BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **31**(2) (2008), 175–183

An Application of Catalan Numbers on Cayley Tree of Order 2: Single Polygon Counting

C. H. PAH
Department of Computational and Theoretical Sciences,
Faculty of Science, International Islamic University Malaysia,
25250 Kuantan, Pahang, Malaysia.
pahchinhee@hotmail.com

Abstract. In this paper, we consider a problem on finding the number of different single connected component containing a fixed root for a given number of vertices on semi-infinite Cayley tree. The solution of this problem is the well known Catalan numbers. The result is then extended to the complete graph. Then, we gave a suitable estimate for the given problem.

2000 Mathematics Subject Classification: 82B20, 11B83

Key words and phrases: Cayley tree, phase transition, contour method, Catalan numbers

1. Introduction

In statistical physics, certain algebraic problems especially on integer lattice Z^2 could be translated in combinatoric problems by a suitable choice of geometric representation. One of the most successful idea was suggested by Peierl i.e. the so called "contour" method [8]. His argument is to consider the boundary of a spin configuration, was later given a more rigorous proof by Dobrusin and Griffith independently [3, 6] (for details, see [7, 4]). In Peierl's study of Ising model, he proposed a geometrical object "contour" i.e. a set of connected edges which separate the different "spin" for two nearest neighbours (sites) in a given configuration. The study is then transformed from configuration space into geometric representation as contour, becoming a combinatoric problem. His investigation proceed by a fundamental mathematical tools: counting, here counts the number of a class of contours. This idea turned out to be very fruitful [10, 11] in the line of research of phase transition, i.e. existence of non-uniqueness of limiting Gibbs measures.

In integer lattice Z^2, it is well known that the number of different non-equivalent polygon is less than $4 \cdot 3^{n-1}$ (the number of all polygonal paths passing through a given point), which could be found in all standard texts [7] discussed on phase

Received: March 15, 2007; *Revised:* December 12, 2007.

transition on Z^2. The Gibbs measures constructed in the Ising model on Z^2 (for details, see [7, 4]), a constant i-configuration, $\Lambda \subset Z^2$, is then

$$P^{i=+1}_{\Lambda,\beta}(V_-) < \sum_{n=4,6,\ldots}^{\infty} 4 \cdot 3^n \exp(-2\beta n). \tag{1.1}$$

where n is the contour length, β is inverse temperature and V_- is the event of the center having the negative spin value. This estimate is required in the line of the proof of existence of phase transition. The advantage of the contour method is that no explicit calculation of the probability is required.

A Cayley tree $\Gamma^k = (V, L)$ (see [1]) of order $k \geq 1$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k + 1$ edges incident to each vertex. Here V is the set of vertices and L that of edges (arcs). In Cayley tree, such an estimate as $4 \cdot 3^{n-1}$ in Z^2 is still not widely used. In [2], a similar result was given as

Lemma 1.1. *Let G be a countable graph of maximal degree k and let $N_m(x)$ be the number of connected subgraph $G' \subset G$ with $x \in V(G')$ and $|E(G')| = m$. Then*

$$N_m(x) \quad \leq \quad (k \cdot e)^m.$$

Our result would be closer compare to this result only for the case $k = 3$. In our investigation, we found a recurrent equation generating the number of polygons according to the number of vertices. The sequence of these numbers is exactly the well known Catalan numbers. Note that Fibonacci numbers appear naturally in nature while Catalan numbers appear naturally in counting. One of the example whose solution is Catalan numbers is the number of orientation to write m brackets:

$$\begin{array}{ll} m = 1 & \{()\}, \\ m = 2 & \{()(), (())\}, \\ m = 3 & \{()()(), ()(()), (())(), (()()), ((()))\}, \\ \vdots & \vdots \end{array}$$

Catalan numbers [12, 5, 9], in term of binomial coefficient is defined as:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \text{ for } n \geq 0. \tag{1.2}$$

The main theorem is then proved based on the properties of Catalan numbers.

