oplus)$ *be an* H_v*-semilattice and let* I *be an ideal of* $(L, \oplus)$. *If* $a \in I$ *and* $b \leq_L a$, *then* $b \in I$.

Proof. If $b \leq_L a$, we have $b \oplus c \cap a \oplus c \neq \phi$ for all $c \in L$. Let $c = b$. Then $b \in b \oplus b \subseteq I$, i.e., $b \in I$. ∎

Proposition 7.4. *Let* $(L, \oplus)$ *be a poset together with a binary hyperoperation* $\oplus$ *defined by* $a \oplus b = \{c \mid c \leq a, c \leq b, c \in L\} \in P^*(L)$ *for all* $a, b \in L$. *Then, the following conditions hold:*

(i) $(L, \oplus)$ *is a* H_v*-semilattice.*
(ii) *For all* $a, b \in L, a \leq_L b$ *if and only if* $a \leq b$.
(iii) *Let* J *be a nonempty subset of* L. *Then* J *is an ideal of* $(L, \oplus)$ *if and only if for all* $j \in J, x \in L$, *if* $x \leq_L j$, *then* $x \in J$.
(iv) J *is an ideal of* $(L, \leq)$ *if and only if* J *is an ideal of* $(L, \oplus)$.

Proof. The proof is straightforward. ∎

Definition 7.1. *Let* $(L, \oplus)$, *be an* H_v*-semilattice and* $a, b \in L$. *If* $a \leq_L b$ *and* $b \leq_L a$, *then say* a *is hyperequal to* b *which is denoted by* $a =_L b$.

Corollary 7.1. *Let* $(L, \oplus)$ *be an* H_v*-semilattice and* $a, b \in L$. *Then* $a =_L b$ *if and only if* $a \oplus b = b \oplus c$ *for all* $c \in L$.

Proposition 7.5. *Let* $(L, \oplus)$ *be an* H_v*-semilattice. Then* $(L, \oplus)$ *is a trivial* H_v*-semilattice if and only if* $a =_L b$ *for all* $a, b \in L$.

Proof. Suppose that $(L, \oplus)$ is a trivial H_v-semilattice, then for all $a, b \in L, a \oplus c = L$ and $b \oplus c = L$ for all $c \in L$, i.e., $a =_L b$. Conversely, $a =_L x$ for all $x \in L$, then $a \oplus b = x \oplus b$ by Corollary 7.1. Therefore, we have $a \oplus b = \cup_{x \in L}(x \oplus b)$. So $a \oplus b = \cup x \in L(x \oplus b) = \cup x \in l(\cup y \in Lx \oplus y) = L \oplus L = L$. That is to say $(L, \oplus)$ is a trivial H_v-semilattice. ∎

Proposition 7.6. *Let* $(L, \oplus)$ *be an* H_v*-semilattice. Then* $=_L$ *is an equivalence relation on* L.

Proof. The proof is straightforward. ∎

Proposition 7.7. *Let* $(L, \oplus)$ *be an* H_v*-semilattice,* $[a] = \{x \in L \mid x =_L a\}$ *and* $C_L = \{[a] \mid a \in L\}$. *We may define a binary hyperoperation on* C_L *by* $[a] * [b] = \{[n] | n \in a \oplus b\}$, *then* $(C_L, *)$ *is also an* H_v*-semilattice.*

Proof. For all $[a], [b], (c] \in C_L$, we have:

(i) Since $a \in a \oplus a$, then $[a] \in \{[n] | n \in a \oplus a\} = [a] \oplus [a]$.

(ii) Since $a \oplus b = b \oplus a$, then $[a] * [b] = \{[n] | n \in a \oplus b\} = \{[n] | n \in b \oplus a\} = [a] \oplus [b]$.
(iii) Since $(a \oplus b) \oplus c \cap a \oplus (b \oplus c) \neq \emptyset$, then

$$\begin{aligned} Q &= ([a] * [b]) * [c] \\ &= \{[n] | n \in a \oplus b\} * [c] \\ &= \cup_{n \in a \oplus b}([n] * [c]) \\ &= \cup_{n \in a \oplus}\{[m] | m \in n \oplus c\} \\ &= \{[m] | m \in (a \oplus b) \oplus c\}, \end{aligned}$$

and

$$\begin{aligned} Q' &= \{[m] | m \in a \oplus (b \oplus c)\} \\ &= \cup_{n \in b \oplus c}([a] * [c]) \\ &= [a] * \{[n] | n \in b \oplus c\} \\ &= [a] * ([b] * [c]) * [c]). \end{aligned}$$

and we have $Q \cap Q' \neq \emptyset$. Therefore $(C_L, *)$ is an H_v-semilattice. ∎

8. Semilattices obtained from H_v-semilattices

Let $(L, *)$ be an H_v-semilattice. We define the relation β^* as the smallest equivalence relation on L such that the quotient L/β^*, the set of all equivalence classes is a semilattice. The β^* is called the fundamental equivalence relation, and L/β^* is called the fundamental semilattice.

Suppose that $\beta^*(a)$ is the equivalence class containing $a \in L$. Then the product $\odot$ on the semilattice L/β^* is defined as follows:

$$\beta^*(a) \odot \beta^*(b) = \beta^*(c) \text{ for all } c \in \beta^*(a) * \beta^*(b).$$

Let $\mathcal{U}$ denote the set of all products of elements of L. We define the relation β on L as follows:

$$a \ \beta \ b \text{ if and only if } \{a, b\} \subseteq u \text{ for some } u \in \mathcal{U}.$$

Let us denote $\hat{\beta}$ the transitive closure of β. Then we can rewrite the definition of $\hat{\beta}$ on L as follows:

$$\begin{aligned} &a \ \hat{\beta} \ b \text{ if and only if there exist } z_1, \ldots, z_{n+1} \in L \text{ with } z_1 = a, \ z_{n+1} = b \\ &\text{and } \ u_1, \ldots, u_n \in \mathcal{U} \ \text{ such that } \ \{z_i, z_{i+1}\} \subseteq u_i \ (i = 1, \ldots, n). \end{aligned}$$

Then we have the following theorem.

Theorem 8.1. *The fundamental relation β^* is the transitive closure of the relation β.*

Proof. The proof is similar to the proof of Theorem 1 [43]. ∎

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