An efficient way to calculate C_n is to write C_n in

$$C_{n+1} = \frac{2(2n+1)}{n+2} C_n. \tag{1.3}$$

It also satisfies

$$C_0 = 1, \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}. \tag{1.4}$$

There are many famous counting problems in combinatorics whose solution is given by the Catalan numbers (see examples given in [12]), including this problem. Since the investigation on Catalan numbers is well established, we do not emphasis on the properties on such sequence. This paper is mainly devoted to finding a suitable estimate for the number of polygons contain a fixed root on a Cayley tree of order 2.

2. Semi-infinite Cayley Tree

We begin the investigation by the study on the semi-infinite Cayley tree (see Figure 1). A semi-infinite Cayley tree of order 2, denoted by Γ^2_{semi} , i.e. a graph without cycles, with exactly 3 edges incident to each vertex, except a root $x^0 \in V$ which only emanates 2 edges from the vertex.

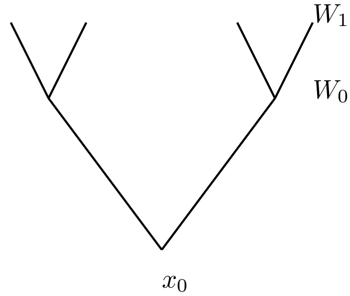

Figure 1. A semi-infinite Cayley tree J^2 where each vertex except x^0 exactly 3 edges issue whereas x^0 emanates 2 edges.

We define distance $d(x, y)$, $x, y \in V$, is the length of (i.e. the number of edges in) the shortest path connecting x with y. Two vertices x, y are called connected (or nearest neighbour) if exist single edge connect x and y, or $d(x, y) = 1$. For the fixed x^0, we set

$$W_n = \{x \in V | d(x, x^0) = n\},$$
$$V_n = \cup_{n=0} W_n$$

where $W_0 = x^0$, and

$$L_n = \{l =< x, y >\in L | x, y \in V_n\}.$$

Denote

$$S(x) = \{y \in W_{n+1} | d(x, y) = 1, x \in W_n\}.$$

The defined set is called the set of direct successors. Observe that any vertex $x \in V_n$ has 2 successors in a semi-finite Cayley of order 2.

Naively, we borrow the term "polygon" as in Z^2.

Definition 2.1. *A polygon* Λ *is a single connected component where* $\Lambda \subset V$.

Geometrically, we actually do not see any polygon as a subset of V.

Let $|\cdot|$ denote the cardinality of a set.

Definition 2.2. $N(m)$ *is the number of different polygons with* m *number of vertices that could be constructed containing a fixed root* x^0.

One could use a number system to represent the semi-infinite Cayley of order 2. We fix root $x^0 = x_1$ as 1, then from each vertex label k, we emanate to two numbers $2k$ and $2k+1$ as two successors, so that any k and $2k$ or k and $2k+1$ are nearest neighbour. In order to generate each polygon containing x^0, here we list each sequence of combinations by following algorithm:

$$\{x_i : x_1 = 1 \text{ and } \forall x_k, \text{exist } x_j, j < k, \text{ such that } x_k = 2x_j \text{ or } x_k = 2x_j + 1.\}$$

The first four terms can be generated as follows:

$$\begin{array}{ll}
N(m=1) = 1, & \{(1)\} \\
N(m=2) = 2, & \{(1,2),(1,3)\} \\
N(m=3) = 5, & \{(1,2,3),(1,2,4),(1,2,5),(1,3,6),(1,3,7)\} \\
N(m=4) = 14, & \{(1,2,3,4),(1,2,3,5),(1,2,3,6),(1,2,3,7),(1,2,4,5), \\
& (1,2,4,8),(1,2,4,9),(1,2,5,10),(1,2,5,11),(1,3,6,7), \\
& (1,3,6,12),(1,3,6,13),(1,3,7,14),(1,3,7,15)\}
\end{array}$$

One could find $N(m)$ easily in the next section.

3. Main Result

3.1. Semi-infinite Cayley tree of order 2.

Lemma 3.1. *The number of polygons with* m *number of vertices which containing a root* x^0, $N(m)$ *is given by recursion below:*

$$N(m) = \sum_{r=0}^{m-1} N(m-r-1)N(r), N(0) = 1 \quad \text{and} \quad N(1) = 1. \tag{3.1}$$

Proof. First we divide the problem of finding m number of vertices that contain a root x^0 into

(i) r number of vertices which containing a root y^0 which is a successor of x^0 , i.e, $N(r)$; and

(ii) $m-r-1$ number of vertices which containing a root z^0 which is another successor of x^0, i.e. $N(m-r-1)$ (see Figure 2).

The total combination is $N(r)\cdot N(m-r-1)$. Then $N(m)$ is directly sum over r from 0 to $m-1$, the multiplication of all combination of $N(m-r-1)N(r)$. We define $N(0) = 1$ and $N(1) = 1$ is simply a result from observation. ∎

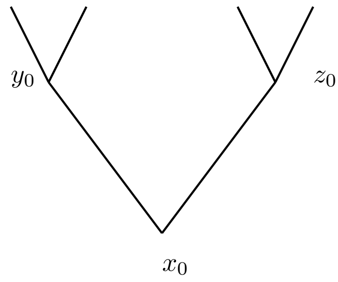

Figure 2. y^0 connect to r vertices and z^0 connect to $m-r-1$ vertices.

The first 10 terms, starting from $N(1)$, are listed as follow based on formula above:

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots.$$

and this is the well-known Catalan numbers. From (1.4), with initial value $N(0) = 1$ and $N(1) = 1$, evidently

Theorem 3.1. *$N(m)$ is the Catalan numbers C_m.*

This is not a surprising result because the applications of a binary operator can be represented in terms of a binary tree. It follows that C_n is the number of rooted ordered binary trees with $n+1$ leaves. This problem is identical to the number of polygons that we defined above, $N(m)$. For any m vertices connected component, there are $m+1$ leaves. Below, we shall find an estimate base on the properties of Catalan numbers given in previous section.

From equation (1.4), we have the following proposition.

Proposition 3.1.

$$\frac{N(m)}{N(m-1)} < 4. \tag{3.2}$$

Proof.

$$\frac{N_{n+1}}{N_n} = \frac{2(2n+1)}{n+2} < \frac{2(2n+4)}{n+2} = 4.$$

∎

The main theorem that we would like to prove is the estimate of $N(m)$.

Theorem 3.2. *The inequality of $N(m)$ is given as*

$$\frac{4^{m-1}}{m^{3/2}} \leq N(m) \leq \frac{4^{m-1}}{m}. \tag{3.3}$$

Proof. We are going to prove this by induction for two cases. For $m = 1$, $1 = N(1) = 1$, the inequality (3.3) holds.

Suppose that $N(m) \leq 4^{m-1}/m$. From (1.4),

$$\begin{aligned} N(m+1) &= \frac{2(2m+1)}{m+2} N(m) \\ &< \frac{2(2m+1)}{m+2} \frac{4^{m-1}}{m} \end{aligned}$$

$$< \quad 2\frac{2m}{m+1}\frac{4^{m-1}}{m}$$
$$= \quad \frac{4^m}{m+1}, m > 1.$$

Suppose that $N(m) \geq \frac{4^{m-1}}{m^{3/2}}$ and multiply both side by $\frac{2(2m+1)}{m+2}$.

$$N(m) \cdot \frac{2(2m+1)}{m+2} \geq \frac{4^{m-1}}{m^{3/2}} \cdot \frac{2(2m+1)}{m+2}$$
$$N(m+1) \geq \frac{4^{m-1}}{m^{3/2}} \cdot \frac{2(2m+1)}{m+2}$$
$$> \frac{4^{m-1}}{m^{3/2}} \cdot \frac{4m}{m+2}$$
$$> \frac{4^m}{(m+1)^{3/2}}.$$ ∎

From (1.3), one could easily verify that

Corollary 3.1.

$$\lim_{m\to\infty} \frac{N(m)}{N(m-1)} = 4. \tag{3.4}$$

We are also include here a less motivated result.

Corollary 3.2. *The number of all possible polygons with maximally* m *vertices which containing a root* x^0, *is*

$$\sum_{i=1}^{m} N(i) < \frac{4^m}{3}. \tag{3.5}$$

Proof. From Proposition 3.1 and geometric progression;

$$\sum_{i=1}^{m} N(i) < \sum_{i=1}^{m} 4^{i-1} = \frac{4^m - 1}{3} < \frac{4^m}{3}.$$ ∎

3.2. Complete graph. Similarly, we derive the $\tilde{N}(m)$ which carry the same meaning as previous section but on the complete graph of Cayley of order 2, Γ^2.

Definition 3.1. $\tilde{N}(m)$ *is the number different polygons* Λ, *where* $\Lambda \subset \Gamma^2$ *with* $|\Lambda| = m$ *number of vertices that could be constructed containing a root* x^0.

Note that $N(m)$ without tilde is merely the Catalan numbers used in previous section.

Lemma 3.2. *The number of different polygons* $\tilde{N}(m)$ *with* m *number of vertices which contain a root* x^0, *is given by recursion below:*

$$\tilde{N}(m) = \sum_{r=1}^{m} N(m-r)N(r), N(0) = 1 \quad \text{and} \quad N(1) = 1. \tag{3.6}$$

Proof. First, we decompose the problem of finding the number of polygons of m number of vertices containing a root x^0 into counting:

(i) r number of vertices which containing x^0 , i.e, $N(r)$ and
(ii) $m-r$ number of vertices which containing a root y which is another successor of x^0, i.e. $N(m-r)$ (see Figure 3).

The former $N(r)$ must always count x^0, then it is not allow to be $N(0)$. The total $\tilde{N}(m)$ is then the sum of all $N(m-r)N(r)$ where r range from 1 to m. ∎

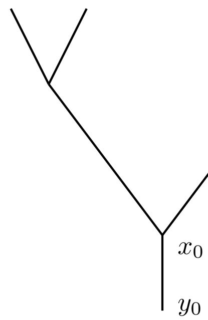

Figure 3. x^0 connect to r vertices (top) and y^0 connect to $m-r$ vertices (bottom).

The first ten terms of $\tilde{N}(m)$ are listed:

$$1, 3, 9, 28, 90, 297, 1001, 3432, 11934, 41990, ...$$

In this case, $\tilde{N}(0)$ is undefined. The recursion (3.6) allow us to state the following lemma.

Lemma 3.3.

$$\tilde{N}(m) = N(m+1) - N(m), N(0) = 1. \tag{3.7}$$

Proof. From Lemma 3.1 and Lemma 3.2,

$$\tilde{N}(m) = \sum_{r=0}^{m} N(m-r)N(r) - N(m) = N(m+1) - N(m).$$ ∎

From lemma above, it is not difficult to prove the result below.

Lemma 3.4.

$$\tilde{N}(m) = \frac{3m}{m+2}N(m). \tag{3.8}$$

Proof. From Equation (1.3) and Lemma 3.3,

$$\begin{aligned}\tilde{N}(m) &= N(m+1) - N(m)\\ &= \frac{2(2m+1)}{m+2}N(m) - N(m)\\ &= \frac{3m}{m+2}N(m).\end{aligned}$$ ∎

From Theorem 3.3 and Lemma 3.4, it is not difficult to deduce the following result:

$$\frac{3}{(m+2)\sqrt{m}}4^{m-1} \leq \tilde{N}(m) \leq \frac{3}{m+2}4^{m-1}. \tag{3.9}$$

and

$$\lim_{m\to\infty} \frac{\tilde{N}(m)}{\tilde{N}(m-1)} = 4.$$

We can see that both $N(m)$ and $\tilde{N}(m)$ proportional to 4^m. As m increase, $N(m)$ increase by not more than 4 times of previous $N(m-1)$. The exact value of $N(m)$ would be close to the form $A_m 4^m$ where A_m are some positive constant depend on m. Note that the Catalan numbers has the order of $O\left(\frac{4^{m-1}}{m^{3/2}}\right)$ which the lower bound is found to be $\frac{4^{m-1}}{m^{3/2}}$ and the asymptote of the Catalan numbers is given in [14] as

$$N(m) \sim \frac{4^m}{m^{3/2}\sqrt{\pi}}.$$

Since $\tilde{N}(m) \leq \frac{3}{m+2}4^{m-1} < 3\cdot 4^{m-1}$, we can choose a simple estimate i.e. $3\cdot 4^{m-1}$ for the problem on Cayley tree of order 2 stated in introduction. Carefully identifying each term in the Lemma 1.1 , the estimate obtained $3\cdot 4^{m-1}$ is less $(3\cdot e)^{m+1}$ from Lemma 1.1 obtained for the same problem. Somehow $3\cdot 4^{m-1}$ is much simple and closer for the purpose in the study discussed in the introduction.

4. Conclusion

We proved that a solution to the problem of finding number of different "polygon" containing a fixed root for a given number of vertices on semi-infinite Cayley tree of order 2 is exactly the well known Catalan numbers. We gave two suitable estimates as

$$N(m) \leq \frac{4^{m-1}}{m}$$

and

$$\tilde{N}(m) \leq 3\cdot\frac{4^{m-1}}{m+2} < 3\cdot 4^{m-1}.$$

Acknowledgements. The author thanks the referee for the comments. This research is funded by the SAGA Fund P77c by the Ministry of Science, Technology and Innovation (MOSTI) through the Academy of Sciences Malaysia(ASM). The author is grateful to the Faculty of Science of IIUM for the facilities provided to him in undertaking his doctorate programme. The author thanks his supervisor Prof. Dr. Nasir Ganikhodjaev for suggesting the problem and also his valuable discussions.

References

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic, London, 1982.
[2] C. Borgs, *Statistical Physics Expansion Methods in Combinatorics and Computer Science*, http://research.microsoft.com/∼borgs/CBMS.pdf, March 22, 2004.
[3] R. L. Dobrushin, Existence of a phase transition in the two dimensional and three-dimensional Ising model. *Soviet Phys. Doklady* **10**(1965), 111–113 .
[4] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, Walte de Gruyter, Berlin, 1998.
[5] R. L. Graham, D. E. Knuth and O. Patashnik, Exercise 9.8 in *Concrete Mathematics: A Foundation for Computer Science*, Second ed., Reading, MA: Addison-Wesley, 1994.

[6] R. B. Griffiths, Peierls' proof of spontaneous magnetization of a two-dimensional Ising ferromagnet. *Phys. Rev.* **A136**(1964), 437–439.
[7] R. A. Minlos, *Introduction To Mathematical Statistical Physics*, Amer. Math. Soc., Providence, RI, 2000.
[8] R. Peierls, On Ising model of ferro magnetism. *Proc. Cambridge Phil. Soc.* **32**(1936), 477–481.
[9] S. Pemmaraju and S. Skiena, *Computational Discrete Mathematics*, Cambridge Univ. Press, Cambridge, 2003
[10] S. A. Pirogov and Ya. G. Sinai, Phase diagrams of classical lattice systems, I. *Theor. Math. Phys.* **25**(1975), 1185–1192 .
[11] S. A. Pirogov and Ya. G. Sinai, Phase diagrams of classical lattice systems, II. *Theor. Math. Phys.* **26**(1976), 39–49 .
[12] R. P. Stanley, *Catalan addendum*, `http://math.mit.edu/~rstan/ec/catadd.pdf`, version of 6 October 2008; 72 pages.
[13] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, Second edition, Cambridge Univ. Press, Cambridge, 2001.
[14] I. Vardi, *Computational recreations in Mathematica*, Addison-Wesley, Redwood City, CA, 1991.